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Fixed Point Theorem in Fuzzy Metric Spaces

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Abstract

In the present paper we are proving a common fixed point theorem for fuzzy metric spaces for weakly commuting mappings.

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1. Introduction and Preliminaries:

The study of common fixed points of mappings in a fuzzy metric space satisfying certain contractive conditions has been at the center of vigorous research activity. In

1965, the concept of fuzzy sets was introduced by Zadeh [36]. With the concept of fuzzy sets, the fuzzy metric space was introduced by O.Kramosil and J. Michalek [25] in 1975. Helpern [19] in 1981first proved a fixed point theorem for fuzzy mappings. Also M.Grabiec [17] in 1988 proved the contraction principle in the setting of the fuzzy metric spaces. Moreover, A. George and P. Veeramani [16] in1994 modified the notion of fuzzy metric spaces with the help of t-norm, by generalizing the concept of probabilistic metric space to fuzzy situation. Consequently in due course of time some metric fixed point results were generalized to fuzzy metric spaces by various authors.

Gahler in a series of papers [13, 14, and 15] investigated 2-metric spaces. Sharma, Sharma and Iseki [30] studied for the first time contraction type mappings in 2-metic space.

We know that that 2-metric space is a real valued function of a point triples on a set X, which abstract properties were suggested by the area function in Euclidean spaces. Now it is natural to expect 3-Metric space, which is suggested by the volume function. In the present paper we are proving a common fixed point theorem for **fuzzy3-metric** spaces for weakly commuting mappings.

Definition (2.A): A binary operation *: $[0, 1]^4 \rightarrow [0, 1]$ is called a continuous t-norm if ([0, 1], *) is an abelian topological monoid with unit 1 such that $a_1 * b_1 * c_1 * d_1 \geq a_2 * b_2 * c_2 * d_2$ Whenever $a_1 \geq a_2$, $b_1 \geq b_2$, $c_1 \geq c_2$ and $d_1 \geq d_2$ for all a_1, a_2, b_1, b_2, c_1, c_2 and d_1, d_2 are in [0,1].

Definition (2.B): The 3-tuple (X, M, *) is called a fuzzy 3-metric space if X is an arbitrary set, * is continuous t-norm monodies and M is a fuzzy set in X^4 x $[0,\infty]$ satisfying the following conductions:

$(FM-1): M(x, y, z, w, 0) = 0$

$(FM-2): M(x, y, z, w, t) = 1, \forall t \succ 0,$

$Only\ when\ the\ three simplex \langle x, y, z, w \rangle \deg enerate$

$(FM-3): M(x, y, z, w, t) = M(x, w, z, y, t) = M(z, w, x, y, t) = ----$

$(FM-4): M(x, y, z, w, t + t_2 + t_3) \geq M(x, y, z, u, t_{1,}) *$

$$M(x, y, u, w, t_{2,}) * M(x, u,, z, w, t_{3,}) * M(u,, y, z, w, t_4)$$

$(FM-5): M(x, y, z, w): [0,1) \rightarrow [0,1]\ is\ left\ continuous,$

$$\forall x, y, z, u, w \varepsilon X, t_1, t_2, t_3, t_4 \succ 0$$

Definition (2.C): Let (X, M, *) be a fuzzy 3-metric space:
(1)A sequence {Xn} in fuzzy 3-metric space X is said to be convergent to a point x $\in$ X, if

$$\lim_{n\to\infty} M(x_n, x, a, b, t) = 1, \text{ for all } a, b\, \varepsilon\, X \text{ and } t \succ 0$$

(2)A sequence $\{x_n\}$ in fuzzy 3-metric space X is called a Cauchy sequence, if

$$\lim_{n\to\infty} M(x_{n+p}, x_n, a, b, t) = 1, \text{ for all } a, b\, \varepsilon\, X \text{ and } t, p \succ 0$$

(3)A fuzzy 3-metric space in which every Cauchy sequence is convergent is said to be complete.

Definition (2.D) A function M is continuous in fuzzy 3-metric space if

$$x_n \to x, y_n \to y, \text{then } \lim_{n\to\infty} M(x_n, y_n, a, b, t) = M(x, y, a, t), \forall\, a, b\; \varepsilon\, X \text{ and } t \succ 0$$

Definition (2.E): Two mappings A and S on fuzzy 3-metric space X are weakly commuting iff,

$$M(ASu, SAu, a, b, t) \geq M(Au, Su, a, t)\;\; \forall u, a, b\, \varepsilon\, X and\, t \succ 0$$

MAIN RESULT

THEOREM 3.1 Let (X, M, *) be a complete fuzzy 3-metric space with the condition (FM-6) and let F and T be continuous mappings of X in X. Let A be a self mapping of X satisfying {A, F} and {A, T} are R-weakly commuting and

(3.1 a) A(X) $\subseteq$ F(X) $\cap$ T (X)

(3.1 b)

$$M(Ax, Ay, a, b, t) \geq r\left[Min\begin{Bmatrix} M(Fx, Ty, a, bt), M(Fx, Ax, a, bt), M(Fx, Ay, a, b, t), \\ M(Ty, Ay, a, b, t), M(Ax, Ty, a, b, t), M(Fy, Ay, a, b, t) \end{Bmatrix}\right]$$

For all x, y $\in$ X, Where r :[0,1]→ [0,1] is a continuous function such that r(t) > t, for each
o ≤ t ≤ 1 and r(t) = 1 for t=1 $a, b\, \varepsilon\, X\; a$ .The sequence { x_n } and $\{y_n\}$ in X are such that

$$x_n \to x, y_n \to y \Rightarrow M(x_n, y_n, a, b, t) \to M(x, y, a, b, t), where\ t \succ 0 \text{ ------}$$

(3.1 c)

Then F, T and A have a unique common fixed point in X.

PROOF: We define a sequence $\{x_n\}$ such that $Fx_{2n+1} = Ax_{2n}$ and $Tx_{2n+2} = Ax_{2n+1}$, n = 1,2,-We shall prove that $\{Ax_n\}$ is a Cauchy sequence for n = 0,1,2----

$$G_n = M(Ax_n, Ax_{n+1}, t) \prec 1;\ n = 0,1,2,3 --$$

$$G_{2n} = M(Ax_{2n+1}, Ax_{2n}, t)$$

$$\geq r\left[Min\begin{Bmatrix} M(Fx_{2n+1}, Tx_{2n}, a,b,t), M(Fx_{2n+1}, Ax_{2n+1}, a,b,t), M(Fx_{2n+1}, Ax_{2n}, a,b,t), \\ M(Tx_{2n}, Ax_{2n}, a,b,t), M(Ax_{2n+1}, Tx_{2n}, a,b,t),\ M(Fx_{2n}, Ax_{2n}, a,b,t) \end{Bmatrix}\right]$$

$$= r\left[Min\begin{Bmatrix} M(Ax_{2n}, Ax_{2n-1}, a,b,t), M(Ax_{2n}, Ax_{2n+1}, a,b,t), M(Ax_{2n}, Ax_{2n}, a,b,t), \\ M(Ax_{2n-1}, Ax_{2n}, a,b,t), M(Ax_{2n+1}, Ax_{2n-1}, a,b,t), M(Ax_{2n-1}, Ax_{2n}, a,b,t) \end{Bmatrix}\right]$$

$$\geq r\left[Min\begin{Bmatrix} M(Ax_{2n}, Ax_{2n-1}, a,b,t), M(Ax_{2n}, Ax_{2n+1}, a,b,t), M(Ax_{2n}, Ax_{2n}, a,b,t), \\ M(Ax_{2n-1}, Ax_{2n}, a,b,t), M(Ax_{2n+1}, Ax_{2n}, a,b,t), M(Ax_{2n-1}, Ax_{2n}, a,b,t)\ ,\ M(Ax_{2n-1}, Ax_{2n}, a,b,t) \end{Bmatrix}\right]$$

$$= r\left[Min\{G_{2n-1}, G_{2n}, 1, G_{2n-1}, G_{2n}, G_{2n-1}, G_{2n-1}\}\right] \qquad \text{-------- (3.1 c)}$$

$$If\ G_{2n-1} \succ G_{2n}, then\ G_{2n} \succ r[G_{2n-1}] \succ G_{2n-1,}$$

$$a\,contradiction, therefore\,G_{2n-1} \leq G_{2n}$$

Therefore $from\ (3.1c), G_{2n} \geq r(G_{2n-1}) \geq G_{2n-1} \qquad ----(3.1d)$

Thus $\{G_{2n} : n \geq 0\}$ *is increasing sequence* of positive real numbers in [0, 1] and therefore tends to a finite limit L ≤ 1.It is clear that L= 1 because if L< 1 then on taking

limit n $\rightarrow\infty$ in (3.1d) we get L≥ r(L) >L, a contradiction. Hence L =1

Now for any integer m,

$$M(Ax_n, Ax_{n+m}, a,b,) \geq M\left(Ax_n, Ax_{n+1}, a,b,\frac{t}{m}\right) * ------ * M\left(Ax_{n+m-1}, Ax_{n+m}, a,b,\frac{t}{m}\right)$$

$$\geq M\left(Ax_n, Ax_{n+1}, a,b,\frac{t}{m}\right) * ------ * M\left(Ax_n, Ax_{n+1}, a,b,\frac{t}{m}\right)$$

$$\lim it_{n\to\infty} M(Ax_n, Ax_{n+1}, t) \geq 1*1*1* ------ *1 = 1, because\ \lim it_{n\to\infty} M(Ax_n, Ax_{n+1}, t) = 1\ for\ t \succ 0$$

Thus $\{Ax_n\}$ is a Cauchy sequence and by the completeness of X, $\{Ax_n\}$ converges to $u \in X$. So subsequence $\{Fx_{2n+1}\}$ and $\{Tx_{2n}\}$ of $\{Ax_n\}$ also converges to same point u.

Since A is R-weakly commuting with F, so

$$M(AFx_{2n+1}, FAx_{2n+1}, a,b,t) \geq M\left(Ax_{2n+1}, Fx_{2n+1}, a,b,\frac{t}{R}\right)$$

On taking limit n $\rightarrow\infty$, $AFx_{2n+1} = FAx_{2n+1}$ = Fu. Now we will prove that Fu = u. First suppose that

Fu ≠ u then there exists t > 0 such that M (Fu, u, t) < 1

Now

$$M\left(AFx_{2n+1}, Ax_{2n}, a,b,t\right) \geq r\left[Min\left\{ \begin{array}{l} M\left(F^2 x_{2n+1}, Tx_{2n}, a,b,t\right), M\left(F^2 x_{2n+1}, AFx_{2n+1}, a,b,t\right), \\ M\left(F^2 x_{2n+1}, Ax_{2n}, a,b,t\right), M\left(Tx_{2n}, Ax_{2n}, a,b,t\right), \\ M\left(AFx_{2n+1}, Tx_{2n}, a,b,t\right), M\left(Fx_{2n}, Ax_{2n}, a,b,t\right) \end{array} \right\} \right]$$

$$M\left(Fu,u,a,b,t\right) \geq r\left[Min\left\{ \begin{array}{l} M\left(Fu,u,a,b,t\right), M\left(Fu,Fu,a,b,t\right), M\left(Fu,u,a,b,t\right), \\ M\left(u,u,a,b,t\right),, M\left(Fu,u,a,b,t\right), M\left(Fu,u,a,b,t\right) \end{array} \right\} \right]$$

$M\left(Fu,u,a,b,t\right) \geq r\left[\left\{M\left(Fu,u,a,b,t\right)\right\}\right] \succ M\left(Fu,u,a,b,t\right), \textit{which is a contradiction.}$

Thus u is a fixed point of F. similarly we can show that u is also fixed point of A. Now we claim that u is fixed point of T. Suppose it is not so then for any t > 0, M (u, Tu, t) < 1

$$M\left(Au, ATx_{2n}, a,b,t\right) \geq r\left[Min\left\{ \begin{array}{l} M\left(Fu, T^2 x_{2n}, a,b,t\right), M\left(Fu, Au, a,b,t\right), M\left(Fu, ATx_{2n}, a,b,t\right), \\ M\left(T^2 x_{2n}, ATx_{2n}, a,b,t\right), M\left(Au, T^2 x_{2n}, a,b,t\right), M\left(FTx_{2n}, ATx_{2n}, a,b,t\right) \end{array} \right\} \right]$$

$$M\left(u,Tu,a,b,t\right) \geq r\left[Min\left\{ \begin{array}{l} M\left(u,Tu,a,b,t\right), M\left(u,u,a,b,t\right), M\left(u,Tu,a,b,t\right), \\ M\left(Tu,Tu,a,b,t\right), M\left(u,Tua,b,,t\right), , M\left(Tu,Tu,a,b,t\right) \end{array} \right\} \right]$$

$$M\left(u,Tu,a,b,t\right) \geq r\left[Min\left\{M\left(u,Tu,a,b,t\right)\right\}\right],$$

$\textit{which is contradiction.So } M\left(Tu,u,a,b,t\right) = 1$

Hence u is also a fixed point of T. That is u is common fixed point of T, F and A.

Uniqueness:

Suppose there is another fixed point v ≠ u, then

$$M\left(Ax, Ay, a,b,t\right) \geq r\left[Min\left\{ \begin{array}{l} M\left(Fx,Ty,a,b,t\right), M\left(Fx,Ax,a,b,t\right), M\left(Fx,Ay,a,b,t\right), \\ M\left(Ty,Ay,a,b,t\right), M\left(Ax,Ty,a,b,t\right), M\left(Fy,Ay,a,b,t\right) \end{array} \right\} \right]$$

$$M\left(Au, Av, a,b,t\right) \geq r\left[Min\left\{ \begin{array}{l} M\left(Fu,Tv,a,b,t\right), M\left(Fu,Au,a,b,t\right), M\left(Fu,Av,a,b,t\right), \\ M\left(Tv,Av,a,b,t\right), M\left(Au,Tv,a,b,t\right), M\left(Fv,Av,a,b,t\right) \end{array} \right\} \right]$$

$$M\left(u,v,a,b,t\right) \geq r\left[Min\left\{ \begin{array}{l} M\left(u,v,a,b,t\right), M\left(u,u,a,b,t\right), M\left(u,v,a,b,t\right), \\ M\left(v,v,a,b,t\right), M\left(u,v,a,b,t\right), M\left(v,v,a,b,t\right) \end{array} \right\} \right]$$

$M\left(u,v,a,b,t\right) \geq r\left[M\left(u,v,a,b,t\right)\right]$, which is a contradiction. So u = v.

Hence A, F and T have unique common fixed point.

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