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Some New Near-Normal Sequences[1]

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Abstract. The normal sequences $NS(n)$ and near-normal sequences $NN(n)$ play an important role in the construction of orthogonal designs and Hadamard matrices. They can be identified with certain base sequences $(A; B; C; D)$, where A and B have length $n + 1$ and C and D length n. C.H. Yang conjectured that near-normal sequences exist for all even n. While this has been confirmed for $n \leq 30$, so far nothing else was known for $n > 30$. Our main result is that $NN(32)$ consists of 8 equivalence classes and we exhibit their representatives. We also construct representatives for two equivalence classes of $NN(34)$. On the other hand, we have shown by exhaustive computer searches that $NS(31)$ and $NS(33)$ are void.

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1. Introduction

We deal with quadruples $(A; B; C; D)$ of binary sequences, i.e., sequences with entries ± 1. Base sequences are such quadruples, with A and B of length m and C and D of length n, such that the sum of their nonperiodic autocorrelation functions is a δ-function. The collection of such sequences is denoted by $BS(m, n)$.

In section 2 we recall the definition of base sequences and how they can be used to construct Hadamard matrices. In section 3 we define normal sequences, $NS(n)$, and near-normal sequences, $NN(n)$, as some special classes of base sequences $BS(n + 1, n)$. We also recall some basic facts about these sequences: their use to construct Yang multiplications, their importance for the construction of T-sequences, and what is known about their existence. The

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T-sequences are not binary; they are quadruples of ternary sequences with entries from $\{0, \pm 1\}$, all of the same length. See the main text for the complete definition.

In section 4 we describe the new results that we have obtained. We have classified the near-normal sequences $NN(32)$ and constructed two non-equivalent near-normal sequences in $NN(34)$. Let us mention that in our recent paper [3] we have classified the near-normal sequences $NN(n)$ for all even $n \leq 30$. We have introduced there two equivalence relations in $NN(n)$: BS- and NN-equivalence. In this note we use only the NN-equivalence. We also report that our exhaustive searches have shown that $NS(31) = NS(33) = \emptyset$. As a consequence of these facts, we deduce that 63 and 67 are not Yang numbers while 69 is such a number. By definition, a Yang number is an odd integer $2s + 1$ such that $NS(s)$ or $NN(s)$ is not empty.

2. Base sequences

We denote finite sequences of integers by capital letters. If, say, A is such a sequence of length n then we denote its elements by the corresponding lower case letters. Thus

$$A = a_1, a_2, \dots, a_n.$$

To this sequence we associate the polynomial

$$A(x) = a_1 + a_2 x + \cdots + a_n x^{n-1},$$

which we view as an element of the Laurent polynomial ring $\mathbf{Z}[x, x^{-1}]$. (As usual, $\mathbf{Z}$ denotes the ring of integers.) The nonperiodic autocorrelation function N_A of A is defined by:

$$N_A(i) = \sum_{j \in \mathbf{Z}} a_j a_{i+j}, \quad i \in \mathbf{Z},$$

where $a_k = 0$ for $k < 1$ and for $k > n$. Note that $N_A(-i) = N_A(i)$ for all $i \in \mathbf{Z}$ and $N_A(i) = 0$ for $i \geq n$. The norm of A is the Laurent polynomial $N(A) = A(x)A(x^{-1})$. We have

$$N(A) = \sum_{i \in \mathbf{Z}} N_A(i) x^i.$$

To the sequence A we associate two other sequences of the same length: the negation

$$-A = -a_1, -a_2, \dots, -a_n$$

and the alternation

$$A^* = a_1, -a_2, a_3, -a_4, \dots, (-1)^{n-1} a_n.$$

By A, B we denote the concatenation of the sequences A and B.

The *base sequences* consist of four $\{\pm 1\}$-sequences $(A; B; C; D)$, with A and B of length m and C and D of length n, such that

$$N(A) + N(B) + N(C) + N(D) = 2(m + n). \tag{2.1}$$

We denote by $BS(m, n)$ the set of such base sequences with m and n fixed.

It is known that $BS(n + 1, n) \neq \emptyset$ for $0 \leq n \leq 35$ (see [5, 7]) and that $BS(2n - 1, n) \neq \emptyset$ for all even $n = 2, 4, \ldots, 36$ (see [5, 7, 4]).

Base sequences can be used to construct Hadamard matrices. Recall that a Hadamard matrix of order m is a $\{\pm 1\}$-matrix H of order m such that $HH^T = mI_m$, where T denotes the transpose and I_m the identity matrix. For instance if $(A; B; C; D) \in BS(n, n)$ then we can construct a Hadamard matrix H of order $4n$ as follows. Let A^c denote the circulant matrix having A as its first row, and define similarly the circulants B^c, C^c and D^c. The construction of H is based on the Goethals–Seidel array

$$\begin{bmatrix} U & XR & YR & ZR \\ -XR & U & -Z^T R & Y^T R \\ -YR & Z^T R & U & -X^T R \\ -ZR & -Y^T R & X^T R & U \end{bmatrix}.$$

To obtain H we just substitute the symbol R with the $n \times n$ matrix having ones on the back-diagonal and all other entries zero, and substitute (in any order) the symbols U, X, Y, Z with the four circulants A^c, B^c, C^c, D^c. The condition (2.1) guarantees that H is indeed a Hadamard matrix.

In connection with this construction, observe that there is a map $BS(m, n) \to BS(m + n, m + n)$ sending

$$(A; B; C; D) \to (A, C;\ A, -C;\ B, D;\ B, -D).$$

3. Normal, near-normal and T-sequences

Normal resp. *near-normal sequences*, originally defined by C.H. Yang [9], can be viewed as a special type of base sequences $BS(n + 1, n)$ (see [5, 2]), namely such that $b_i = a_i$ resp. $b_i = (-1)^{i-1} a_i$ for $1 \leq i \leq n$. We denote by $NS(n)$ resp. $NN(n)$ the subset of $BS(n + 1, n)$ consisting of normal resp. near-normal sequences.

Very little is known about the existence of normal sequences $NS(n)$. *Golay sequences* of length n are two $\{\pm 1\}$-sequences $(A; B)$ of length n such that $N(A) + N(B) = 2n$. If such sequences of length n exist, we say that n is a *Golay number*. The known Golay numbers are $n = 2^a 10^b 26^c$, where a, b, c are arbitrary nonnegative integers. If n is a Golay number, then $NS(n) \neq \emptyset$. Indeed, if $(A; B)$ are Golay sequences of length n, then $(A, +; A, -; B; B) \in NS(n)$. For $n \leq 30$ it is known (see [2, 1]) that $NS(n) = \emptyset$ iff

$$n \in \{6, 14, 17, 21, 22, 23, 24, 27, 28, 30\}.$$

The case of near-normal sequences $NN(n)$ is apparently more promising. We mention that if $n > 1$ and $NN(n) \neq \emptyset$, then n must be even. The following question (now known as Yang's conjecture) was raised about twenty years ago.

Conjecture 3.1. *(Yang [9]) $NN(n) \neq \emptyset$ for all positive even n's.*

It has been known since 1994 that near-normal sequences exist for even $n \leq 30$ (see [5]), but nothing else was known for larger values of n (see [1]).

Some of the most powerful methods for constructing orthogonal designs and Hadamard matrices are based on T-sequences (see [1]). Let us recall that *T-sequences* are quadruples $(A; B; C; D)$ of $\{0, \pm 1\}$-sequences of the same length n such that $N(A) + N(B) + N(C) + N(D) = n$ and, for each i, exactly one of a_i, b_i, c_i, d_i is nonzero. We denote by $TS(n)$ the set of T-sequences of length n. It is known that $TS(n) \neq \emptyset$ for all odd $n < 100$ different from $73, 79$ and 97. It has been conjectured that $TS(n) \neq \emptyset$ for all odd integers n.

Normal and near-normal sequences are important for the construction of T-sequences. If $NN(s)$ and $BS(m, n)$ are nonempty then there is a map, called *Yang multiplication* [9, 5]

$$NN(s) \times BS(m, n) \to TS((2s + 1)(m + n)), \tag{3.1}$$

and a similar statement is valid for $NS(s)$. For that reason it is customary to refer to the odd integer $2s + 1$ as a *Yang number* if $NS(s)$ or $NN(s)$ is nonempty.

Remark 3.2. While implementing in Maple the Theorems 1-4 of [9] we discovered two misprints: In the definition of τ_k on p. 770 one should replace the two f_k^*'s with f_k's, and in the definition of β_k on p. 773 one should replace A with A^*. The asterisk is used in [9], and in this remark, to denote the reversed sequence. These errors were not easy to locate and correct, the same errors appear in [6].

It is well known that there are infinitely many Yang numbers. Indeed, if s is a Golay number then $NS(2s + 1) \neq \emptyset$, and so $2s + 1$ is a Yang number. The known Yang numbers up to 100 are;

$$1, 3, 5, \ldots, 31, 33, 37, 39, 41, 45, 49, 51, 53, 57, 59, 61, 65, 81.$$

It is also known that $35, 43, 47$ and 55 are not Yang numbers.

4. New results

We show first that near-normal sequences $NN(32)$ and $NN(34)$ exist, and thereby confirm Yang's conjecture for $n = 32$ and $n = 34$. These sequences have been discovered by using the same algorithm as in our paper [2].

Proposition 4.1. *The sets $NN(32)$ and $NN(34)$ are nonempty.*

Proof. To prove this, it suffices to verify that $(A;B;C;D) \in NN(32)$, where

$$\begin{array}{rcl} A &=& +,-,+,-,+,-,-,-,+,-,-,-,-,-,+,+,-,+,+,-,\\ && -,+,+,-,+,-,-,+,-,+,-,-,+;\\ B &=& +,+,+,+,+,+,-,+,+,+,-,+,-,+,+,-,-,-,+,+,\\ && -,-,+,+,+,+,-,-,-,-,-,+,-;\\ C &=& +,+,+,+,-,-,-,-,+,+,+,-,+,-,-,+,+,+,-,+,\\ && +,+,-,-,+,+,-,+,+,+,-,+;\\ D &=& +,+,+,+,-,+,+,-,+,-,-,+,-,-,+,+,+,-,-,-,\\ && -,-,+,-,+,-,+,+,+,+,-,+, \end{array}$$

and $(P;Q;R;S) \in NN(34)$, where

$$\begin{array}{rcl} P &=& +,-,+,+,+,-,+,+,+,-,+,+,-,+,+,+,+,-,-,-,\\ && +,+,+,+,+,+,-,-,-,-,+,-,-,-,+;\\ Q &=& +,+,+,-,+,+,+,-,+,+,+,-,-,-,+,-,+,+,-,+,\\ && +,-,+,-,+,-,-,+,-,+,+,+,-,+,-;\\ R &=& +,+,-,-,-,-,+,+,-,-,+,-,+,+,-,-,-,-,+,+,\\ && -,-,+,+,-,+,+,-,+,-,+,-,-,+;\\ S &=& +,+,+,-,-,+,-,+,-,+,-,-,-,-,-,-,+,+,+,+,\\ && +,+,+,-,+,+,+,+,-,-,+,+,-,+. \end{array}$$

The signs "+" and "−" stand for +1 and −1, respectively. It is tedious to verify by hand that $(A;B;C;D)$ and $(P;Q;R;S)$ are base sequences, but this can be easily done on a computer. The additional requirements for near-normality can be checked by inspection. □

Proposition 4.2. *The number* 69 *is a (new) Yang number. The numbers* 63 *and* 67 *are not Yang numbers.*

Proof. The first assertion holds since $NN(34) \neq \emptyset$. Our exhaustive computer searches showed that $NS(31) = NS(33) = \emptyset$. This implies the second assertion. □

As 32 is a Golay number, we know that $NS(32) \neq \emptyset$. Hence, the first unresolved case for the existence question of normal sequences $NS(n)$ is now $n = 34$.

In our paper [3], we have introduced two equivalence relations for near-normal sequences $NN(n)$: The BS-equivalence and the NN-equivalence. The former is finer than the latter. An NN-equivalence class may contain 1,2 or 4 BS-equivalence classes. In this note we use only the NN-equivalence.

In the case $n = 32$ we have carried out an exhaustive search and found that $NN(32)$ consists of 8 NN-equivalence classes. In the case $n = 34$ our search was not complete and we constructed only two non-equivalent near-normal sequences. We list in Table 1 the representatives of these 10 NN-equivalence classes. The representatives are written in the compact encoded form. For the description of our encoding scheme see [2, 3]. The sequences $(A;B;C;D)$ and $(P;Q;R;S)$ displayed above are the first sequences in Table 1 for $n=32$ and $n=34$, respectively. The numbers a,b,c,d resp. a^*,b^*,c^*,d^* are the sums of the corresponding sequences A,B,C,D resp. A^*,B^*,C^*,D^*. Note that

$$(A;B;C;D)\in NN(n) \;\Rightarrow\; (A^*;B^*;C^*;D^*)\in NN(n).$$

Table 1: Near-normal sequences $NN(n)$

	A & B	C & D	a,b,c,d	a^*,b^*,c^*,d^*
		$n=32$		
1	07656587173587123	1611375364252851	$-5,5,8,4$	$7,-7,-4,4$
2	07643217328262853	1657222254564485	$3,-7,6,-6$	$-5,1,-10,2$
3	07841512343414140	1663752642548557	$9,7,0,0$	$9,7,0,0$
4	07651732153537650	1767258654155337	$3,9,-2,6$	$11,1,-2,2$
5	07156434121787153	1867665578785216	$3,9,-6,2$	$11,1,-2,2$
6	05671462321465123	1166547238573585	$11,1,-2,2$	$3,9,-6,-2$
7	05128282658784653	1653815347277422	$-1,-11,2,2$	$-9,-3,-2,6$
8	05126417143285123	1657686527418862	$11,1,-2,-2$	$3,9,6,2$
		$n=34$		
1	076417646512321462	16738541372344337	$7,7,-2,6$	$9,5,4,-4$
2	076782178767646231	17621532262576812	$-5,3,10,-2$	$5,-7,0,-8$

The $NN(32)$ above show that 65 is a Yang number. However this fact is already known since $NS(32)\neq\emptyset$. Nevertheless, each of the above ten near-normal sequences provides infinitely many (probably new) Hadamard matrices by using the Yang multiplication (3.1) and the infinite supply of known Williamson-type matrices (see e.g. [8]).

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