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Common Right Multiples in $\mathbb{K}P$ of Linear Combinations of Generators of the Positive Monoid P of Thompson's Group F

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Abstract

We show that any two elements of $\mathbb{K}P$ which can be written as a linear combination of generators of the positive monoid P of the group F have nonzero common right multiples in $\mathbb{K}P$.

1 Introduction

Let S be a semigroup, and let $B(S)$ denote the set of all bounded, real valued functions on S. For each $f \in B(S)$ and for each $a \in S$, define $f_a \in B(S)$ by $f_a(x) = f(ax)$ for each $x \in S$. We say that S is *left amenable* if there exists a function $\mu : B(S) \to \mathbb{R}$ such that for all $f, g \in B(S)$, for each $a \in S$, and for all $r \in \mathbb{R}$:

1. $\sup_{x \in S} f(x) \geq \mu(f) \geq \inf_{x \in S} f(x)$
2. $\mu(f_a) = \mu(f)$
3. $\mu(f + g) = \mu(f) + \mu(g)$
4. $\mu(rf) = r\mu(f)$.

We say that a group G is *amenable* if it is left amenable.

We say that a semigroup S *has common right multiples* if for each pair of elements $b_1, b_2 \in S$, there exist elements $d_1, d_2 \in S$ such that $b_1 d_1 = b_2 d_2$. One can analogously define the notion of *common right multiples* in a ring. A theorem of Ore states that if M is a cancellative monoid which has common right multiples, then M embeds as a submonoid into a group G such that the following two conditions are satisfied:

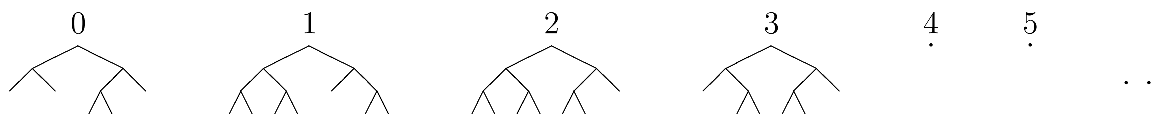
(i) For each $g \in G$, there exists $x, y \in M$ such that $g = xy^{-1}$.

(ii) If M has monoid presentation $\langle A \mid R \rangle$, then G is isomorphic to the group defined by the presentation $\langle A \mid R \rangle$.

The group G is called the *group of right fractions* of M, and M is called the *positive monoid* of G. We define Richard Thompson's group F to be the group of right fractions of the monoid P which is given by the presentation

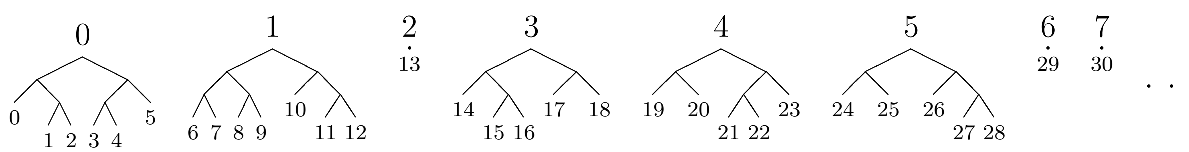
$$\langle x_0, x_1, x_2, \ldots \mid x_n x_m = x_m x_{n+1} \ \text{ for } \ n > m \rangle \,.$$

We define a (k, m)-*binary forest* $\mathcal{F}$ to be a sequence (starting the count at zero) of rooted binary trees such that $\mathcal{F}$ has a total of exactly m carets, and such that for each $i \geq k+1$, the i^{th} tree of $\mathcal{F}$ is the trivial tree, which consists of just a single point. The following is an example of a (3,21)-binary forest.

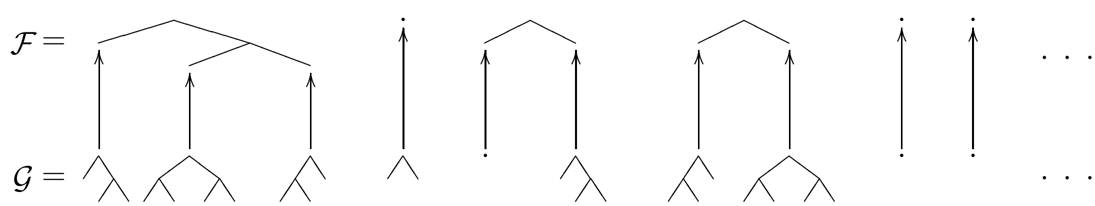


The set of all (k, m)-binary forests is denoted by $S_{k,m}$. If $\mathcal{F}$ is a binary forest, then we denote the i^{th} tree of $\mathcal{F}$ by $\tau_{\mathcal{F}}(i)$.

We enumerate the leaves of a binary forest $\mathcal{F}$ from left to right. The following example shows the enumeration on the leaves of a (5,23)-binary forest.

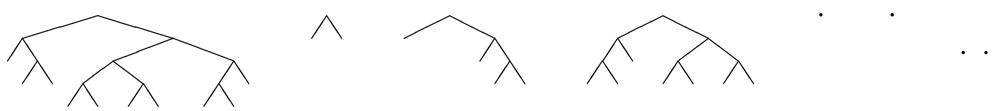


Using this enumeration on the leaves of a binary forest, we define a multiplication on the set of all binary forests. Given two binary forests $\mathcal{F}$ and $\mathcal{G}$, construct the binary forest $\mathcal{F}\mathcal{G}$ by attaching the i^{th} tree of $\mathcal{G}$ to the i^{th} leaf of $\mathcal{F}$. For example, we multiply a (3,4)-binary forest $\mathcal{F}$ with a (7,15)-binary forest $\mathcal{G}$:



to get the following binary forest $\mathcal{FG}$:

$\mathcal{FG}$ =



With this multiplication, the set of all binary forests forms a monoid which is isomorphic to the monoid P. More specifically, for $n \geq 0$, the generator x_n of P is the binary forest all of whose trees are trivial except for the n^{th} tree, which consists of a single caret.

In [7], Tamari proves the following theorem. A proof is also given by Engel in [4].

Theorem 1. *Let M be a left amenable, cancellative monoid, and let $\mathbb{K}$ be a field. Then for each pair of nonzero elements $\alpha, \beta \in \mathbb{K}M$ there exist nonzero elements $\gamma, \rho \in \mathbb{K}M$ such that $\alpha\gamma = \beta\rho$.*

Geoghegan has conjectured that the group F is an example of a finitely presented, nonamenable group which has no free subgroup on two generators [5]. In [1], Brin and Squier show that F has no free subgroup on two generators. However, the question of whether or not the group F is amenable is still open, and has resisted the efforts of mathematicians for over twenty years [2]. Thus, one can ask if for any field $\mathbb{K}$, does the group ring $\mathbb{K}F$ have nonzero common right multiples, thereby satisfying the conclusion of Theorem 1. In Section 2, we show that the group ring $\mathbb{K}F$ has nonzero common right multiples if and only if the monoid ring $\mathbb{K}P$ has nonzero common right multiples. In Section 4 we show that any two nonzero elements $\alpha, \beta \in \mathbb{K}P$ which can be written as a linear combination of generators of the positive monoid P have nonzero common right multiples in $\mathbb{K}P$. Moreover, we show that that if $\gamma, \rho \in \mathbb{K}P$ are such that $\alpha\gamma = \beta\rho$, then γ and ρ are linear combinations of elements from $S_{k+1,m}$.

2 The Relationship Between $\mathbb{K}G$ and $\mathbb{K}M$

Throughout this section, we assume that M is a cancellative monoid which has common right multiples, that G is the group of right fractions of the monoid M, and that $\mathbb{K}$ is a field such that $\mathbb{K}M$ has no zero divisors. Using induction on $|H|$, together with the fact that G is the group of right fractions of the monoid M, one can show that if H is a finite subset of G, then there exists $q \in M$ such that $Hq \subseteq M$. The proof of the following lemma is left to the reader.

Lemma 1. *Let G be the group of right fractions of a monoid M. Let $z_1, \ldots, z_n \in \mathbb{K}G$. Then there exists $q \in M$ such that for each $i \in \{1, \ldots, n\}$, we have that $z_i q \in \mathbb{K}M$.*

Lemma 2. *Let G be the group of right fractions of a monoid M. Then $\mathbb{K}G$ has nonzero common right multiples if and only if $\mathbb{K}M$ has nonzero common right multiples.*

Proof. Assume that $\mathbb{K}G$ has nonzero common right multiples. Let z_1 and z_2 be nonzero elements of $\mathbb{K}M$. Since $\mathbb{K}M \subseteq \mathbb{K}G$, and since $\mathbb{K}G$ has nonzero common right multiples, then there exist $u_1, u_2 \in \mathbb{K}G$ such that $z_1 u_1 = z_2 u_2 \neq 0$. By Lemma 1, there exists $q \in M$ such that $u_1 q, u_2 q \in \mathbb{K}M$. Since $z_1 u_1 \neq 0$, $z_2 u_2 \neq 0$, and $q \neq 0$, and since $\mathbb{K}G$ has no zero divisors, then $z_1 u_1 q \neq 0$ and $z_2 u_2 q \neq 0$. Thus, $u_1 q$ and $u_2 q$ are elements of $\mathbb{K}M$ such that $z_1(u_1 q) = z_2(u_2 q) \neq 0$. Hence, $\mathbb{K}M$ has nonzero common right multiples.

Conversely, assume that $\mathbb{K}M$ has nonzero common right multiples. Let y_1 and y_2 be nonzero elements of $\mathbb{K}G$. By Lemma 1, there exists $b \in M$ such that $y_1 b, y_2 b \in \mathbb{K}M$. Since $y_1 b, y_2 b \in \mathbb{K}M$, and since $\mathbb{K}M$ has nonzero common right multiples, then there exist $v_1, v_2 \in \mathbb{K}M \subseteq \mathbb{Z}G$ such that $y_1 b v_1 = y_2 b v_2 \neq 0$. Hence, $\mathbb{K}G$ has nonzero common right multiples. ☐

3 Properties of $S_{k,m}$

The set $S_{0,m}$ is essentially the set of all rooted binary trees, each of which has exactly m carets. It is well known that the number of elements in this set is the m^{th} catalan number $\dfrac{(2m)!}{m!(m+1)!}$ [6]. The following lemma is proven in [3].

Lemma 3. *Let $m \geq 0$.*

(i) $|S_{1,m}| = |S_{0,m+1}|$.

(ii) *For each $k \geq 1$,* $|S_{k+1,m}| = |S_{k,m+1}| - |S_{k-1,m+1}|$.

Theorem 2. *For each $k \geq 0$, and for each $m \geq 0$,* $|S_{k,m}| = \dfrac{(k+1)(2m+k)!}{(m+k+1)!m!}$.

Proof. We prove this by induction on k. If $k = 0$, then we see that

$$|S_{0,m}| = \frac{(2m)!}{m!(m+1)!} = \frac{(0+1)(2m+0)!}{m!(m+0+1)!}$$

Assume that $k = 1$. It follows by Lemma 3 that $|S_{1,m}| = |S_{0,m+1}|$. Thus, we see that

$$|S_{1,m}| = |S_{0,m+1}| = \frac{(2(m+1))!}{(m+1)!(m+1+1)!} = \frac{(2m+2)!}{(m+1)!(m+2)!}.$$

Since

$$2 = 2\left(\frac{m+1}{m+1}\right) = \frac{2m+2}{m+1} = \frac{(2m+2)(2m+1)!m!}{(m+1)(2m+1)!m!} = \frac{(2m+2)!m!}{(2m+1)!(m+1)!}$$

then

$$\frac{(2m+2)!}{(m+1)!} = \frac{2(2m+1)!}{m!}$$

which implies that

$$|S_{1,m}| = \frac{(2m+2)!}{(m+1)!(m+2)!} = \frac{2(2m+1)!}{m!(m+2)!} = \frac{(1+1)(2m+1)!}{(m+1+1)!m!}.$$

Now assume that there exists $k \geq 1$, such that for each $j \in \{0, \ldots, k\}$ and for each $m \geq 0$,

$$|S_{j,m}| = \frac{(j+1)(2m+j)!}{(m+j+1)!m!}$$

Since $k \geq 1$, then it follows by Lemma 3 that $|S_{k+1,m}| = |S_{k,m+1}|$ - $|S_{k-1,m+1}|$. Thus, we see that

$$|S_{k+1,m}| = |S_{k,m+1}| - |S_{k-1,m+1}|$$

$$= \frac{(k+1)(2(m+1)+k)!}{((m+1)+k+1)!(m+1)!} - \frac{((k-1)+1)(2(m+1)+(k-1))!}{((m+1)+(k-1)+1)!(m+1)!}$$

$$= \frac{(k+1)(2m+k+2)! - (k)(2m+k+1)!(m+k+2)}{(m+k+2)!(m+1)!}$$

$$= \frac{(2m+k+1)!(km+k+2m+2)}{(m+k+2)!m!(m+1)}$$

$$= \frac{(2m+k+1)!(k+2)(m+1)}{(m+k+2)!m!(m+1)}$$

$$= \frac{((k+1)+1)(2m+(k+1))!}{(m+(k+1)+1)!m!}.$$

□

Theorem 3. *For each $k \geq 0$,* $\lim_{m\to\infty} \frac{|S_{k+1,m}|}{|S_{k,m}|} = \frac{2(k+2)}{k+1}$.

Proof. We see that

$$\lim_{m\to\infty} \frac{|S_{k+1,m}|}{|S_{k,m}|} = \lim_{m\to\infty} \frac{\left(\frac{(k+2)(2m+k+1)!}{(m+k+2)!m!}\right)}{\left(\frac{(k+1)(2m+k)!}{(m+k+1)!m!}\right)} = \lim_{m\to\infty} \frac{(k+2)(2m+k+1)}{(k+1)(m+k+2)} = \frac{2(k+2)}{k+1}.$$

□

Theorem 4. *For each $k \geq 0$,* $\lim_{m\to\infty} \frac{|S_{k,m}|}{|S_{0,m}|} = (k+1)2^k$.

Proof. We prove this by induction on k. When $k = 0$, we see that

$$\lim_{m\to\infty} \frac{|S_{0,m}|}{|S_{0,m}|} = 1 = (0+1)2^0$$

Assume that there exists $k \geq 1$ such that $\lim_{m\to\infty} \frac{|S_{k,m}|}{|S_{0,m}|} = (k+1)2^k$. Thus, by Lemma 3, together with our induction hypothesis, we see that

$$\lim_{m\to\infty} \frac{|S_{k+1,m}|}{|S_{0,m}|} = \left(\lim_{m\to\infty} \frac{|S_{k+1,m}|}{|S_{k,m}|}\right)\left(\lim_{m\to\infty} \frac{|S_{k,m}|}{|S_{0,m}|}\right) = \left(\frac{2(k+2)}{k+1}\right)\left((k+1)2^k\right) = (k+2)(2^{k+1}).$$

□

Theorem 5. *For each $k \geq 0$,* $\lim_{m\to\infty} \frac{|S_{0,m+k}|}{|S_{0,m}|} = 4^k$.

Proof. We prove this by induction on k. When $k = 0$, we see that

$$\lim_{m\to\infty} \frac{|S_{0,m+0}|}{|S_{0,m}|} = \lim_{m\to\infty} \frac{|S_{0,m}|}{|S_{0,m}|} = 1 = 4^0.$$

Assume that $k = 1$. Then we see that

$$\lim_{m\to\infty} \frac{|S_{0,m+1}|}{|S_{0,m}|} = \lim_{m\to\infty} \frac{\left(\frac{(2(m+1))!}{(m+1)!(m+2)!}\right)}{\left(\frac{(2m)!}{m!(m+1)!}\right)} = \lim_{m\to\infty} \frac{4m^2+6m+2}{m^2+3m+2} = 4 = 4^1.$$

Now assume that there exists $k \geq 1$ such that

$$\lim_{m\to\infty} \frac{|S_{0,m+k}|}{|S_{0,m}|} = 4^k.$$

As $m \to \infty$, then $(m+k) \to \infty$. Therefore, by the base case for $k = 1$, we see that

$$\lim_{m\to\infty} \frac{|S_{0,m+k+1}|}{|S_{0,m+k}|} = \lim_{m\to\infty} \frac{|S_{0,m+1}|}{|S_{0,m}|} = 4.$$

Hence, we see that

$$\lim_{m\to\infty} \frac{|S_{0,m+k+1}|}{|S_{0,m}|} = \left(\lim_{m\to\infty} \frac{|S_{0,m+k+1}|}{|S_{0,m+k}|}\right)\left(\lim_{m\to\infty} \frac{|S_{0,m+k}|}{|S_{0,m}|}\right) = (4)(4^k) = 4^{k+1}.$$

□

4 The Main Theorem

The following theorem is a version of Theorem 1. However, we have reworded the statement of the theorem to fit the context here more appropriately. In particular, we do not assume that the monoid M is amenable, but instead focus on one specific nonempty, finite subset H of M. Note that if M is amenable, then the theorem can be applied to any nonempty, finite subset of M. The proof is identical to that of Theorem 1. However, we include the proof here for the sake of completeness.

Theorem 6. *Let M be a monoid, and let H be a nonempty, finite subset of M. Assume that H consists precisely of the z elements $h_1, \ldots, h_z$. Let α and β be nonzero elements of $\mathbb{K}M$ such that $\alpha = \sum_{k=1}^{z} a_k h_k$ and $\beta = \sum_{k=1}^{z} b_k h_k$, where for each $k \in \{1, \ldots, z\}$, we have that $a_k, b_k \in \mathbb{K}$. If there exists a nonempty, finite subset $E \subseteq M$ such that $|HE| < 2|E|$, then there exist nonzero elements $\gamma, \rho \in \mathbb{K}M$ such that $\alpha\gamma = \beta\rho$.*

Proof. Assume that there exists a nonempty, finite subset $E \subseteq M$ such that $|HE| < 2|E|$. Let $r = |E|$, and let $t = |HE|$. Let HE consist of the elements $g_1, g_2, \ldots, g_t$. We assume that the order of these elements is fixed. Since $|HE| < 2|E|$, then $t < 2r$. We need to show that there exist nonzero elements $\gamma, \rho \in \mathbb{K}M$ such that $\alpha\gamma = \beta\rho$. Let E consist precisely of the r elements $e_1, e_2, \ldots, e_r$. Again, we assume that the order of these elements is fixed. Let $\gamma = \sum_{l=1}^{r} u_l e_l$ and $\rho = \sum_{l=1}^{r} w_l e_l$, where for each $l \in \{1, \ldots, r\}$, the coefficients u_l and w_l are unknown.

Let V_γ denote the $r \times 1$ vector whose entries are the unknown coefficients $u_1, u_2, \ldots, u_r$ of γ. That is,

$$V_\gamma = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_r \end{bmatrix}$$

Given $g_i \in HE$, then there exist $h_{k_i} \in H$ and $e_{l_i} \in E$ such that $g_i = h_{k_i}e_{l_i}$. Let $\mathcal{M}_\gamma = [m_{i,j}]$ be the $t \times r$ matrix whose ij^{th} entry $m_{i,j}$ is given by

$$m_{i,j} = \begin{cases} a_{k_i} & \text{if } g_i = h_{k_i}e_j \\ 0 & \text{otherwise} \end{cases}$$

Similarly, let V_ρ denote the $r \times 1$ vector whose entries are the unknown coefficients $w_1, w_2, \ldots, w_r$ of ρ. That is,

$$V_\rho = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_r \end{bmatrix}$$

Also, let $\mathcal{M}_\rho = [q_{i,j}]$ be the $t \times r$ matrix whose ij^{th} entry $q_{i,j}$ is given by

$$q_{i,j} = \begin{cases} b_{k_i} & \text{if } g_i = h_{k_i}e_j \\ 0 & \text{otherwise} \end{cases}$$

Since $\mathcal{M}_\gamma V_\gamma = \mathcal{M}_\rho V_\rho$ is a linear system with t equations and $2r$ unknowns, and since $t < 2r$, then it has a nontrivial solution. Thus, we have coefficients $c_1, c_2, \ldots, c_r, d_1, d_2, \ldots, d_r$, not all of which are zero, such that for each $i \in \{1, \ldots, t\}$,

$$\sum_{h_{k_i}e_j = g_i} a_{k_i}c_j = \sum_{h_{k_i}e_j = g_i} b_{k_i}d_j$$

Thus, we have elements $\gamma, \rho \in \mathbb{K}P$ such that $\gamma = \sum_{l=1}^{r} c_l e_l$ and $\rho = \sum_{l=1}^{r} d_l e_l$, and moreover, that $\alpha\gamma = \left(\sum_{k=1}^{z} a_k h_k\right)\left(\sum_{l=1}^{r} c_l e_l\right) = \sum_{i=1}^{z}\sum_{l=1}^{r}(a_k c_l)h_k e_l = \sum_{i=1}^{t}\sum_{h_{k_i}e_j = g_i}(a_{k_i}c_j)g_i = \sum_{i=1}^{t}\sum_{h_{k_i}e_j = g_i}(b_{k_i}d_j)g_i = \sum_{i=1}^{z}\sum_{l=1}^{r}(b_k d_l)h_k e_l = \left(\sum_{k=1}^{z} b_k h_k\right)\left(\sum_{l=1}^{r} d_l e_l\right) = \beta\rho.$

□

Let $k \in \mathbb{Z}^+$. Let $A_k = \{x_0, x_1, \ldots, x_k\}$.

Theorem 7. *Let $k \in \mathbb{Z}^+$. Then for each $m \in \mathbb{Z}$, with $m \geq 0$, we have that $A_k S_{k+1,m} = S_{k,m+1}$.*

Proof. Let $m \in \mathbb{Z}$, with $m \geq 0$. Let $y \in A_k S_{k+1,m}$. Thus, there exist $x_i \in A_k$ and $z \in S_{k+1,m}$ such that $y = x_i z$. We form the product $x_i z$ by attaching a caret λ to the forest z from above by attaching the left leaf of λ to the root of the i^{th} tree $\tau_z(i)$ of z, and by attaching the right leaf of λ to the root of the $(i+1)^{st}$ tree $\tau_z(i+1)$ of

z. Since z has a total of m carets overall, and since we get $x_i z$ by attaching a single caret to z, then $x_i z$ has a total of $m+1$ carets overall. The i^{th} and $(i+1)^{st}$ trees $\tau_z(i)$ and $\tau_z(i+1)$, respectively, of z become subtrees of the i^{th} tree $\tau_{x_i z}(i)$ of $x_i z$. The roots of these trees bcome the the left and right children of the root caret λ of $\tau_{x_i z}(i)$. Moreover, for each $j \in \mathbb{Z}$, with $j \geq i+2$, the j^{th} tree $\tau_z(j)$ of z becomes the $(j-1)^{st}$ tree $\tau_{x_i z}(j-1)$ of $x_i z$. Since for each $b \in \mathbb{Z}$, with $b \geq k+2$, the b^{th} tree $\tau_z(b)$ of z is trivial, then for each $b \in \mathbb{Z}$, with $b \geq k+1$, the b^{th} tree $\tau_{x_i z}(b)$ of $x_i z$ is trivial. Therefore, $x_i z \in S_{k,m+1}$. Thus, $A_k S_{k+1,m} \subseteq S_{k,m+1}$.

Conversely, assume that $w \in S_{k,m+1}$. Since w has at least one caret, then at least one of the trees $\tau_w(0), \tau_w(1), \dots, \tau_w(k)$ is nontrivial. Let $j \in \{0, \dots, k\}$ be such that the j^{th} tree $\tau_w(j)$ of w is nontrivial. Remove the top caret λ of $\tau_w(j)$ to get a forest which is equal to $x_j^{-1}w$. Since w has a total of $m+1$ carets overall, and since we get $x_j^{-1}w$ by removing a single caret from w, then $x_j^{-1}w$ has a total of m carets overall. The subtrees of $\tau_w(j)$ whose roots are the left and right children of the root of $\tau_w(j)$ become the j^{th} and $(j+1)^{st}$ trees $\tau_{x_j^{-1}w}(j)$ and $\tau_{x_j^{-1}w}(j+1)$, respectively, of $x_j^{-1}w$. Also, for each $i \in \mathbb{Z}$, with $i \geq j+1$, we have that $\tau_{x_j^{-1}w}(i+1)$ and $\tau_w(i)$. Note that if $j \geq 1$, then for each $i \in \{0, \dots, j-1\}$, we have that $\tau_{x_j^{-1}w}(i)$ and $\tau_w(i)$. Since for each $i \in \mathbb{Z}$, with $i \geq k+1 \geq j+1$, $\tau_w(i)$ is trivial then for each $d \in \mathbb{Z}$, with $d \geq k+2$, $\tau_{x_j^{-1}w}(d)$ is trivial. Therefore $x_j^{-1}w$ is a forest which has a total of m carets overall such that for each $i \in \mathbb{Z}$, with $i \geq k+2$, $\tau_{x_j^{-1}w}(i)$ is trivial. Thus, $x_j^{-1}w \in S_{k+1,m}$. Since $j \in \{0, \dots, k\}$, then $x_j \in A_k$. Since we get w by multiplying $x_j^{-1}w$ on the left by x_j, then $w \in A_k S_{k+1,m}$. Hence, $S_{k,m+1} \subseteq A_k S_{k+1,m}$. □

Theorem 8. *Let $k \in \mathbb{Z}^+$. There exists $m \in \mathbb{Z}$, with $m \geq 0$, such that*

$$\frac{|A_k S_{k+1,m}|}{|S_{k+1,m}|} < 2.$$

Proof. By Theorems 4 and 5, we see that

$$\lim_{m\to\infty} \frac{|S_{k,m+1}|}{|S_{k+1,m}|} = \left(\lim_{m\to\infty} \frac{|S_{k,m+1}|}{|S_{0,m+1}|}\right)\left(\lim_{m\to\infty} \frac{|S_{0,m+1}|}{|S_{0,m}|}\right)\left(\lim_{m\to\infty} \frac{|S_{0,m}|}{|S_{k+1,m}|}\right)$$

$$= \left((k+1)(2^k)\right)(4)\left(\frac{1}{((k+1)+1)(2^{k+1})}\right) < 2.$$

Therefore, there exists $b \in \mathbb{Z}$, with $b \geq 0$, such that

$$\frac{|S_{k,b+1}|}{|S_{k+1,b}|} < 2.$$

By Theorem 7, we see that $A_k S_{k+1,b} = S_{k,b+1}$. Thus, there exists $b \in \mathbb{Z}$, with $b \geq 0$, such that

$$\frac{|A_k S_{k+1,b}|}{|S_{k+1,b}|} < 2.$$

□

Let $\alpha = \sum_{j=1}^{k} a_j x_j$ and $\beta = \sum_{j=1}^{k} b_j x_j$, where for each $j \in \{1, \ldots, k\}$, we have that $a_j, b_j \in \mathbb{K}$, and $x_j \in A_k$. Let $q = |S_{k+1,m}|$. By Theorem 8, there exists $m \in \mathbb{Z}$, with $m \geq 0$, such that $|A_k S_{k+1,m}| < 2|S_{k+1,m}|$. Thus, by Theorem 6, there exist $\gamma, \rho \in \mathbb{K}M$ such that $\alpha\gamma = \beta\rho$. Moreover, by Theorems 6 and 8, we see that $\gamma = \sum_{l=1}^{q} c_l z_l$, and $\rho = \sum_{l=1}^{q} d_l z_l$, where for each $l \in \{1, \ldots, q\}$, we have that $c_j, d_j \in \mathbb{K}$, and $z_l \in S_{k+1,m}$.

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