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Global Asymptotic Stability for a Fourth-order Rational Difference Equation

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Abstract

In this work, we investigate the global asymptotic stability of the following fourth-order rational difference equation:

$$x_{n+1} = \frac{x_n x_{n-1}^b + x_{n-2}^b + x_{n-3}^b + a}{x_n x_{n-2}^b + x_{n-1}^b + x_{n-3}^b + a} \qquad (1)$$

where $a, b \in [0, \infty)$ and the initial values x_{-3}, x_{-2}, x_{-1}, x_0 are arbitrary positive real numbers.

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1 Introduction

Rational difference equations can be look very simple; but in fact their global behaviors are mostly very complicated.
Li and Zhu[1] are obtained a sufficient condition to guarantee the global asymptotic stability of the following recursive sequence:

$$x_{n+1} = \frac{x_n x_{n-1}^b + x_{n-2}^b + a}{x_{n-1}^b + x_n x_{n-2}^b + a} \qquad (2)$$

where $a, b \in [0, \infty)$ and the initial values $x_{-2}, x_{-1}, x_0 \in (0, \infty)$.
Li[2] use a new method to investigate the qualitative properties of the following rational difference equation:

$$x_{n+1} = \frac{x_n x_{n-1} x_{n-3} + x_n + x_{n-1} + x_{n-3} + a}{x_n x_{n-1} + x_n x_{n-3} + x_{n-1} x_{n-3} + 1 + a} \quad , \quad n = 0, 1, ... \qquad (3)$$

Li [3] use a new method to investigate the global behavior of the following rational difference equation:

$$x_{n+1} = \frac{x_{n-1}x_{n-2}x_{n-3} + x_{n-1} + x_{n-2} + x_{n-3} + a}{x_{n-1}x_{n-2} + x_{n-1}x_{n-3} + x_{n-2}x_{n-3} + 1 + a} \quad , \quad n = 0, 1, ... \qquad (4)$$

Bayram and Daş [4] investigate the global behavior of the following nonlinear recursive sequence:

$$x_{n+1} = \frac{x_n x_{n-2}^b + x_{n-3}^b + x_{n-1}^b + a}{x_n x_{n-3}^b + x_{n-2}^b + x_{n-1}^b + a} \quad , \quad n = 0, 1, 2, \ldots \qquad (5)$$

To be motivated by the above studies, in this paper, we consider the following nonlinear difference equation:

$$x_{n+1} = \frac{x_n x_{n-1}^b + x_{n-2}^b + x_{n-3}^b + a}{x_n x_{n-2}^b + x_{n-1}^b + x_{n-3}^b + a} \quad , \quad n = 0, 1, 2, \ldots \qquad (6)$$

where $a, b \in [0, \infty)$ and the initial values $x_{-3}, x_{-2}, x_{-1}, x_0$ are arbitrary positive real numbers.
We review some results which will be useful in our investigation.

Definition 1.1 *Let $I \subset R$ and $f : I^{k+1} \to I$ be a continuously differentiable function. Then for every set of initial conditions $x_{-k}, x_{-k+1}, ..., x_0 \in I$, the difference equation*

$$x_{n+1} = f(x_n, x_{n-1}, ..., x_{n-k}) \quad , \quad n = 0, 1, ... \qquad (7)$$

has an unique solution $\{x_n\}_{n=-k}^{\infty}$.
A point $\bar{x} \in I$ is called an equilibrium point of Eq(7) if $\bar{x} = f(\bar{x}, \bar{x}, ..., \bar{x})$.

Definition 1.2 *Let $\bar{x}$ be the equilibrium point of the Eq(7).*

(i) The equilibrium point $\bar{x}$ of Eq(7) is called locally stable if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x_{-k}, x_{-k+1}, ..., x_0 \in I$ with $|x_{-k} - \bar{x}| + |x_{-k+1} - \bar{x}| + ... + |x_0 - \bar{x}| < \delta$, we have

$$|x_n - \bar{x}| < \varepsilon \quad for\ all\ n \geq -k \qquad (8)$$

(ii) The equilibrium point $\bar{x}$ of Eq(7) is called a global attractor if for every $x_{-k}, x_{-k+1}, ..., x_0 \in I$, we have

$$\lim_{n\to\infty} x_n = \bar{x} \qquad (9)$$

(iii) The equilibrium point $\bar{x}$ of Eq(7) is called global asymptotically stable if it is locally stable and a global attractor.

Definition 1.3 *Let $\bar{x}$ be an equilibrium point of Eq (7). A positive semicycle of a solution $\{x_n\}_{n=-k}^{\infty}$ of Eq (7) consists of a "string" of terms $\{x_\ell, x_{\ell+1}, \ldots, x_m\}$ all greater than or equal to $\bar{x}$, with $\ell \geq -k$ and $m \leq \infty$ s*

$$\textit{either} \quad \ell = -k \quad \textit{or} \quad \ell > -k \quad \textit{and} \quad x_{\ell-1} < \bar{x}$$
$$\textit{and}$$
$$\textit{either} \quad m = \infty \ \textit{or} \ m < \infty \quad \textit{and} \quad x_{m+1} < \bar{x}$$

A negative semicycle of a solution $\{x_n\}_{n=-k}^{\infty}$ of Eq (7) consists of a "string" of terms $\{x_\ell, x_{\ell+1}, \ldots, x_m\}$ all less than $\bar{x}$, with $\ell \geq -k$ and $m \leq \infty$ such that

$$\textit{either} \quad \ell = -k \quad \textit{or} \quad \ell > -k \quad \textit{and} \quad x_{\ell-1} \geq \bar{x}$$
$$\textit{and}$$
$$\textit{either} \quad m = \infty \ \textit{or} \ m < \infty \quad \textit{and} \quad x_{m+1} \geq \bar{x}$$

2 Several Lemmas

It is easy to see that the positive equilibrium point $\bar{x}$ of Eq (6) satisfies

$$x_{n+1} = \frac{\bar{x}^{b+1} + 2\bar{x}^b + a}{\bar{x}^{b+1} + 2\bar{x}^b + a} \tag{10}$$

from which one can see that Eq (6) has an unique positive equilibrium $\bar{x} = 1$.

Lemma 2.1 *A positive solution $\{x_n\}_{n=-3}^{\infty}$ of Eq (6) is eventually equal to 1 if and only if*

$$(x_0 - 1)(x_{-1} - x_{-2}) = 0 \tag{11}$$

Proof. Assume that (11) holds. Then, according to Eq (6), it is easy to see that $x_n = 1$ for $n \geq 1$.
Conversely, assume that

$$(x_0 - 1)(x_{-1} - x_{-2}) \neq 0 \tag{12}$$

Then, we must show that

$$x_n \neq 1 \quad \textit{for any} \quad n \geq 1 \tag{13}$$

Assume the contrary that for some $N \geq 1$,

$$x_N = 1 \quad \textit{and} \quad x_n \neq 1 \quad \textit{for} \quad -1 \leq n \leq N-1 \tag{14}$$

Clearly,

$$1 = x_N = \frac{x_{N-1}x_{N-2}^b + x_{N-3}^b + x_{N-4}^b + a}{x_{N-1}x_{N-3}^b + x_{N-2}^b + x_{N-4}^b + a} \tag{15}$$

which implies $x_{N-2} = x_{N-3}$ and by (12), $N \geq 2$. Thus, from

$$\begin{aligned} x_{N-3} &= x_{N-2} = \frac{x_{N-3}x^b_{N-4} + x^b_{N-5} + x^b_{N-6} + a}{x_{N-3}x^b_{N-5} + x^b_{N-4} + x^b_{N-6} + a} \\ &\Rightarrow (x_{N-3} - 1)(x^b_{N-5}(x_{N-3} + 1) + x^b_{N-6} + a) = 0 \end{aligned} \tag{16}$$

one can obtain that from $(x^b_{N-5}(x_{N-3} + 1) + x^b_{N-6} + a) \neq 0$, $x_{N-3} = 1$ which contradicts (14).

Lemma 2.2 *Let* $\{x_n\}_{n=-3}^{\infty}$ *be a positive solution of Eq (6) which is not eventually equal to 1. Then the following statements are true:*

(i) $(x_{n+1} - x_n)(x_n - 1) < 0 \ , \ for \ \ n \geq 0$

(ii) $(x_{n+1} - 1)(x_n - 1)(x_{n-1} - x_{n-2}) > 0 \ , \ for \ \ n \geq 0$

(iii) $(x_{n+1} - 1)(x_n - 1)(x_{n-2} - 1) > 0 \ , \ for \ \ n \geq 2$

Lemma 2.3 *If* $x_{-2} < x_{-1} < x_0 < 1$*, then* $\{x_n\}_{n=-3}^{\infty}$ *has a negative semicycle with an infinite number of terms and it monotonically tends to the positive equilibrium point* $\bar{x} = 1$*.*

Proof. If $x_{-2} < x_{-1} < x_0 < 1$, from Lemma 2.2.(i) and (ii), for $n \geq -3$

$$x_0 < x_1 < ... < x_{n-1} < x_n < 1 \tag{17}$$

Clearly,$\{x_n\}_{n=-3}^{\infty}$ has a negative semicycle with an infinite number of terms. Furthermore, we know that the positive solution is strictly increasing for $n \geq 0$. So the limit

$$\lim_{n\to\infty} x_n = L \tag{18}$$

exist and finite. Taking the limit on both sides of Eq (6), we have

$$L = \frac{L^{b+1} + 2L^b + a}{L^{b+1} + 2L^b + a} = 1 \tag{19}$$

We can easily see that $\{x_n\}_{n=-3}^{\infty}$ tends to the positive equilibrium point $\bar{x} = 1$.

Lemma 2.4 *Let* $\{x_n\}_{n=-3}^{\infty}$ *be a positive solution of Eq (6) which is not eventually less than or equal to 1. Then, with the possible exception of the first semicycle, the following affirmations hold.*

(a) Every positive semicycle consists of three or one terms;
Every negative semicycle consists of two or one terms.

(b) *The positive and negative semicycles of Eq (6) has the form* $3^{+}, 1^{-}, 1^{+}, 2^{-}$.

Theorem 2.5 *Let* $a \in [0, \infty)$ *and* $b > 0$. *Then the positive equilibrium point of Eq (6) is globally asymptotically stable.*

Proof. We must show that the positive equilibrium point is locally asymptotically stable and global attractor. The linearized equation of Eq (6) is

$$z_{n+1} = 0.z_n + 0.z_{n-1} + 0.z_{n-2} + 0.z_{n-3} \tag{20}$$

By virtue of [[4],Remark 1.3.7], $\bar{x}$ is locally asymptotically stable. Now we must show that every positive solution $\{x_n\}_{n=-3}^{\infty}$ of Eq (6) converges to 1 as $n \to \infty$. That is,

$$\lim_{n\to\infty} x_n = \bar{x} \tag{21}$$

If the solution is nonoscillatory about the positive equilibrium point $\bar{x}$ of Eq (6), then from Lemma 2.1 and Lemma 2.2, the solution is either equal to 1 or eventually negative one which has an infinite number of terms and monotonically tends to the positive equilibrium point $\bar{x}$ of Eq (6), and so Eq (21) holds. Therefore, it suffices to prove that Eq (21) holds for the solution to be strictly oscillatory. Consider now $\{x_n\}_{n=-3}^{\infty}$ to be strictly oscillatory about the positive equilibrium point $\bar{x}$ of Eq (6). By virtue of Lemmas 2.2(ii) and Lemmas 2.4, the $\{x_n\}_{n=-3}^{\infty}$ solution of Eq (6) has the positive and negative semicycles of the form $3^{+}, 1^{-}, 1^{+}, 2^{-}$. So we have the following sequences:

$$\{x_{p+7n}, x_{p+7n+1}, x_{p+7n+2}\}^{+}, \{x_{p+7n+3}\}^{-}, \{x_{p+7n+4}\}^{+}, \{x_{p+7n+5}, x_{p+7n+6}\}^{-}$$

We now have the following assertions:

(i) $x_{p+7n} > x_{p+7n+1} > x_{p+7n+2}$, $x_{p+7n+6} > x_{p+7n+5}$

(ii) $x_{p+7n+2}x_{p+7n+3} > 1$, $x_{p+7n+3}x_{p+7n+4} < 1$, $x_{p+7n+4}x_{p+7n+5} > 1$, $x_{p+7n+6}x_{p+7n+7} < 1$

inequality (i) can be easily seen from Lemma 2.2.(i) for $n = 0, 1, ...$

From the observations of

$$\begin{aligned} x_{p+7n+3} &= \frac{x_{p+7n+2}x^{b}_{p+7n+1} + x^{b}_{p+7n} + x^{b}_{p+7n-1} + a}{x_{p+7n+2}x^{b}_{p+7n} + x^{b}_{p+7n+1} + x^{b}_{p+7n-1} + a} \\ &> \frac{x_{p+7n+2}x^{b}_{p+7n+1} + x^{b}_{p+7n} + x^{b}_{p+7n-1} + a}{x^{2}_{p+7n+2}x^{b}_{p+7n} + x^{b}_{p+7n+1}x_{p+7n+2} + x^{b}_{p+7n+1}x_{p+7n+2} + ax_{p+7n+2}} \\ &= \frac{1}{x_{p+7n+2}} \end{aligned}$$

$x_{p+7n+3}x_{p+7n+4} < 1$, $x_{p+7n+4}x_{p+7n+5} > 1$ and $x_{p+7n+6}x_{p+7n+7} < 1$ can be easily shown. From inequality (i) and (ii),

$$\begin{aligned} x_{p+7n+7} &< \frac{1}{x_{p+7n+6}} < \frac{1}{x_{p+7n+5}} < x_{p+7n+4} < \\ \frac{1}{x_{p+7n+3}} &< x_{p+7n+2} < x_{p+7n+1} < x_{p+7n} \end{aligned} \tag{22}$$

From equation (22), we can see that $\{x_{p+7n}\}_{n=0}^{\infty}$ is decreasing with lower bound 1. So the limit

$$\lim_{n\to\infty} x_{p+7n} = L \tag{23}$$

exist and are finite. From equation(22), we obtain

$\lim_{n\to\infty} x_{p+7n+7} = \lim_{n\to\infty} x_{p+7n+4} = \lim_{n\to\infty} x_{p+7n+2} = \lim_{n\to\infty} x_{p+7n+1} = L$
$\lim_{n\to\infty} x_{p+7n+6} = \lim_{n\to\infty} x_{p+7n+5} = \lim_{n\to\infty} x_{p+7n+3} = \frac{1}{L}$,

$$x_{p+7n+7} = \frac{x_{p+7n+6}x^b_{p+7n+5} + x^b_{p+7n+4} + x^b_{p+7n+3} + a}{x_{p+7n+6}x^b_{p+7n+4} + x^b_{p+7n+5} + x^b_{p+7n+3} + a} \tag{24}$$

If we take the limits on both sides of the equation (24),

$$L = \frac{\frac{1}{L^{b+1}} + L^b + \frac{1}{L^b} + a}{L^{b-1} + \frac{2}{L^b} + a} \tag{25}$$

which imply that $L = 1$.

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