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Improved Homotopy Perturbation Method

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Abstract

In this paper a new treatment for homotopy perturbation method (**HPM**) is introduced. The new treatment is called improved homotopy perturbation method (**IHPM**) which improves the results obtained from **HPM**. At first, the principle of the **HPM** is described. Then **IHPM** is proposed, which yields the analytic approximate solution. To illustrate the methods some experiments are provided. In some experiments different approximate solutions are obtained by using **HPM**. By applying **IHPM**, we obtain exact solutions. The results show the efficiency, accuracy, and superiority of the new method. Furthermore, the convergency of the method is also considered.

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1 Introduction

Recently a great deal of interest has been focused on the application of **HPM** for the solution of many different problems. The **HPM** has been applied with great success to obtain the approximate solution of large variety of linear and nonlinear problems in ordinary differential equations (**ODEs**),

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partial differential equations (**PDEs**), and integral equations [1-15]. In some **PDEs** different approximate solutions can be obtained by using **HPM**. In this paper we improve **HPM** to obtain exact solutions for these **PDEs**.

The present study consists of the following sections. In section 2, we introduce **HPM** and **IHPM**. In section 3, to illustrate the method, some experiments are provided. In section 4, the convergency of the method is considered and finally in section 5 a short conclusion is given.

2 HPM and its improvement (IHPM)

He presented a homotopy perturbation technique based on the introduction of a homotopy and an artificial parameter for the solution of algebraic equations and **ODEs** [1]. To describe **HPM**, we consider the following nonlinear differential equation:

$$A(\upsilon) - f(r) = 0, r \in \Omega, \tag{1}$$

with the boundary conditions:

$$B(\upsilon, \frac{\partial \upsilon}{\partial n}) = 0, r \in \Gamma, \tag{2}$$

where A is a differential operator, B is a boundary operator, $f(r)$ is a known analytic function and Γ is the boundary of the domain Ω. The operator A can be divided into two parts L and N, where L is a linear operator and N is a nonlinear operator. Therefore, equation (1) can be rewritten as:

$$L(\upsilon) + N(\upsilon) - f(r) = 0. \tag{3}$$

Now we construct a homotopy $\upsilon(r,p) : \Omega \times [0,1] \longrightarrow R$ which satisfies:

$$H(\upsilon, p) = (1-p)[L(\upsilon) - L(u_0)] + p[A(\upsilon) - f(r)] =$$

$$L(\upsilon) - (1-p)L(u_0) + p[N(\upsilon) - f(r)] = 0, p \in [0,1], r \in \Omega, \tag{4}$$

where $p \in [0,1]$ is an embedding parameter, u_0 is an initial approximation which satisfies the boundary conditions. Therefore, obviously we have:

$$H(\upsilon, 0) = L(\upsilon) - L(u_0) = 0,$$

$$H(\upsilon, 1) = A(\upsilon) - f(r) = 0.$$

Changing the process of p from zero to unity is just that of $\upsilon(r,p)$ from $u_0(r)$ to $\upsilon(r)$. We assume that the solution of equation (4) can be written as a power series in p; i.e. $\upsilon = \upsilon_0 + p\upsilon_1 + p^2\upsilon_2 + p^3\upsilon_3 + ...$. By setting $p = 1$, the approximation solution of (1) is obtained.

In some **PDEs** we can select the operator L in different ways. According to the definition of $H(\upsilon, p)$, the approximate solution that is obtained from $H(\upsilon, p)$ depends on operator L and initial conditions. Therefore, in some problems different approximate solutions are obtained. To show the subject consider the following problem

$$u_t + [f(u)]_x = [g(u)]_{xx} + h(u), \tag{5}$$

with the following conditions:

$$u(x, 0) = \varphi(x), 0 \leq x \leq 1, \tag{6}$$

$$u(0, t) = p(t), 0 \leq t \leq 1, \tag{7}$$

$$u(1, t) = q(t), 0 \leq t \leq T, \tag{8}$$

where $0 \leq u(x, t) \leq C$; C and T are given constant numbers. Besides, $[f(u)]_x$, $[g(u)]_{xx}$, and $h(u)$ represent nonlinear advection, nonlinear diffusion, and nonlinear reaction terms respectively. This problem is widely used to describe many important phenomena and dynamics, mechanics, chemistry, biology, and ect. By defining the $L = L_t$, we can seek the solution of problem based on **HPM** as:

$$\widehat{u(x, t)} = \sum_{n=0}^{\infty} \widehat{u_n(x, t)}. \tag{9}$$

Likewise, by defining the $L = L_{xx}$, we can seek another approximate solution of the problem based on **HPM** as:

$$\widetilde{u(x, t)} = \sum_{n=0}^{\infty} \widetilde{u_n(x, t)}. \tag{10}$$

Let us define $\widehat{u_n(x, t)} = \sum_{n=0}^{n-1} \widehat{u_n(x, t)}$ and $\widetilde{u_n(x, t)} = \sum_{n=0}^{n-1} \widetilde{u_n(x, t)}$. By combining these two approximate solutions, the better approximation solution can be obtained. For this purpose, let:

$$u_n(x, t) = \alpha_n \widehat{u_n} + \beta_n \widetilde{u_n}. \tag{11}$$

Some authors used $\alpha_n = \beta_n = \frac{1}{2}$ in their works to combine two approximate solutions that are obtained by Adomian decomposition method [16]. The optimum values of α_n and β_n are discussed in [17] for the approximate solutions that are obtained by Adomian decomposition method. The best values for α_n

and β_n in (11) can be obtained for each n. For instance, consider problem (5)-(8). The residual function for this problem is defined as:

$$J_n = \|u_n(0,t) - p(t)\|^2 + \|u_n(1,t) - q(t)\|^2 + \|u_n(x,0) - \varphi(x)\|^2. \tag{12}$$

The values of α_n and β_n are chosen such that this residual is minimized. The initial approximation for this problem (i.e. (5)-(8)), can be obtained by using $u_0 = (1-x)p(t) + xq(t)$.

3 Numerical experiments

To show the efficiency of the methods that is described in section 2 (i.e. **HPM** and **IHPM**), we test with the following experiments. The convergency of the method is also considered.

Experiment 1. Consider linear **PDE** $u_{tt} + u_{xx} + u_{xt} = 2(x+t)$, with the following conditions: $u(x,0) = ax$, $u_t(x,0) = x^2$, $u(0,t) = 0$, and $u_x(0,t) = a$. First we set $L = L_{tt} = \frac{\partial^2}{\partial t^2}$. According to the **HPM** that is defined in section 2, we have:

$$H(\upsilon,p) = \frac{\partial^2}{\partial t^2}(\upsilon_0 + p\upsilon_1 + p^2\upsilon_2 + p^3\upsilon_3 + p^4\upsilon_4 + p^5\upsilon_5 + ...) - (1-p)\frac{\partial^2 u_0}{\partial t^2}$$

$$+p[\frac{\partial^2}{\partial x^2}(\upsilon_0 + p\upsilon_1 + p^2\upsilon_2 + p^3\upsilon_3 + p^4\upsilon_4 + p^5\upsilon_5 + ...) + \frac{\partial^2}{\partial x \partial t}(\upsilon_0 + p\upsilon_1 + p^2\upsilon_2$$
$$+p^3\upsilon_3 + p^4\upsilon_4 + p^5\upsilon_5 + ...) - 2(x+t)] = 0.$$

The initial approximation (i.e. u_0) can be obtained from initial and boundary conditions ($u_0 = tu_t(x,0) + xu_x(0,t)$). By equating the coefficients of p to zero, we obtain:

coefficient of p^0 : $\frac{\partial^2 \upsilon_0}{\partial t^2} - \frac{\partial^2 u_0}{\partial t^2} = 0, \Rightarrow \upsilon_0 = u_0 = ax + x^2 t$,
coefficient of p^1 : $\frac{\partial^2 \upsilon_1}{\partial t^2} + \frac{\partial^2 u_0}{\partial t^2} + \frac{\partial^2 \upsilon_0}{\partial x^2} + \frac{\partial^2 \upsilon_0}{\partial xt} - 2(x+t) = 0, \Rightarrow \upsilon_1 = 0$,
coefficient of p^2 : $\frac{\partial^2 \upsilon_2}{\partial t^2} + \frac{\partial^2 \upsilon_1}{\partial x^2} + \frac{\partial^2 \upsilon_1}{\partial x \partial t} = 0, \Rightarrow \upsilon_2 = 0$,
$\vdots$
coefficient of p^n : $\frac{\partial^2 \upsilon_n}{\partial t^2} + \frac{\partial^2 \upsilon_{n-1}}{\partial x^2} + \frac{\partial^2 \upsilon_{n-1}}{\partial x \partial t} = 0, \Rightarrow \upsilon_n = 0$.

Therefore, we obtain $u\widehat{(x,t)} = ax + x^2 t$, which is the exact solution of the problem.

Now we set $L = L_{xx} = \frac{\partial^2}{\partial x^2}$. According to the **HPM** that is defined in section 2, we have:

$$H(\upsilon,p) = \frac{\partial^2}{\partial x^2}(\upsilon_0 + p\upsilon_1 + p^2\upsilon_2 + p^3\upsilon_3 + p^4\upsilon_4 + p^5\upsilon_5 + ...) - (1-p)\frac{\partial^2 u_0}{\partial x^2}$$

$$+p[\frac{\partial^2}{\partial t^2}(\upsilon_0 + p\upsilon_1 + p^2\upsilon_2 + p^3\upsilon_3 + p^4\upsilon_4 + p^5\upsilon_5 + ...) + \frac{\partial^2}{\partial x \partial t}(\upsilon_0 + p\upsilon_1 + p^2\upsilon_2$$
$$+p^3\upsilon_3 + p^4\upsilon_4 + p^5\upsilon_5 + ...) - 2(x+t)] = 0.$$

By equating the coefficients of p to zero, we obtain:

coefficient of p^0 : $\frac{\partial^2\upsilon_0}{\partial x^2} - \frac{\partial^2 u_0}{\partial x^2} = 0, \Rightarrow \upsilon_0 = u_0 = ax + x^2t,$

coefficient of p^1 : $\frac{\partial^2\upsilon_1}{\partial x^2} + \frac{\partial^2\upsilon_0}{\partial x^2} + \frac{\partial^2\upsilon_0}{\partial t^2} + \frac{\partial^2\upsilon_0}{\partial xt} - 2(x+t) = 0, \Rightarrow \upsilon_1 = 0,$

coefficient of p^2 : $\frac{\partial^2\upsilon_2}{\partial x^2} + \frac{\partial^2\upsilon_1}{\partial t^2} + \frac{\partial^2\upsilon_1}{\partial x \partial t} = 0, \Rightarrow \upsilon_2 = 0,$

$\vdots$

coefficient of p^n : $\frac{\partial^2\upsilon_n}{\partial x^2} + \frac{\partial^2\upsilon_{n-1}}{\partial t^2} + \frac{\partial^2\upsilon_{n-1}}{\partial x \partial t} = 0, \Rightarrow \upsilon_n = 0.$

Therefore, we obtain $\widetilde{u(x,t)} = ax + x^2t$, which is the exact solution of the problem. Since finite numbers of υ_i are not zero, the convergency of the solution is trivial and **HPM** is equivalent to **IHPM**. Each number in $[0, 1]$, can be the optimum value for α_n. In this case we set $\beta_n = 1 - \alpha_n$.

In this experiment, by using $u_0 = x^2t$ (initial condition), and $L = L_{tt} = \frac{\partial^2}{\partial t^2}$, we obtain:

$$\upsilon_0 = x^2t,$$
$$\upsilon_1 = 0,$$
$$\upsilon_2 = 0,$$
$$\upsilon_3 = 0,$$
$$\vdots$$
$$\upsilon_n = 0.$$

Therefore, we obtain $\widehat{u(x,t)} = x^2t$. In the same way, by using $u_0 = ax$ (initial condition), and $L = L_{xx} = \frac{\partial^2}{\partial x^2}$, we obtain:

$$\upsilon_0 = ax,$$
$$\upsilon_1 = \frac{1}{3}x^3 + x^2t,$$
$$\upsilon_2 = -\frac{1}{3}x^3,$$
$$\upsilon_3 = 0,$$
$$\vdots$$
$$\upsilon_n = 0.$$

Therefore, we obtain $\widetilde{u(x,t)} = ax + x^2t$. The optimum values of α_n and β_n are $\alpha_n = 0$ and $\beta_n = 1$. So, the exact solution is obtained. This experiment shows that **IHPM** with these initial conditions (i.e. $u_0 = x^2t$ and $u_0 = ax$) can obtain exact solution but **HPM** with $u_0 = x^2t$ can not obtain the exact solution.

Experiment 2. Consider linear **PDE** $u_t = u_{xx} - u_x$, with the following conditions: $u(x,0) = e^x - x$, $u(0,t) = 1+t$, and $u_x(1,t) = e-1$. First we set $L = L_t = \frac{\partial}{\partial t}$. According to the **HPM**, we have:

$$H(\upsilon,p) = \frac{\partial}{\partial t}(\upsilon_0 + p\upsilon_1 + p^2\upsilon_2 + p^3\upsilon_3 + p^4\upsilon_4 + p^5\upsilon_5 + ...) - (1-p)\frac{\partial u_0}{\partial t}$$

$$+p[\frac{\partial}{\partial x}(\upsilon_0 + p\upsilon_1 + p^2\upsilon_2 + p^3\upsilon_3 + p^4\upsilon_4 + p^5\upsilon_5 + ...) - \frac{\partial^2}{\partial x^2}(\upsilon_0 + p\upsilon_1$$

$$+p^2\upsilon_2 + p^3\upsilon_3 + p^4\upsilon_4 + p^5\upsilon_5 + ...)] = 0.$$

By equating the coefficients of p to zero, we obtain:

coefficient of p^0 : $\frac{\partial \upsilon_0}{\partial t} - \frac{\partial u_0}{\partial t} = 0, \Rightarrow \upsilon_0 = u_0 = e^x - x$,
coefficient of p^1 : $\frac{\partial \upsilon_1}{\partial t} + \frac{\partial u_0}{\partial t} + \frac{\partial \upsilon_0}{\partial x} - \frac{\partial^2 \upsilon_0}{\partial x^2} = 0, \Rightarrow \frac{\partial \upsilon_1}{\partial t} = 1, \Rightarrow \upsilon_1 = t$,
coefficient of p^2 : $\frac{\partial \upsilon_2}{\partial t} + \frac{\partial \upsilon_1}{\partial x} - \frac{\partial^2 \upsilon_1}{\partial x^2} = 0, \Rightarrow \frac{\partial \upsilon_2}{\partial t} = 0, \Rightarrow \upsilon_2 = 0$,
coefficient of p^3 : $\frac{\partial \upsilon_3}{\partial t} + \frac{\partial \upsilon_2}{\partial x} - \frac{\partial^2 \upsilon_2}{\partial x^2} = 0, \Rightarrow \frac{\partial \upsilon_3}{\partial t} = 0, \Rightarrow \upsilon_3 = 0$,
$\vdots$
coefficient of p^n : $\frac{\partial \upsilon_n}{\partial t} + \frac{\partial \upsilon_{n-1}}{\partial x} - \frac{\partial^2 \upsilon_{n-1}}{\partial x^2} = 0, \Rightarrow \frac{\partial \upsilon_n}{\partial t} = 0, \Rightarrow \upsilon_n = 0$.

Therefore, we obtain $\widehat{u(x,t)} = e^x - x + t$, which is the exact solution of the problem.

Now we set $L = L_x = \frac{\partial}{\partial x}$. According to the **HPM**, we have:

$$H(\upsilon,p) = \frac{\partial}{\partial x}(\upsilon_0 + p\upsilon_1 + p^2\upsilon_2 + p^3\upsilon_3 + p^4\upsilon_4 + p^5\upsilon_5 + ...) - (1-p)\frac{\partial u_0}{\partial x}$$

$$+p[\frac{\partial}{\partial t}(\upsilon_0 + p\upsilon_1 + p^2\upsilon_2 + p^3\upsilon_3 + p^4\upsilon_4 + p^5\upsilon_5 + ...) - \frac{\partial^2}{\partial x^2}(\upsilon_0 + p\upsilon_1$$

$$+p^2\upsilon_2 + p^3\upsilon_3 + p^4\upsilon_4 + p^5\upsilon_5 + ...)] = 0.$$

By equating the coefficients of p to zero, we obtain:

coefficient of p^0 : $\frac{\partial \upsilon_0}{\partial x} - \frac{\partial u_0}{\partial x} = 0, \Rightarrow \upsilon_0 = u_0 = 1 + t$,
coefficient of p^1 : $\frac{\partial \upsilon_1}{\partial x} + \frac{\partial u_0}{\partial x} + \frac{\partial \upsilon_0}{\partial t} - \frac{\partial^2 \upsilon_0}{\partial x^2} = 0, \Rightarrow \frac{\partial \upsilon_1}{\partial x} = -1 \Rightarrow \upsilon_1 = -x$,
coefficient of p^2 : $\frac{\partial \upsilon_2}{\partial x} + \frac{\partial \upsilon_1}{\partial t} - \frac{\partial^2 \upsilon_1}{\partial x^2} = 0, \Rightarrow \upsilon_2 = 0$,
coefficient of p^3 : $\frac{\partial \upsilon_3}{\partial x} + \frac{\partial \upsilon_2}{\partial t} - \frac{\partial^2 \upsilon_2}{\partial x^2} = 0, \Rightarrow \upsilon_3 = 0$,
$\vdots$
coefficient of p^n : $\frac{\partial \upsilon_n}{\partial x} + \frac{\partial \upsilon_{n-1}}{\partial t} - \frac{\partial^2 \upsilon_{n-1}}{\partial x^2} = 0, \Rightarrow \upsilon_n = 0$.

Therefore, we obtain $\widetilde{u(x,t)} = 1 + t - x$. Consequently **HPM** by using $u_0 = 1 + t$ can not obtain the exact solution of the problem. The optimum values of α_n and β_n for this problem, according to the (12), are $\alpha_n = 1$ and

$\beta_n = 0$. So, the exact solution is obtained.

Experiment 3. Consider linear **PDE** (*Klein-Gordon equation*) $u_{tt} - u_{xx} + u = 0$, with the following conditions: $u(x,0) = x + e^{-x}, u_t(x,0) = 0$. First we set $L = L_{tt} = \frac{\partial^2}{\partial x^2}$. According to the **HPM**, we have:

$$H(\upsilon,p) = \frac{\partial^2}{\partial t^2}(\upsilon_0 + p\upsilon_1 + p^2\upsilon_2 + p^3\upsilon_3 + p^4\upsilon_4 + p^5\upsilon_5 + ...) - (1-p)\frac{\partial^2 u_0}{\partial t^2}$$

$$-p[\frac{\partial^2}{\partial x^2}(\upsilon_0 + p\upsilon_1 + p^2\upsilon_2 + p^3\upsilon_3 + p^4\upsilon_4 + p^5\upsilon_5 + ...) - (\upsilon_0 + p\upsilon_1 + p^2\upsilon_2$$

$$+p^3\upsilon_3 + p^4\upsilon_4 + p^5\upsilon_5 + ...)] = 0.$$

By equating the coefficients of p to zero, we obtain:

coefficient of p^0 : $\frac{\partial^2\upsilon_0}{\partial t^2} - \frac{\partial^2 u_0}{\partial t^2} = 0, \Rightarrow \upsilon_0 = u_0 = e^{-x} + x$,

coefficient of p^1 : $\frac{\partial^2\upsilon_1}{\partial t^2} + \frac{\partial^2 u_0}{\partial t^2} - \frac{\partial^2\upsilon_0}{\partial x^2} + \upsilon_0 = 0, \Rightarrow \upsilon_1 = -\frac{t^2}{2!}x$,

coefficient of p^2 : $\frac{\partial^2\upsilon_2}{\partial t^2} - \frac{\partial^2\upsilon_1}{\partial x^2} + \upsilon_1 = 0, \Rightarrow \upsilon_2 = \frac{t^4}{4!}x$,

$\vdots$

coefficient of p^n : $\frac{\partial^2\upsilon_n}{\partial t^2} - \frac{\partial^2\upsilon_{n-1}}{\partial x^2} + \upsilon_{n-1} = 0, \Rightarrow \upsilon_n = \frac{(-1)^n t^{2n}}{(2n)!}x$.

Therefore, by using Maclaurin's formula, we obtain $\widehat{u(x,t)} = e^{-x} + x(1 - \frac{t^2}{2!} + \frac{t^4}{4!} - + ... + \frac{(-1)^n t^{2n}}{(2n)!} + ...) = e^{-x} + xcos(t)$, which is the exact solution of the problem. In this experiment, there is a unique way to choose L.

Experiment 4. Consider nonlinear **PDE** $u_{tt} + u_{xx} + u_x^2 = 2x + t^4$, with the following conditions: $u(x,0) = 0$, $u_t(x,0) = a$, $u(0,t) = at$, and $u_x(0,t) = t^2$. First, we set $L = L_{tt} = \frac{\partial^2}{\partial t^2}$. According to the **HPM**, we have:

$$H(\upsilon,p) = \frac{\partial^2}{\partial t^2}(\upsilon_0 + p\upsilon_1 + p^2\upsilon_2 + p^3\upsilon_3 + p^4\upsilon_4 + p^5\upsilon_5 + ...) - (1-p)\frac{\partial^2 u_0}{\partial t^2}$$

$$+p[\frac{\partial^2}{\partial x^2}(\upsilon_0 + p\upsilon_1 + p^2\upsilon_2 + p^3\upsilon_3 + p^4\upsilon_4 + p^5\upsilon_5 + ...) + (\frac{\partial}{\partial x}(\upsilon_0 + p\upsilon_1 + p^2\upsilon_2$$

$$+p^3\upsilon_3 + p^4\upsilon_4 + p^5\upsilon_5 + ...))^2 - 2x - t^4] = 0.$$

The initial approximation can be obtained from initial and boundary conditions (i.e. $u_0 = tu_t(x,0) + xu_x(0,t) = at + xt^2$). By equating the coefficients of p to zero, we obtain:

coefficient of p^0 : $\frac{\partial^2\upsilon_0}{\partial t^2} - \frac{\partial^2 u_0}{\partial t^2} = 0, \Rightarrow \upsilon_0 = u_0 = at + xt^2$,

coefficient of p^1 : $\frac{\partial^2\upsilon_1}{\partial t^2} + 2x + t^4 - 2x - t^4 = 0, \Rightarrow \upsilon_1 = 0$,

coefficient of p^2 : $\frac{\partial^2\upsilon_2}{\partial t^2} + \frac{\partial^2\upsilon_1}{\partial x^2} + (\frac{\partial}{\partial x}2\upsilon_0\upsilon_1)^2 = 0, \Rightarrow \upsilon_2 = 0$,

$\vdots$

Therefore, we obtain $\widehat{u(x,t)} = at + xt^2$, which is the exact solution of the problem.

Now we set $L = L_{xx} = \frac{\partial^2}{\partial x^2}$. According to the **HPM**, we have:

$$H(\upsilon, p) = \frac{\partial^2}{\partial x^2}(\upsilon_0 + p\upsilon_1 + p^2\upsilon_2 + p^3\upsilon_3 + p^4\upsilon_4 + p^5\upsilon_5 + ...) - (1-p)\frac{\partial^2 u_0}{\partial x^2}$$

$$+p[\frac{\partial^2}{\partial t^2}(\upsilon_0 + p\upsilon_1 + p^2\upsilon_2 + p^3\upsilon_3 + p^4\upsilon_4 + p^5\upsilon_5 + ...) + (\frac{\partial}{\partial x}(\upsilon_0 + p\upsilon_1 + p^2\upsilon_2$$

$$+p^3\upsilon_3 + p^4\upsilon_4 + p^5\upsilon_5 + ...))^2 - 2x - t^4] = 0.$$

By equating the coefficients of p to zero, we obtain:

coefficient of p^0 : $\frac{\partial^2 \upsilon_0}{\partial x^2} - \frac{\partial^2 u_0}{\partial x^2} = 0, \Rightarrow \upsilon_0 = u_0 = at + xt^2$,

coefficient of p^1 : $\frac{\partial^2 \upsilon_1}{\partial x^2} + \frac{\partial^2 u_0}{\partial x^2} + \frac{\partial^2 \upsilon_0}{\partial t^2} + (\frac{\partial \upsilon_0}{\partial x})^2 - 2x - t^4 = 0, \Rightarrow \upsilon_1 = 0$,

coefficient of p^2 : $\frac{\partial^2 \upsilon_2}{\partial x^2} + \frac{\partial^2 \upsilon_1}{\partial x^2} + (\frac{\partial}{\partial x} 2\upsilon_0\upsilon_1)^2 = 0, \Rightarrow \upsilon_2 = 0$,

$\vdots$

Therefore, we obtain $\widetilde{u(x,t)} = at + xt^2$, which is the exact solution of the problem. Since finite numbers of υ_i are not zero, the convergency of the solution is trivial and **HPM** is equivalent to **IHPM** in this case. Each number in $[0,1]$, can be the optimum value of α_n. In this case, we set $\beta_n = 1 - \alpha_n$.

In this experiment, by using $u_0 = at$, $L = L_{tt} = \frac{\partial^2}{\partial t^2}$, and **HPM**, we obtain:

$$\upsilon_0 = at,$$

$$\upsilon_1 = xt^2 + \frac{1}{30}t^6,$$

$$\upsilon_2 = 0,$$

$$\upsilon_3 = -\frac{1}{30}t^6,$$

$$\upsilon_4 = 0,$$

$$\vdots$$

$$\upsilon_n = 0.$$

Therefore, we obtain $\widehat{u(x,t)} = at + xt^2$. In the same way, by using $u_0 = xt^2$, and **HPM**, we obtain:

$$\upsilon_0 = xt^2,$$

$$\upsilon_1 = 0,$$

$$\upsilon_2 = 0,$$

$$\vdots$$

$$\upsilon_n = 0.$$

Therefore, we obtain $\widetilde{u(x,t)} = xt^2$. The optimum values of α_n and β_n are $\alpha_n = 1$ and $\beta_n = 0$. So, the exact solution of the problem is obtained. By using $L = L_{tt}$, and $u_0 = xt^2$ **HPM** can not obtain exact solution but **IHPM** can obtain the exact solution.

Experiment 5. Consider nonlinear **PDE** (*Burger equation*) $u_t = u_{xx} + uu_x$, with the following conditions: $u(x,0) = 1 - x$, $u(0,t) = \frac{1}{1+t}$, $u(1,t) = 0$. First we set $L = L_t = \frac{\partial}{\partial t}$. According to the **HPM**, we have:

$$H(\upsilon,p) = \frac{\partial}{\partial t}(\upsilon_0 + p\upsilon_1 + p^2\upsilon_2 + p^3\upsilon_3 + p^4\upsilon_4 + p^5\upsilon_5 + ...) - (1-p)\frac{\partial u_0}{\partial t}$$

$$+p[-\frac{\partial^2}{\partial x^2}(\upsilon_0 + p\upsilon_1 + p^2\upsilon_2 + p^3\upsilon_3 + p^4\upsilon_4 + p^5\upsilon_5 + ...) - (\upsilon_0 + p\upsilon_1 + p^2\upsilon_2$$

$$+p^3\upsilon_3 + p^4\upsilon_4 + p^5\upsilon_5 + ...)(\frac{\partial}{\partial x}(\upsilon_0 + p\upsilon_1 + p^2\upsilon_2 + p^3\upsilon_3 + p^4\upsilon_4 + p^5\upsilon_5 + ...))] = 0.$$

By equating the coefficients of p to zero, we obtain:

coefficient of p^0 : $\frac{\partial \upsilon_0}{\partial t} - \frac{\partial u_0}{\partial t} = 0, \Rightarrow \upsilon_0 = u_0 = 1 - x$,

coefficient of p^1 : $\frac{\partial \upsilon_1}{\partial t} + \frac{\partial u_0}{\partial x} - \frac{\partial^2 \upsilon_0}{\partial x^2} - \upsilon_0\frac{\partial \upsilon_0}{\partial x} = 0, \Rightarrow \upsilon_1 = -t(1-x)$,

coefficient of p^2 : $\frac{\partial \upsilon_2}{\partial t} - \frac{\partial^2 \upsilon_1}{\partial x^2} - \upsilon_0\frac{\partial \upsilon_1}{\partial x} - \upsilon_1\frac{\partial \upsilon_0}{\partial x} = 0, \Rightarrow \upsilon_2 = t^2(1-x)$,

coefficient of p^3 : $\frac{\partial \upsilon_3}{\partial t} - \frac{\partial^2 \upsilon_2}{\partial x^2} - \upsilon_0\frac{\partial \upsilon_2}{\partial x} - \upsilon_1\frac{\partial \upsilon_1}{\partial x} - \upsilon_2\frac{\partial \upsilon_0}{\partial x} = 0, \Rightarrow \upsilon_2 = -t^3(1-x)$,

$\vdots$

Therefore, we obtain $\widehat{u(x,t)} = (1 - t + t^2 - t^3 + ...)(1-x) = \frac{1-x}{1+t}$, which is the exact solution of the problem.

Now, we set $L = L_{xx} = \frac{\partial^2}{\partial x^2}$. According to the **HPM**, we have:

$$H(\upsilon,p) = \frac{\partial^2}{\partial x^2}(\upsilon_0 + p\upsilon_1 + p^2\upsilon_2 + p^3\upsilon_3 + p^4\upsilon_4 + p^5\upsilon_5 + ...) - (1-p)\frac{\partial^2 u_0}{\partial x^2}$$

$$+p[-\frac{\partial}{\partial t}(\upsilon_0 + p\upsilon_1 + p^2\upsilon_2 + p^3\upsilon_3 + p^4\upsilon_4 + p^5\upsilon_5 + ...) + (\upsilon_0 + p\upsilon_1 + p^2\upsilon_2 + p^3\upsilon_3$$

$$+p^4\upsilon_4 + p^5\upsilon_5 + ...)(\frac{\partial}{\partial x}(\upsilon_0 + p\upsilon_1 + p^2\upsilon_2 + p^3\upsilon_3 + p^4\upsilon_4 + p^5\upsilon_5 + ...))] = 0.$$

By equating the coefficients of p to zero, we obtain:

coefficient of p^0 : $\frac{\partial^2 \upsilon_0}{\partial x^2} - \frac{\partial^2 u_0}{\partial x^2} = 0, \Rightarrow \upsilon_0 = u_0 = \frac{1-x}{1+t}$,

coefficient of p^1 : $\frac{\partial^2 \upsilon_1}{\partial x^2} + \frac{\partial^2 u_0}{\partial x^2} - \frac{\partial \upsilon_0}{\partial t} + \upsilon_0\frac{\partial \upsilon_0}{\partial x} = 0, \Rightarrow \upsilon_1 = 0$,

coefficient of p^2 : $\frac{\partial^2 \upsilon_2}{\partial x^2} - \frac{\partial \upsilon_1}{\partial t} + \upsilon_0\frac{\partial \upsilon_1}{\partial x} + \upsilon_1\frac{\partial \upsilon_0}{\partial x} = 0, \Rightarrow \upsilon_2 = 0$,

$\vdots$

Therefore, we obtain $\widetilde{u(x,t)} = \frac{1-x}{1+t}$, which is the exact solution of the problem. Since finite numbers of υ_i are not zero, the convergency of the solution is trivial and **HPM** is equivalent to **IHPM**. Each number in $[0,1]$, can be the optimum value of α_n. In this case, we set $\beta_n = 1 - \alpha_n$.

Experiment 6. Consider nonlinear **PDE** $u_t = uu_{xx} + u_x^2 + u$, with the following conditions: $u(x,0) = \sqrt{x}, u(0,t) = 0, u(1,t) = e^t$. First we set $L = L_t = \frac{\partial}{\partial t}$. According to the **HPM**, we have:

$$H(\upsilon,p) = \frac{\partial}{\partial t}(\upsilon_0 + p\upsilon_1 + p^2\upsilon_2 + p^3\upsilon_3 + p^4\upsilon_4 + p^5\upsilon_5 + ...) - (1-p)\frac{\partial u_0}{\partial t}$$

$$-p[(\upsilon_0 + p\upsilon_1 + p^2\upsilon_2 + p^3\upsilon_3 + p^4\upsilon_4 + p^5\upsilon_5 + ...)(\frac{\partial^2}{\partial x^2}(\upsilon_0 + p\upsilon_1 + p^2\upsilon_2 + p^3\upsilon_3 + p^4\upsilon_4$$

$$+p^5\upsilon_5 + ...)) + (\frac{\partial}{\partial x}(\upsilon_0 + p\upsilon_1 + p^2\upsilon_2 + p^3\upsilon_3 + p^4\upsilon_4 + p^5\upsilon_5 + ...))^2 + (\upsilon_0 + p\upsilon_1 + p^2\upsilon_2 + p^3\upsilon_3$$

$$+p^4\upsilon_4 + p^5\upsilon_5 + ...)] = 0.$$

By equating the coefficients of p to zero, we obtain:
coefficient of p^0 : $\frac{\partial \upsilon_0}{\partial t} - \frac{\partial u_0}{\partial t} = 0, \Rightarrow \upsilon_0 = u_0 = \sqrt{x}$,
coefficient of p^1 : $\frac{\partial \upsilon_1}{\partial t} + \frac{\partial u_0}{\partial t} - \upsilon_0 \frac{\partial^2 \upsilon_0}{\partial x^2} - (\frac{\partial \upsilon_0}{\partial x})^2 + \upsilon_0 = 0, \Rightarrow \upsilon_1 = t\sqrt{x}$,
coefficient of p^2 : $\frac{\partial \upsilon_2}{\partial t} - [\upsilon_0 \frac{\partial^2 \upsilon_1}{\partial x^2} + \upsilon_1 \frac{\partial^2 \upsilon_0}{\partial x^2} + 2\frac{\partial \upsilon_0}{\partial x}\frac{\partial \upsilon_1}{\partial x} + \upsilon_1] = 0, \Rightarrow \upsilon_2 = \frac{t^2}{2}\sqrt{x}$,
$\vdots$

Therefore, by using Maclaurin's formula, we obtain $\widehat{u(x,t)} = \sqrt{x}(1 + t + \frac{t^2}{2} + ...) = \sqrt{x}e^t$, which is the exact solution of the problem. In this experiment, there is a unique way to choose L.

Experiment 7. Consider nonlinear **PDE** $u_t + u^2 u_x = 0$, with the following condition: $u(x,0) = 3x$. We set $L = L_t = \frac{\partial}{\partial t}$. According to the **HPM**, we have:

$$H(\upsilon,p) = \frac{\partial}{\partial t}(\upsilon_0 + p\upsilon_1 + p^2\upsilon_2 + p^3\upsilon_3 + p^4\upsilon_4 + p^5\upsilon_5 + ...) - (1-p)\frac{\partial u_0}{\partial t}$$

$$+p[(\upsilon_0 + p\upsilon_1 + p^2\upsilon_2 + p^3\upsilon_3 + p^4\upsilon_4 + p^5\upsilon_5 + ...)^2(\frac{\partial}{\partial x}(\upsilon_0 + p\upsilon_1 + p^2\upsilon_2$$

$$+p^3\upsilon_3 + p^4\upsilon_4 + p^5\upsilon_5 + ...))] = 0.$$

By equating the coefficients of p to zero, we obtain:
coefficient of p^0 : $\frac{\partial \upsilon_0}{\partial t} - \frac{\partial u_0}{\partial t} = 0, \Rightarrow \upsilon_0 = u_0 = 3x$,
coefficient of p^1 : $\frac{\partial \upsilon_1}{\partial t} + \frac{\partial u_0}{\partial t} + \upsilon_0^2 \frac{\partial \upsilon_0}{\partial x} = 0, \Rightarrow \upsilon_1 = -27x^2 t$,
coefficient of p^2 : $\frac{\partial \upsilon_2}{\partial t} + \upsilon_0^2 \frac{\partial \upsilon_1}{\partial x} + 2\upsilon_0\upsilon_1 \frac{\partial \upsilon_0}{\partial x} = 0, \Rightarrow \upsilon_2 = 486x^3 t^2$,
coefficient of p^3 : $\frac{\partial \upsilon_3}{\partial t} + \upsilon_0^2 \frac{\partial \upsilon_2}{\partial x} + 2\upsilon_0\upsilon_1 \frac{\partial \upsilon_1}{\partial x} + \upsilon_1^2 \frac{\partial \upsilon_0}{\partial x} + 2\upsilon_0\upsilon_2 \frac{\partial \upsilon_0}{\partial x} = 0, \Rightarrow$

$$\upsilon_3 = -10935x^4 t^3,$$

$\vdots$

Therefore, by using Maclaurin's formula, we obtain:

$$\widehat{u(x,t)} = (3x - 27x^2 t + 486x^3 t^2 - 10935x^4 t^3 + ...)$$

$$= \begin{cases} 3x, & for \quad (t=0), \\ \frac{1}{6t}(\sqrt{1+36xt}-1, & for \quad (t>0), \end{cases}$$

which is the exact solution of the problem. In this experiment, there is a unique way to choose L.

4 Convergency of the method

An approximate solution of problem (1) is expressed as:

$$\upsilon = \sum_{i=0}^{\infty} \upsilon_i. \tag{13}$$

To consider the convergency of this series, we can apply classical test such as the root and the ratio tests [18].

Theorem .1 (Root Test) Given (13), put $\alpha = \limsup_{n\to\infty} \sqrt[n]{|\upsilon_n|}$. Then:
(a) if $\alpha < 1$, then (13) converges;
(b) if $\alpha > 1$, then (13) diverges;
(a) if $\alpha = 1$, the test gives no information.
Proof. See [18].

Theorem .2 (Ratio Test) The series (13),
(a) converges if $\limsup_{n\to\infty} |\frac{\upsilon_{n+1}}{\upsilon_n}| < 1$,
(b) diverges if $|\frac{\upsilon_{n+1}}{\upsilon_n}| \geq 1$ for all $n \geq n_0$, where n_0 is some fixed integer.
Proof. See [18].

For expressed experiments in section 3, with infinite non-zero υ_i, the convergency of the approximate solutions can be obtained easily by ratio or root tests.

5 Conclusions

According to the approximate solutions that we have obtained, we infer that **IHPM** is a powerful tool for solving the linear, and nonlinear equations. Expressed experiments show that **IHPM** is superior to **HPM**. This method is easy to implement and the convergency shows that the method can gives good approximate solution for different types of evolution equations.

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