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On Median of Linear Combination of Order Statistics

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Abstract

Bounds for medians of L-statistics are consider. In nonparametric statistical model with all continuous and strictly increasing distribution function, they obtained by Zielinski. In this paper, as a corollary under beta distribution we change the inequality.

Keywords: L-Statistics; Median; Order Statistics; Inequality

1 Introduction

Let $X_1, ..., X_n$ be a sample from a distribution $F \in \mathcal{F}$, where $\mathcal{F}$ is the class of all continuous and strictly increasing distribution function on their supports. Let $X_{1:n}, ..., X_{n:n}$ be the order statistics, let $T = \sum_{j=1}^{n} \lambda_j X_{j:n}$; $\lambda_j \geq 0$, $j = 1, 2, ..., n$; $\sum_{j=1}^{n} \lambda_k = 1$, be a nontrivial L-statistics (at least two λ's are positive). Let $S = S(X_1, ..., X_n)$ be any function of observations $X_1, ..., X_n$ and let $Med(F, S)$ denote a median (of a distribution) of S if the sample comes from the distribution F.

The L-statistics are very important in nonparametric methods which are linear combinations of order statistics. However, L-statistics may produce very large errors in estimating quantiles in the nonparametric model $\mathcal{F}$ with all continuous increasing distribution function. In some parametric families of distribution L-statistics may perform excellent[3].

In Theorem 1, the sharp bounds for medians of L-statistics in the nonparametric model with all continuous and strictly increasing function are obtained by Zielinski (2007)[4].

Theorem 1: If $T = \sum_{j=k}^{m} \lambda_j X_{j:n}$ is an L-statistics such that $\lambda_k > 0$, $\lambda_m > 0$, $k < m$, and $\lambda_k + \cdots + \lambda_m = 1$, then

$$m(U_{k:n}) \leq Med(F, F(T)) \leq m(U_{m:n}), \quad (1)$$

where $m(U_{k:n})$ and $m(U_{m:n})$ is median of order statistics $U_{k:n}$ and $U_{m:n}$ from a sample of size n from the uniform $(0,1)$ distribution. The bounds are sharp in the sense that for every $\varepsilon > 0$ there exists $F \in \mathcal{F}$ such that $Med(F, F(T)) > m(U_{m:n}) - \varepsilon$ and for every $\eta > 0$ there exists $G \in \mathcal{F}$ such that $Med(G, G(T)) > m(U_{k:n}) - \eta$.

Inequality (1) will be change with determining of distribution. In this paper, we show that how the relation between $m(U_{k:n})$ and $m(U_{m:n})$ induces the new version of inequality (1), under some conditions. The conditions, which we take, concentrate properties of beta distribution.

2 Results

The following results may be satisfy for some random variables which they symmetric distribution about $\frac{1}{2}$, on $(0,1)$. The beta distribution with α and α parameters has this property, that is X has identically distribution as $1 - X$. [1,2]

Corollary 2: For random sample from beta distribution with α and α parameters, then

$$m(X_{k:n}) + m(X_{n-k+1:n}) = 1,$$

where $m(X_{k:n})$ and $m(X_{n-k+1:n})$ is median of $X_{k:n}$ and $X_{n-k+1:n}$ respectively.

Proof. For beta distribution with α and α parameters,

$$X_{k:n} = 1 - X_{n-k+1:n}; \quad k = 1, 2, ..., n,$$

otherhand, by definition of median we have

$$P(X_{k:n} < m(X_{k:n})) = \frac{1}{2},$$

or

$$P(1 - X_{n-k+1:n} < m(X_{k:n})) = \frac{1}{2}.$$

Therefore

$$P(X_{n-k+1:n} > 1 - m(X_{k:n})) = \frac{1}{2} = P(X_{n-k+1:n} < m(X_{n-k+1:n})).$$

Conclude that

$$1 - m(X_{k:n}) = m(X_{n-k+1:n}),$$

or

$$m(X_{k:n}) + m(X_{n-k+1:n}) = 1.$$

Corollary 3. For random variables with conditions (1) in beta distribution with α and α parameters, then

$$m(\Sigma_{k=1}^{n} X_{k:n}) = \Sigma_{k=1}^{n} m(X_{k:n}) = \frac{n}{2}.$$

Proof. If n is even number, then

$$\begin{aligned}\Sigma_{k=1}^{n} m(X_{k:n}) &= [m(X_{1:n}) + m(X_{n:n})] + \cdots + [m(X_{\frac{n}{2}:n}) + m(X_{\frac{n}{2}+1:n})] \\ &= 1 + \cdots + 1 \\ &= \frac{n}{2}.\end{aligned}$$

If n is odd number, then

$$\begin{aligned}\Sigma_{k=1}^{n} m(X_{k:n}) &= [m(X_{1:n}) + m(X_{n:n})] + \cdots + [m(X_{\frac{n-1}{2}:n}) + m(X_{\frac{n+3}{2}:n})] \\ &\quad + m(X_{\frac{n+1}{2}:n}) \\ &= 1 + \cdots + 1 + \frac{1}{2} \\ &= \frac{n-1}{2} + \frac{1}{2} \\ &= \frac{n}{2}.\end{aligned}$$

Now result of corollary 2, changes Theorem 1.

Corollary 4. For random variables with conditions (1) and $m+k = n+1$, then for $k < \frac{n+1}{2}$ if n is odd, or $k < \frac{n}{2}$ if n is even,

$$m(U_{k:n}) \leq Med(F, F(T)) \leq 1 - m(U_{k:n}),$$

for $m > \frac{n+1}{2}$ if n is odd, or $m > \frac{n}{2}$ if n is even,

$$m(U_{m:n}) \leq Med(F, F(T)) \leq 1 - m(U_{m:n}).$$

The lengths of above intervals are $d_1 = 1-2m(U_{k:n})$ and $d_2 = 2m(U_{m:n})-1$ respectively. Obviously,$d_1 = d_2$.

Another property about beta distribution with two parameters obtained. In median subject, the rule of two parameters change with each other, if we apply $1 - X_i$ instead of X_i. If Y obtain from beta distribution with β and α parameters, which it has the the same rule as X. The following corollary show that the translation has affect on parameters on beta distribution.

Corollary 5. With above notation,

$$m(X_{k:n}) + m(Y_{n-k+1:n}) = 1.$$

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