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Self-Duality of Tridiagonal Pairs of q-Serre Type

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Abstract

Let $\mathcal{F}$ denote an algebraically closed field. Let V denote a vector space over $\mathcal{F}$ with finite positive dimension, and let A, A^* be a tridiagonal pair on V. Let $\{\theta_i\}_{i=0}^d$ (resp. $\{\theta_i^*\}_{i=0}^d$) denote a standard ordering of the eigenvalues of A (resp. A*). We assume $\theta_i = q^{2i-d}$ (resp. $\theta_i^* = q^{d-2i}$) for $(0 \leq i \leq d)$. We show that A, A^* is isomorphic to A^*, A.

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1 Introduction

Throughout this paper $\mathcal{F}$ denotes an algebraically closed field. In this section we recall the notion of a Tridiagonal pair. Let V denote a vector space over $\mathcal{F}$ with finite positive dimension. Let End(V) denote the $\mathcal{F}$-algebra consisting of all $\mathcal{F}$-linear transformations from V to V.

Definition 1.1 An ordered pair A, A^* of elements from End(V) is said to be a *tridiagonal pair (TDP)* on V whenever the following four conditions are satisfied.

(i) Each of A and A^* is diagonalizable over $\mathcal{F}$.

(ii) There exists an ordering $V_0, V_1, \ldots, V_d$ of the eigenspaces of A such that

$$A^*V_i \subseteq V_{i-1} + V_i + V_{i+1} \quad (0 \leq i \leq d), \tag{1}$$

where $V_{-1} = 0$, $V_{d+1} = 0$.

(iii) There exists an ordering $V_0^*, V_1^*, \ldots, V_\delta^*$ of the eigenspaces of A^* such that

$$AV_i^* \subseteq V_{i-1}^* + V_i^* + V_{i+1}^* \quad (0 \leq i \leq \delta), \tag{2}$$

where $V_{-1}^* = 0$, $V_{\delta+1}^* = 0$.

(iv) There is no proper nonzero subspace H of V such that both $AH \subseteq H$ and $A^*H \subseteq H$.

Tridiagonal pairs in which the eigenspaces V_i and V_i^* have dimension one are called *Leonard pairs*, and have been extensively studied in [11, 12, 13, 14, 15]. In this paper we focus on TDP's of q-Serre type. TDP's of q-Serre type have received a great deal of attention [1, 2, 3, 5, 6, 7, 8], in part because of their connection to finite-dimensional representations of the quantum affine algebra $U_q(\widehat{sl_2})$. We recall the q-Serre property now.

Definition 1.2 Fix a nonzero scalar $q \in \mathcal{F}$ which is not a root of unity. A TDP A, A^*on V is said to be of *q-Serre type* whenever the following hold:

$$A^3A^* - [3]A^2A^*A + [3]AA^*A^2 - A^*A^3 = 0, \tag{3}$$

$$A^{*3}A - [3]A^{*2}AA^* + [3]A^*AA^{*2} - AA^{*3} = 0, \tag{4}$$

where $[3] = (q^3 - q^{-3})/(q - q^{-1})$.

Equations (3) and (4) are called the *cubic q-Serre relations* and are among the defining relations of the quantum affine algebra $U_q(\widehat{sl_2})$.

2 Main result

To state the main result, we recall some facts.

Definition 2.1 Let A, A^* and B, B^* be TDP's over $\mathcal{F}$. By an isomorphism of TDP's from A, A^* to B, B^* we mean a vector space isomorphism γ from the vector space underlying A, A^* to the vector space underlying B, B^* such that $\gamma A = B\gamma$ and $\gamma A^* = B^*\gamma$.

Lemma 2.2 *[6] Let A, A^* denote a TDP on V. The scalars d and δ from Definition 1.1 are equal; we refer to this common value as the* diameter of A, A^*.

Definition 2.3 [6] Let A, A^* denote a TDP on V of diameter d.

(i) An ordering $V_0, V_1, \ldots, V_d$ of the eigenspaces of A is said to be *standard* whenever it satisfies (1). An *eigenvalue sequence* of A, A^* is an ordering $\theta_0, \theta_1, \ldots, \theta_d$ of the eigenvalues of A such that the induced ordering of the eigenspaces of A is standard.

(ii) An ordering $V_0^*, V_1^*, \ldots, V_d^*$ of the eigenspaces of A^* is said to be *standard* whenever it satisfies (2). A *dual eigenvalue sequence* of A, A^* is an ordering $\theta_0^*, \theta_1^*, \ldots, \theta_d^*$ of the eigenvalues of A^* such that the induced ordering of the eigenspaces of A^* is standard.

Definition 2.4 Let A, A^* denote TDP over $\mathcal{F}$ and let d denote the diameter. We say that A, A^* is a *normalized* TDP of q-Serre type whenever the sequence $\theta_i = q^{2i-d}$ (resp. $\theta^*_i = q^{d-2i}$) for $(0 \leq i \leq d)$ is a standard ordering of the eigenvalues of A (resp. A^*).

The main result in this paper is the following theorem

Theorem 2.5 *Let V denote a vector space over $\mathcal{F}$ with finite positive dimension, and let A, A^* be a normalized TDP of q-Serre type on V. Then A, A^* and A^*, A are isomorphic TDP's.*

We remark that in [10] Ito and Terwilliger proved the same result for tridiagonal pairs of Krawtchouk type.

3 The quantum affine algebra $U_q(\widehat{sl_2})$

In this section we recall the quantum affine algebra $U_q(\widehat{sl_2})$ and some facts concerning its irreducible modules, also will proof some results. Fix a nonzero scalar $q \in \mathcal{F}$ which is not a root of unity. For all integers k and for all positive integers n write

$$[k] = \frac{q^k - q^{-k}}{q - q^{-1}} \quad \text{and} \quad [n]! = [1][2]\dots[n].$$

Definition 3.1 [4] The *quantum affine algebra* $U_q(\widehat{sl_2})$ is the associative $\mathcal{F}$-algebra with generators $e_i^{\pm}$, K_i, K_i^{-1} $(i = 0, 1)$ and relations:

$$\begin{aligned} K_i K_i^{-1} &= K_i^{-1} K_i = 1, \\ K_0 K_1 &= K_1 K_0, \\ K_i e_i^{\pm} K_i^{-1} &= q^{\pm 2} e_i^{\pm}, \\ K_i e_j^{\pm} K_i^{-1} &= q^{\mp 2} e_j^{\pm} \quad (i \neq j), \\ \left[e_i^+, e_i^-\right] &= \frac{K_i - K_i^{-1}}{q - q^{-1}}, \\ \left[e_0^{\pm}, e_1^{\mp}\right] &= 0, \end{aligned}$$

$$(e_i^{\pm})^3 e_j^{\pm} - [3](e_i^{\pm})^2 e_j^{\pm} e_i^{\pm} + [3] e_i^{\pm} e_j^{\pm} (e_i^{\pm})^2 - e_j^{\pm} (e_i^{\pm})^3 = 0 \qquad (i \neq j). \tag{5}$$

The equations of (5) are cubic q-Serre relations, just as (3) and (4) are.

In the next Lemma we define an automorphism on $U_q(\widehat{sl_2})$.

Lemma 3.2 *Let* $\sigma : U_q(\widehat{sl_2}) \longrightarrow U_q(\widehat{sl_2})$ *such that for* $i = \{0, 1\}$

$$\sigma(e_i^{\pm}) = e_{1-i}^{\pm}, \qquad \sigma(K_i) = K_{i-1}, \quad \text{and} \qquad \sigma(K_i^{-1}) = K_{i-1}^{-1}.$$

Then σ *is an automorphism of* $U_q(\widehat{sl_2})$.

Proof. Straightforward.

Lemma 3.3 *[4]* $U_q(\widehat{sl_2})$ *is a Hopf algebra with co-multiplication* $\triangle(e_i^+) = e_i^+ \otimes K_i + 1 \otimes e_i^+$, $\triangle(e_i^-) = e_i^- \otimes 1 + K_i^{-1} \otimes e_i^-$, $\triangle(K_i) = K_i \otimes K_i$ $(i = 0, 1)$.

Lemma 3.4 *[4] Let* M *denote a finite-dimensional irreducible* $U_q(\widehat{sl_2})$*-module. Then there exist unique scalars* ε_0, ε_1 *in* $\{1, -1\}$ *and a unique decomposition* $U_0, U_1, \ldots, U_d$ *of* M *such that for all* i $(0 \le i \le d)$

$$\begin{aligned} (K_0 - \varepsilon_0 q^{2i-d})U_i = 0, &\qquad (K_1 - \varepsilon_1 q^{d-2i})U_i = 0, \\ e_0^+ U_i \subseteq U_{i+1}, &\qquad e_1^- U_i \subseteq U_{i+1}, \\ e_0^- U_i \subseteq U_{i-1}, &\qquad e_1^+ U_i \subseteq U_{i-1}. \end{aligned}$$

We refer to the ordered pair ε_0, ε_1 as the type of M.

Lemma 3.5 *[4] For all nonnegative integers* d *and for all nonzero scalars* $a \in \mathcal{F}$, *there is a* $(d+1)$*-dimensional irreducible* $U_q(\widehat{sl_2})$*-module of type (1,1) with basis* $v_0, v_1, \ldots, v_d$ *and action*

$$\begin{aligned} e_0^+ \cdot v_{i-1} &= q^{-1}a[i]v_i, & e_1^- \cdot v_{i-1} &= [i]v_i, \\ e_0^- \cdot v_i &= qa^{-1}[d-i+1]v_{i-1}, & e_1^+ \cdot v_i &= [d-i+1]v_{i-1}, \\ K_0 \cdot v_i &= q^{-d+2i}v_i, & K_1 \cdot v_i &= q^{d-2i}v_i. \end{aligned}$$

Such a module is called an evaluation module*.*

In light of Lemma 3.3, the tensor product of evaluation modules is a $U_q(\widehat{sl_2})$-module.

Theorem 3.6 *[4] Every finite-dimensional irreducible* $U_q(\widehat{sl_2})$*-modules of type (1,1) is isomorphic to a tensor product of evaluation modules.*

There is a second action $\cdot_\sigma$ of $U_q(\widehat{sl_2})$ on each $U_q(\widehat{sl_2})$-module V given by

$$u \cdot_\sigma v = \sigma(u) \cdot v, \qquad for\ all\ v \in V \quad and\ u \in U_q(\widehat{sl_2}),$$

where σ is the automorphism in Lemma 3.2 and the $\cdot$ is the action in Lemma 3.5, this new action will help us to proof our main result.

Lemma 3.7 *Let M denote a finite dimensional irreducible $U_q(\widehat{sl_2})$-module of type $(1,1)$. Then the module structures on M defined by $\cdot$ and $\cdot_\sigma$ are isomorphic.*

Proof. The finite dimensional irreducible $U_q(\widehat{sl_2})$-module M may be evaluation module or according to Theorem 3.6 isomorphic to tensor product of evaluation modules. If M is an evaluation module with basis $v_0, v_1, \ldots, v_d$ as in Lemma 3.5, then for $0 \leq i \leq d$, define $w_i = (qa^{-1})^i v_{d-i}$. Observe that for each $u \in U_q(\widehat{sl_2})$ the $\cdot_\sigma$ action on $\{w_i\}_{i=0}^d$ agrees with the $\cdot$ action on $\{v_i\}_{i=0}^d$. If M tensor product of evaluation modules, then the result hold by induction and Lemma 3.3.

4 TD pairs of q-Serre type

In this section, we recall some facts related to TDP of q-Serre type and use them to show that A, A^* and A^*, A are isomorphic.

Lemma 4.1 [12, Lemma 4.8] *Let A, A^* denote a TDP on V of diameter d. Then the following are equivalent.*

(i) *A, A^* is of q-Serre type.*

(ii) *There exist eigenvalue and dual eigenvalue sequences for A, A^* which satisfy*

$$\theta_i = q^{2i}\theta, \quad \theta_i^* = q^{2d-2i}\theta^* \qquad (0 \leq i \leq d) \tag{6}$$

for some nonzero scalars θ, $\theta^* \in \mathcal{F}$.

Lemma 4.2 *If A, A^* is a TDP on V of q-Serre type with diameter d, then so is A^*, A.*

Proof. It is clear that A^*, A is a TDP since Definition 1.1 is symmetric in A and A^*. The q-Serre type follows directly by replacing q by q^{-1}, θ by $q^{2d}\theta$ and θ^* by $q^{2d}\theta^*$ in Lemma 4.1.

Lemma 4.3 *Let A, A^* be a TDP on V of q-Serre type, and let θ and θ^* be as in Lemma 4.1. Then $q^{-d}\theta^{-1}A$, $q^{-d}\theta^{*-1}A^*$ is a normalized TDP on V of q-Serre type.*

Proof. Straightforward.

Theorem 4.4 *[9] Let A, A^* be a normalized TDP on V of q-Serre type. Then there exists a unique $U_q(\widehat{sl_2})$-module structure on V such that A and A^* act as $e_0^+ + K_0$ and $e_1^+ + K_1$, respectively. This $U_q(\widehat{sl_2})$-module is irreducible of type (1,1).*

Proof of Theorem 2.5. Let $\gamma : V \longrightarrow V$ be the $U_q(\widehat{sl_2})$-module isomorphism from V with $\cdot$ to V with $\cdot_\sigma$. By construction and since $\sigma^2 = 1$,

$$\gamma u \cdot v = u \cdot_\sigma \gamma v = \sigma(u) \cdot \gamma v, \qquad \forall u \in U_q(\widehat{sl_2}), \;\; v \in V.$$

Compute:

$$\begin{aligned} \gamma A v &= \gamma(e_0^+ + K_0) \cdot v \\ &= \sigma(e_0^+ + K_0) \cdot \gamma v \\ &= e_1^+ + K_1 \cdot \gamma v \\ &= A^* \gamma v. \end{aligned}$$

And

$$\begin{aligned} \gamma A^* v &= \gamma(e_1^+ + K_1) \cdot v \\ &= \sigma(e_1^+ + K_1) \cdot \gamma v \\ &= e_0^+ + K_0 \cdot \gamma v \\ &= A \gamma v. \end{aligned}$$

Thus by Def 2.1, γ is an isomorphism of TDP's from A, A^* to A^*, A.**.**

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