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Identities in (X(YZ))Z with Reverse Arc Graph Varieties of Type (2,0)

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Abstract

Graph algebras establish a connection between directed graphs without multiple edges and special universal algebras of type (2,0). We say that a graph G satisfies a term equation $s \approx t$ if the corresponding graph algebra $\underline{A(G)}$ satisfies $s \approx t$. A class of graph algebras V is called a graph variety if $V = Mod_g\Sigma$ where Σ is a subset of $T(X) \times T(X)$. A graph variety $V' = Mod_g\Sigma^{'}$ is called an $(x(yz))z$ with reverse arc graph variety if $\Sigma^{'}$ is a set of $(x(yz))z$ with reverse arc term equations.

In this paper we characterize identities in each $(x(yz))z$ with reverse arc graph variety.

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1 Introduction

Graph algebras have been invented in [10] to obtain examples of nonfinitely based finite algebras. To recall this concept, let $G = (V, E)$ be a (directed) graph with the vertex set V and the set of edges $E \subseteq V \times V$. Define the *graph algebra* $\underline{A(G)}$ corresponding to G with the underlying set $V \cup \{\infty\}$, where ∞ is a symbol outside V, and with two basic operations, namely a nullary operation pointing to ∞ and a binary one denoted by juxtaposition, given for $u, v \in V \cup \{\infty\}$ by

$$uv = \begin{cases} u, & \text{if} \quad (u, v) \in E, \\ \infty, & \text{otherwise.} \end{cases}$$

In [9] graph varieties had been investigated for finite undirected graphs in order to get graph theoretic results (structure theorems) from universal algebra via graph algebras. In [8] these investigations are extended to arbitrary (finite) directed graphs where the authors ask for a graph theoretic characterization of graph varieties, i.e., of classes of graphs which can be defined by term equations for their corresponding graph algebras. The answer is a theorem of **Birkhoff-type**, which uses graph theoretic closure operations. *A class of finite directed graphs is equational* (i.e., a graph variety) *if and only if it is closed with respect to finite restricted pointed subproducts and isomorphic copies.*

In [2] J. Khampakdee and T. Poomsa-ard characterized identities in the class of $x(yx) \approx x(yy)$ - graph algebras. In [4] T. Poomsa-ard characterized identities in the class of associative graph algebras. In [5], [6] T. Poomsa-ard, J. Wetweerapong and C. Samartkoon characterized identities in the class of idempotent graph algebras and in the class of transitive graph algebras respectively. In [1] A. Apinantpinitwatna and T. Poomsa-ard characterize identities in all biregular leftmost graph varieties.

In this paper we characterize identities in each $(x(yz))z$ with reverse arc graph variety.

2 Terms, identities and graph varieties

In [7] R. Pöschel was introduced terms for graph algebras, the underlying formal language has to contain a binary operation symbol (juxtaposition) and a symbol for the constant ∞ (denoted by ∞, too).

Definition 2.1. The set $T(X)$ of all terms over the alphabet

$$X = \{x_1, x_2, x_3, ...\}$$

is defined inductively as follows:

(i) every variable $x_i, i = 1, 2, 3, ...$, and ∞ are terms;
(ii) if t_1 and t_2 are terms, then t_1t_2 is a term;
(iii) $T(X)$ is the set of all terms which can be obtained from (i) and (ii) in finitely many steps.

Terms built up from the two-element set $X_2 = \{x_1, x_2\}$ of variables are thus binary terms. We denote the set of all binary terms by $T(X_2)$. The leftmost variable of a term t is denoted by $L(t)$ and rightmost variable of a term t is denoted by $R(t)$. A term, in which the symbol ∞ occurs is called a *trivial term.*

Definition 2.2. For each non-trivial term t of type $\tau = (2, 0)$ one can define a directed graph $G(t) = (V(t), E(t))$, where the vertex set $V(t)$ is the set of all variables occurring in t and the edge set $E(t)$ is defined inductively by

$$E(t) = \phi \text{ if } t \text{ is a variable and } E(t_1t_2) = E(t_1) \cup E(t_2) \cup \{(L(t_1), L(t_2))\}$$

where $t = t_1t_2$ is a compound term.

$L(t)$ is called the *root* of the graph $G(t)$, and the pair $(G(t), L(t))$ is the *rooted graph* corresponding to t. Formally, we assign the empty graph ϕ to every trivial term t.

Definition 2.3. A non-trivial term t of type $\tau = (2, 0)$ is called $(x(yz))z$ with reverse arc term if and only if $G(t)$ is a graph with $V(t) = \{x, y, z\}$ and $E(t) = E \cup E'$, where $E = \{(x, y), (x, z), (y, z)\}$, $E' \subseteq \{(y, x), (z, x), (z, y)\}$, $E' \neq \phi$. A term equation $s \approx t$ is called $(x(yz)))z$ with reverse arc equation if s and t are $(x(yz))z$ with reverse arc terms.

Definition 2.4. We say that a graph $G = (V, E)$ *satisfies a term equation* $s \approx t$ if the corresponding graph algebra $\underline{A(G)}$ satisfies $s \approx t$ (i.e., we have $s = t$ for every assignment $V(s) \cup V(t) \to \overline{V \cup \{\infty\}}$), and in this case, we write $G \models s \approx t$. Given a class $\mathcal{G}$ of graphs and a set Σ of term equations (i.e., $\Sigma \subseteq T(X) \times T(X)$) we introduce the following notation:

$G \models \Sigma$ if $G \models s \approx t$ for all $s \approx t \in \Sigma$,

$\mathcal{G} \models s \approx t$ if $G \models s \approx t$ for all $G \in \mathcal{G}$,

$\mathcal{G} \models \Sigma$ if $G \models \Sigma$ for all $G \in \mathcal{G}$,

$Id\mathcal{G} = \{s \approx t \mid s, t \in T(X),\ \mathcal{G} \models s \approx t\}$,

$Mod_g\Sigma = \{G \mid G$ is a graph and $G \models \Sigma\}$,

$\mathcal{V}_g(\mathcal{G}) = Mod_gId\mathcal{G}$.

$\mathcal{V}_g(\mathcal{G})$ is called the *graph variety generated by* $\mathcal{G}$ and $\mathcal{G}$ is called *graph variety* if $\mathcal{V}_g(\mathcal{G}) = \mathcal{G}$. $\mathcal{G}$ is called *equational* if there exists a set Σ' of term equations such that $\mathcal{G} = \mathrm{Mod}_g\Sigma'$. Obviously $\mathcal{V}_g(\mathcal{G}) = \mathcal{G}$ if and only if $\mathcal{G}$ is an equational class.

Definition 2.5. Let $G = (V, E)$ and $G' = (V', E')$ be graphs. A *homomorphism* h from G into G' is a mapping $h : V \to V'$ carrying edges to edges ,that is, for which $(u, v) \in E$ implies $(h(u), h(v)) \in E'$.

3 Identities in $(x(yz))z$ with reverse arc graph varieties

All $(x(yz))z$ with reverse arc graph varieties were characterized in [11] as the following table:

Table 1. $(x(yz))z$ with reverse arc graph varieties and the property of graph.

Graph variety	Property of graph For any $a, b \in V$ if $(a,b),(b,c),$ $(a,c) \in E$
$\mathcal{K}_1 = Mod_g\{(x(yx)z))z \approx (x(y(zy)))z\}$	then $(b,a) \in E$ if and only if $(c,b) \in E$.
$\mathcal{K}_2 = Mod_g\{(x(yx)z))z \approx (x(y(zx)))z\}$	then $(b,a) \in E$ if and only if $(c,a) \in E$.
$\mathcal{K}_3 = Mod_g\{(x(yx)z))z \approx (x((yx)(zy)))z\}$	and $(b,a) \in E$, then $(c,b) \in E$.
$\mathcal{K}_4 = Mod_g\{(x(y(zy)))z \approx (x(y(zx)))z\}$	then $(c,b) \in E$ if and only if $(c,a) \in E$.
$\mathcal{K}_5 = Mod_g\{(x(y(zy)))z \approx (x((yx)(zy)))z\}$	and $(c,b) \in E$, then $(b,a) \in E$.
$\mathcal{K}_6 = Mod_g\{(x(y(zx)))z \approx (x((yx)(zx)))z\}$	and $(c,a) \in E$, then $(b,a) \in E$.
$\mathcal{K}_7 = Mod_g\{(x((yx)(zy)))z \approx (x((yx)(zx)))z\}$	and $(b,a) \in E$, then $(c,b) \in E$ if and only if $(c,a) \in E$.
$\mathcal{K}_8 = Mod_g\{(x(yx)z))z \approx (x(y(zy)))z,$ $(x(yx)z))z \approx (x(y(zx)))z\}$	(i) and $(c,b) \in E$ or $(c,a) \in E$, then $(b,a) \in E$, (ii) and $(b,a) \in E$, then $(c,a),(c,b) \in E$.

Further let T be the set of all $(x(yz))z$ with reverse arc term equations. Since for any $\Sigma \subseteq T$ the $(x(yz))z$ with reverse arc graph variety $Mod_g\Sigma = \bigcap_{s\approx t\in\Sigma} Mod_g\{s \approx t\}$. Let $\mathcal{K}_8 = Mod_g\{(x((yx)z))z \approx (x(y(zy)))z,\ (x(yx)z))$ $z \approx (x(y(zx)))z\} = \mathcal{K}_1 \cap \mathcal{K}_2$. Then, they check that $\mathcal{K} = \{\mathcal{K}_0, \mathcal{K}_1, \mathcal{K}_3, ..., \mathcal{K}_8\}$ is the set of all $(x(yz))z$ with reverse arc graph varieties, where $\mathcal{K}_0 = Mod_g\{(x((yx)z))z \approx (x((yx)z))z\}$ is the class of all graph algebras.

Graph identities were characterized in [3] by the following proposition:

Proposition 3.1. *A non-trivial equation $s \approx t$ is an identity in the class of all graph algebras iff either both terms s and t are trivial or none of them is trivial, $G(s) = G(t)$ and $L(s) = L(t)$.*

Further it was proved.

Proposition 3.2. *Let $G = (V, E)$ be a graph and let $h : X \cup \{\infty\} \longrightarrow V \cup \{\infty\}$ be an evaluation of the variables such that $h(\infty) = \infty$. Consider the canonical extension of h to the set of all terms. Then there holds: if t is a trivial term then $h(t) = \infty$. Otherwise, if $h : G(t) \longrightarrow G$ is a homomorphism of graphs, then $h(t) = h(L(t))$, and if h is not a homomorphism of graphs, then $h(t) = \infty$.*

Proposition 3.3. *Let s and t be non-trivial terms from $T(X)$ with variables $V(s) = V(t) = \{x_0, x_1, ..., x_n\}$ and $L(s) = L(t)$. Then a graph $G = (V, E)$ satisfies $s \approx t$ if and only if the graph algebra $\underline{A(G)}$ has the following property:*

A mapping $h : V(s) \longrightarrow V$ is a homomorphism from $G(s)$ into G iff it is a homomorphism from $G(t)$ into G.

Now we apply our results to characterize all identities in each $(x(yz))z$ with reverse arc graph variety. Clearly, if s and t are trivial, then $s \approx t$ is an identity in each $(x(yz))z$ with reverse arc graph variety and $x \approx x$, $(x \in X)$ is an identity in each $(x(yz))z$ with reverse arc graph variety, too. Further, if s is a trivial term and t is a non-trivial term or both of them are non-trivial with $L(s) \neq L(t)$ or $V(s) \neq V(t)$, then $s \approx t$ is not an identity in every $(x(yz))z$ with reverse arc graph variety, since for a complete graph G we have an evaluation of the variables h such that $h(s) = \infty$ and $h(t) \neq \infty$. So we consider the case that s and t are non-trivial with $L(s) = L(t)$, $V(s) = V(t)$ and different from variables. Before we do this let us introduce some notation. For any non-trivial term t and $x \in V(t)$, let

$A^0_{(x,y)}(t)$=$\{(x, y)\}$, $A^1_{(x,y)}(t)$ be the set of edges in $G(t)$ which join between an out-neighborhood of x and y or join between and in-neighborhood of y and x, $A^2_{(x,y)}(t)$ be the set of edge $(u', v') \in E(t)$ whenever $(u', v') \in A^1_{(u,v)}(t)$ for some $(u, v) \in A^1_{(x,y)}(t)$,...,$A^n_{(x,y)}(t)$ be the set of edge $(u'', v'') \in E(t)$ whenever $(u'', v'') \in A^1_{(u,v)}(t)$ for some $(u, v) \in A^{n-1}_{(x,y)}(t)$. Let $A^*_{(x,y)}(t) = \bigcup\limits_{i=0}^{\infty} A^i_{(x,y)}(t)$.

$B^0_{(x,y)}(t)$=$\{(x, y)\}$, $B^1_{(x,y)}(t)$ be the set of edges in $G(t)$ which join between an out-neighborhood of y and x or join between an in-neighborhood of y and x, $B^2_{(x,y)}(t)$ be the set of edge $(u', v') \in E(t)$ whenever $(u', v') \in B^1_{(u,v)}(t)$ for some $(u, v) \in B^1_{(x,y)}(t)$,...,$B^n_{(x,y)}(t)$ be the set of edge $(u'', v'') \in E(t)$ whenever $(u'', v'') \in B^1_{(u,v)}(t)$ for some $(u, v) \in B^{n-1}_{(x,y)}(t)$. Let $B^*_{(x,y)}(t) = \bigcup\limits_{i=0}^{\infty} B^i_{(x,y)}(t)$.

$C^0_{(x,y)}(t)$=$\{(x, y)\}$, $C^1_{(x,y)}(t)$ be the set of edges in $G(t)$ which join between an in-neighborhood of y and x, $C^2_{(x,y)}(t)$ be the set of edge $(u', v') \in E(t)$ whenever $(u', v') \in C^1_{(u,v)}(t)$ for some $(u, v) \in C^1_{(x,y)}(t)$,...,$C^n_{(x,y)}(t)$ be the set of edge $(u'', v'') \in E(t)$ whenever $(u'', v'') \in C^1_{(u,v)}(t)$ for some $(u, v) \in C^{n-1}_{(x,y)}(t)$. Let $C^*_{(x,y)}(t) = \bigcup\limits_{i=0}^{\infty} C^i_{(x,y)}(t)$.

$D^0_{(x,y)}(t)$=$\{(x, y)\}$, $D^1_{(x,y)}(t)$ be the set of edges in $G(t)$ which join between an out-neighborhood of x and y or join between an in-neighborhood of x and y, $D^2_{(x,y)}(t)$ be the set of edge $(u', v') \in E(t)$ whenever $(u', v') \in D^1_{(u,v)}(t)$ for some $(u, v) \in D^1_{(x,y)}(t)$,...,$D^n_{(x,y)}(t)$ be the set of edge $(u'', v'') \in E(t)$ whenever $(u'', v'') \in D^1_{(u,v)}(t)$ for some $(u, v) \in D^{n-1}_{(x,y)}(t)$. Let $D^*_{(x,y)}(t) = \bigcup\limits_{i=0}^{\infty} D^i_{(x,y)}(t)$.

$F^0_{(x,y)}(t)$=$\{(x, y)\}$, $F^1_{(x,y)}(t)$ be the set of edges in $G(t)$ which join between an out-neighborhood of x and y, $F^2_{(x,y)}(t)$ be the set of edge $(u', v') \in E(t)$ whenever $(u', v') \in F^1_{(u,v)}(t)$ for some $(u, v) \in F^1_{(x,y)}(t)$,...,$F^n_{(x,y)}(t)$ be the set of edge $(u^{''}, v^{''}) \in E(t)$ whenever $(u^{''}, v^{''}) \in F^1_{(u,v)}(t)$ for some $(u, v) \in F^{n-1}_{(x,y)}(t)$. Let $F^*_{(x,y)}(t) = \bigcup\limits_{i=0}^{\infty} F^i_{(x,y)}(t)$.

$H^0_{(x,y)}(t)$=$\{(x, y)\}$, $H^1_{(x,y)}(t)$ be the set of edges in $G(t)$ which join between an out-neighborhood of y and x, $H^2_{(x,y)}(t)$ be the set of edge $(u', v') \in E(t)$ whenever $(u', v') \in H^1_{(u,v)}(t)$ for some $(u, v) \in H^1_{(x,y)}(t)$,...,$H^n_{(x,y)}(t)$ be the set of edge $(u^{''}, v^{''}) \in E(t)$ whenever $(u^{''}, v^{''}) \in H^1_{(u,v)}(t)$ for some $(u, v) \in H^{n-1}_{(x,y)}(t)$. Let $H^*_{(x,y)}(t) = \bigcup\limits_{i=0}^{\infty} H^i_{(x,y)}(t)$.

$I^0_{(x,y)}(t)$=$\{(x, y)\}$, $I^1_{(x,y)}(t)$ be the set of edge in $G(t)$ which join between a both out-neighborhood and in-neighborhood of x and y, $I^2_{(x,y)}(t)$ be the set of edge $(u', v') \in E(t)$ whenever $(u', v') \in I^1_{(u,v)}(t)$ for some $(u, v) \in I^1_{(x,y)}(t)$,...,$I^n_{(x,y)}(t)$ be the set of edge $(u^{''}, v^{''}) \in E(t)$ whenever $(u^{''}, v^{''}) \in I^1_{(u,v)}(t)$ for some $(u, v) \in I^{n-1}_{(x,y)}(t)$. Let $I^*_{(x,y)}(t) = \bigcup\limits_{i=0}^{\infty} I^i_{(x,y)}(t)$.

$J^0_{(x,y)}(t)$=$\{(x, y)\}$, $J^1_{(x,y)}(t)$ be the set of edges in $G(t)$ which join between an in-neighborhood of y and x or join between an out-neighborhood of x and y or join between an out-neighborhood of y and x, $J^2_{(x,y)}(t)$ be the set of edge $(u', v') \in E(t)$ whenever $(u', v') \in J^1_{(u,v)}(t)$ for some $(u, v) \in J^1_{(x,y)}(t)$,...,$J^n_{(x,y)}(t)$ be the set of edge $(u^{''}, v^{''}) \in E(t)$ whenever $(u^{''}, v^{''}) \in J^1_{(u,v)}(t)$ for some $(u, v) \in J^{n-1}_{(x,y)}(t)$. Let $J^*_{(x,y)}(t) = \bigcup\limits_{i=0}^{\infty} J^i_{(x,y)}(t)$.

Then all identities in each $(x(yz))z$ with reverse arc graph variety are characterized by the following theorems:

Theorem 3.1. *Let s and t be non-trivial terms and difference from variables. Then $s \approx t \in Id\mathcal{K}_1$ if and only if*

(i) for any $x \in V(s)$, $(x, x) \in E(s)$ if and only if $(x, x) \in E(t)$,

(ii) for any $x, y \in V(s)$ with $x \neq y$, $(x, y) \in E(s)$ or $(y, x) \in E(s)$ and there exists $(u, v) \in A^*_{(y,x)}(s)$ such that $(v, u) \in E(s)$ if and only if $(x, y) \in E(t)$ or $(y, x) \in E(t)$ and there exists $(u', v') \in A^*_{(y,t)}(t)$ such that $(v', u') \in E(t)$.

Proof. Suppose that there exists $x \in V(s)$ such that $(x, x) \in E(s)$ but $(x, x) \notin E(t)$. Consider the graph $G = (V, E)$ such that $V = \{0, 1\}$, $E = \{(0, 1), (1, 0), (1, 1)\}$. By Table 1, we see that $G \in \mathcal{K}_1$. Let $h : V(s) \longrightarrow V$ such that $h(x) = 0$, $h(y) = 1$ for all other $y \in V(s)$. We see that $h(s) = \infty$, $h(t) = h(L(t))$. Hence $s \approx t \notin Id\mathcal{K}_1$. Suppose that there exist $x, y \in V(s)$ with $x \neq y$, $(x, y) \in E(s)$ or $(y, x) \in E(s)$ and there exists $(u, v) \in A^*_{(y,x)}(s)$ such that $(v, u) \in E(s)$ but $(x, y) \notin E(t)$ and $(y, x) \notin E(t)$ or there exists no $(u', v') \in A^*_{(y,x)}(t)$ such that $(v', u') \in E(t)$. If $(x, y) \notin E(t)$ and $(y, x) \notin E(t)$, then consider the graph $G = (V, E)$ such that $V = \{0, 1, 2\}$, $E = \{(0, 0), (0, 1), (1, 0), (1, 1), (0, 2), (2, 0), (2, 2)\}$. By Table 1, we see that $G \in \mathcal{K}_1$. Let $h : V(s) \longrightarrow V$ such that $h(x) = 1$, $h(y) = 2$ and $h(z) = 0$ for all other $z \in V(s)$. We see that $h(s) = \infty$, $h(t) = h(L(t))$. Hence $s \approx t \notin Id\mathcal{K}_1$. Suppose that $(y, x) \in E(t)$ and there exists no $(u', v') \in A^*_{(y,x)}(t)$ such that $(v', u') \in E(t)$. We see that the subgraph of $G(t)$ induced by A where A is the set of end vertices of edges in $A^*_{(y,x)}(t)$ belong to $\mathcal{K}_1$. Consider the graph $G = (V, E)$ which obtains from $G(t)$ by adding minimum edges to $G(t)$ until $G \in \mathcal{K}_1$. Let $h : V(s) \longrightarrow V$ such that $h(x) = x$ for all $x \in V(s)$. We have that $h(s) = \infty$, $h(t) = h(L(t))$. Hence, $s \approx t \notin Id\mathcal{K}_1$.

Conversely, suppose that s and t are non-trivial terms satisfying (i) and (ii). Let $G = (V, E)$ be a graph in $\mathcal{K}_1$ and let $h : V(s) \longrightarrow V$ be a function. Suppose that h is a homomorphism from $G(s)$ into G and let $(x, y) \in E(t)$. If $x = y$, then by (i), we have $(x, x) \in E(s)$. Hence $(h(x), h(x)) \in E$. If $x \neq y$, then by (ii), we have $(x, y) \in E(s)$ or $(y, x) \in E(s)$ and there exists $(u, v) \in A^*_{(y,x)}(s)$ such that $(v, u) \in E(s)$. If $(x, y) \in E(s)$, then $(h(x), h(y)) \in E$. Suppose that $(y, x) \in E(s)$, since $(x, y) \notin E(s)$, hence there exists $(u, v) \in A^*_{(y,x)}(s)$ such that $(v, u) \in E(s)$. We have $(u, v) \in A^n_{(y,x)}(s)$ for some n with $n \geq 1$. Let $(u, v) \in A^1_{(u',v')}(s)$ where $(u', v') \in A^{n-1}_{(y,x)}(s)$. We have $(u', v'), (v', v') \in E(s)$ or $(u', v'), (u', u') \in E(s)$ or $(u', v'), (u', v), (v', v), (v, v') \in E(s)$ or $(u', v'), (u', v), (v, u'), (v, v') \in E(s)$. Hence $(h(u'), h(v')), (h(v'), h(v')) \in E$ or $(h(u'), h(v')), (h(u'), h(u')) \in E$ or $(h(u'), h(v')), (h(u'), h(v)), (h(v'), h(v)), (h(v), h(v')) \in E$ or $(h(u'), h(v')), (h(u'), h(v)), (h(v), h(u')), (h(v), h(v')) \in E$.

Since $G \in \mathcal{K}_1$, by Table 1, we have $(h(v'), h(u')) \in E$ for each case. Let $(u', v') \in A^1_{(u'',v'')}(s)$ where $(u'', v'') \in A^{n-2}_{(y,x)}(s)$. Consider in the similar way, we have $(h(v''), h(u'')) \in E$. Continue this process we get $(h(x), h(y)) \in E$. Therefore h is a homomorphism from $G(t)$ into G. In the same way, we can prove that if h is a homomorphism from $G(t)$ into G, then it is a homomorphism from $G(s)$ into G. Hence, by Proposition 3.3, we get $s \approx t \in Id\mathcal{K}_1$. □

Theorem 3.2. *Let s and t be non-trivial terms and difference from variables. Then $s \approx t \in Id\mathcal{K}_2$ if and only if*

(i) for any $x \in V(s)$, $(x, x) \in E(s)$ if and only if $(x, x) \in E(t)$,

(ii) for any $x, y \in V(s)$ with $x \neq y$, $(x, y) \in E(s)$ or $(y, x) \in E(s)$ and there exists $(u, v) \in B^*_{(y,x)}(s)$ such that $(v, u) \in E(s)$ if and only if $(x, y) \in E(t)$ or $(y, x) \in E(t)$ and there exists $(u', v') \in B^*_{(y,x)}(t)$ such that $(v', u') \in E(t)$.

Proof. The proof is similar to the proof of Theorem 3.1. □

Theorem 3.3. *Let s and t be non-trivial terms and difference from variables. Then $s \approx t \in Id\mathcal{K}_3$ if and only if*

(i) for any $x \in V(s)$, $(x, x) \in E(s)$ if and only if $(x, x) \in E(t)$,

(ii) for any $x, y \in V(s)$ with $x \neq y$, $(x, y) \in E(s)$ or $(y, x) \in E(s)$ and there exists $(u, v) \in C^*_{(y,x)}(s)$ such that $(v, u) \in E(s)$ if and only if $(x, y) \in E(t)$ or $(y, x) \in E(t)$ and there exists $(u', v') \in C^*_{(y,x)}(t)$ such that $(v', u') \in E(t)$.

Proof. Suppose that there exists $x \in V(s)$ such that $(x,x) \in E(s)$ but $(x,x) \notin E(t)$. Consider the graph $G = (V,E)$ such that $V = \{0,1\}$, $E = \{(0,1),(1,0),(1,1)\}$. By Table 1, we see that $G \in \mathcal{K}_3$. Let $h : V(s) \longrightarrow V$ such that $h(x) = 0$, $h(y) = 1$ for all other $y \in V(s)$. We see that $h(s) = \infty$, $h(t) = h(L(t))$. Hence $s \approx t \notin Id\mathcal{K}_3$. Suppose that there exist $x, y \in V(s)$ with $x \neq y$, $(x,y) \in E(s)$ or $(y,x) \in E(s)$ and there exists $(u,v) \in C^*_{(y,x)}(s)$ such that $(v,u) \in E(s)$ but $(x,y) \notin E(t)$ and $(y,x) \notin E(t)$ or there exists no $(u',v') \in C^*_{(y,x)}(t)$ such that $(v',u') \in E(t)$. If $(x,y) \notin E(t)$ and $(y,x) \notin E(t)$, then consider the graph $G = (V,E)$ such that $V = \{0,1,2\}$, $E = \{(0,0),(0,1),(1,0),(1,1),(0,2),(2,0),(2,2)\}$. By Table 1, we see that $G \in \mathcal{K}_3$. Let $h : V(s) \longrightarrow V$ such that $h(x) = 1$, $h(y) = 2$ and $h(z) = 0$ for all other $z \in V(s)$. We see that $h(s) = \infty$, $h(t) = h(L(t))$. Hence $s \approx t \notin Id\mathcal{K}_3$. Suppose that $(y,x) \in E(t)$ and there exists no $(u',v') \in C^*_{(y,x)}(t)$ such that $(v',u') \in E(t)$. We see that the subgraph of $G(t)$ induced by C where C is the set of end vertices of edges in $C^*_{(y,x)}(t)$ belong to $\mathcal{K}_3$. Consider the graph $G = (V,E)$ which obtains from $G(t)$ by adding minimum edges to $G(t)$ until $G \in \mathcal{K}_3$. Let $h : V(s) \longrightarrow V$ such that $h(x) = x$ for all $x \in V(s)$. We have that $h(s) = \infty$, $h(t) = h(L(t))$. Hence, $s \approx t \notin Id\mathcal{K}_3$.

Conversely, suppose that s and t are non-trivial terms satisfying (i) and (ii). Let $G = (V,E)$ be a graph in $\mathcal{K}_3$ and let $h : V(s) \longrightarrow V$ be a function. Suppose that h is a homomorphism from $G(s)$ into G and let $(x,y) \in E(t)$. If $x = y$, then by (i), we have $(x,x) \in E(s)$. Hence $(h(x),h(x)) \in E$. If $x \neq y$, then by (ii), we have $(x,y) \in E(s)$ or $(y,x) \in E(s)$ and there exists $(u,v) \in C^*_{(y,x)}(s)$ such that $(v,u) \in E(s)$. If $(x,y) \in E(s)$, then $(h(x),h(y)) \in E$. Suppose that $(y,x) \in E(s)$, since $(x,y) \notin E(s)$, hence there exists $(u,v) \in C^*_{(y,x)}(s)$ such that $(v,u) \in E(s)$. We have $(u,v) \in C^n_{(y,x)}(s)$ for some n with $n \geq 1$. Let $(u,v) \in C^1_{(u',v')}(s)$ where $(u',v') \in C^{n-1}_{(y,x)}(s)$. We have $(u',v'),(u',u') \in E(s)$ or $(u',v'),(v,u'),(u',v),(v,v') \in E(s)$. Hence $(h(u'),h(v')),(h(u'),h(u')) \in E$ or $(h(u'),h(v')),(h(v),h(u')),(h(u'),h(v)),(h(v),h(v')) \in E$.

Since $G \in \mathcal{K}_3$, by Table 1, we have $(h(v'),h(u')) \in E$ for each case. Let $(u',v') \in C^1_{(u'',v'')}(s)$ where $(u'',v'') \in C^{n-2}_{(y,x)}(s)$. Consider in the similar way, we have $(h(v''),h(u'')) \in E$. Continue this process we get $(h(x),h(y)) \in E$. Therefore h is a homomorphism from $G(t)$ into G. In the same way, we can prove that if h is a homomorphism from $G(t)$ into G, then it is a homomorphism from $G(s)$ into G. Hence by Proposition 3.3, we get $s \approx t \in Id\mathcal{K}_3$. □

Theorem 3.4. *Let s and t be non-trivial terms and difference from variables. Then $s \approx t \in Id\mathcal{K}_4$ if and only if*

(i) for any $x \in V(s)$, $(x,x) \in E(s)$ if and only if $(x,x) \in E(t)$,

(ii) for any $x, y \in V(s)$ with $x \neq y$, $(x,y) \in E(s)$ or $(y,x) \in E(s)$ and there exists $(u,v) \in D^*_{(y,x)}(s)$ such that $(v,u) \in E(s)$ if and only if $(x,y) \in E(t)$ or $(y,x) \in E(t)$ and there exists $(u',v') \in D^*_{(y,x)}(t)$ such that $(v',u') \in E(t)$.

Proof. The proof is similar to the proof of Theorem 3.1. □

Theorem 3.5. *Let s and t be non-trivial terms and difference from variables. Then $s \approx t \in Id\mathcal{K}_5$ if and only if*

(i) for any $x \in V(s)$, $(x,x) \in E(s)$ if and only if $(x,x) \in E(t)$,

(ii) for any $x, y \in V(s)$ with $x \neq y$, $(x,y) \in E(s)$ or $(y,x) \in E(s)$ and there exists $(u,v) \in F^*_{(y,x)}(s)$ such that $(v,u) \in E(s)$ if and only if $(x,y) \in E(t)$ or $(y,x) \in E(t)$ and there exists $(u',v') \in F^*_{(y,x)}(t)$ such that $(v',u') \in E(t)$.

Proof. The proof is similar to the proof of Theorem 3.3. □

Theorem 3.6. *Let s and t be non-trivial terms and difference from variables. Then $s \approx t \in Id\mathcal{K}_6$ if and only if*

(i) for any $x \in V(s)$, $(x,x) \in E(s)$ if and only if $(x,x) \in E(t)$,

(ii) for any $x, y \in V(s)$ with $x \neq y$, $(x,y) \in E(s)$ or $(y,x) \in E(s)$ and there exists $(u,v) \in H^*_{(y,x)}(s)$ such that $(v,u) \in E(s)$ if and only if $(x,y) \in E(t)$ or $(y,x) \in E(t)$ and there exists $(u',v') \in H^*_{(y,x)}(t)$ such that $(v',u') \in E(t)$.

Proof. Suppose that there exists $x \in V(s)$ such that $(x,x) \in E(s)$ but $(x,x) \notin E(t)$. Consider the graph $G = (V,E)$ such that $V = \{0,1\}$, $E = \{(0,1),(1,0),(1,1)\}$. By Table 1, we see that $G \in \mathcal{K}_6$. Let $h : V(s) \longrightarrow V$ such that $h(x) = 0$, $h(y) = 1$ for all other $y \in V(s)$. We see that $h(s) = \infty$, $h(t) = h(L(t))$. Hence $s \approx t \notin Id\mathcal{K}_6$. Suppose that there exist $x, y \in V(s)$ with $x \neq y$, $(x,y) \in E(s)$ or $(y,x) \in E(s)$ and there exists $(u,v) \in H^*_{(y,x)}(s)$ such that $(v,u) \in E(s)$ but $(x,y) \notin E(t)$ and $(y,x) \notin E(t)$ or there exists no $(u',v') \in H^*_{(y,x)}(t)$ such that $(v',u') \in E(t)$. If $(x,y) \notin E(t)$ and $(y,x) \notin E(t)$, then consider the graph $G = (V,E)$ such that $V = \{0,1,2\}$, $E = \{(0,0),(0,1),(1,0),(1,1),(0,2),(2,0),(2,2)\}$. By Table 1, we see that $G \in \mathcal{K}_6$. Let $h : V(s) \longrightarrow V$ such that $h(x) = 1$, $h(y) = 2$ and $h(z) = 0$ for all other $z \in V(s)$. We see that $h(s) = \infty$, $h(t) = h(L(t))$. Hence $s \approx t \notin Id\mathcal{K}_6$. Suppose that $(y,x) \in E(t)$ and there exists no $(u',v') \in H^*_{(y,x)}(t)$ such that $(v',u') \in E(t)$. We see that the subgraph of $G(t)$ induced by H where H is the set of end vertices of edges in $H^*_{(y,x)}(t)$ belong to $\mathcal{K}_6$. Consider the graph $G = (V,E)$ which obtains from $G(t)$ by adding minimum edges to $G(t)$ until $G \in \mathcal{K}_6$. Let $h : V(s) \longrightarrow V$ such that $h(x) = x$ for all $x \in V(s)$. We have that $h(s) = \infty$, $h(t) = h(L(t))$. Hence $s \approx t \notin Id\mathcal{K}_6$.

Conversely, suppose that s and t are non-trivial terms satisfying (i) and (ii). Let $G = (V,E)$ be a graph in $\mathcal{K}_6$ and let $h : V(s) \longrightarrow V$ be a function. Suppose that h is a homomorphism from $G(s)$ into G and let $(x,y) \in E(t)$. If $x = y$, then by (i), we have $(x,x) \in E(s)$. Hence $(h(x),h(x)) \in E$. If $x \neq y$, then by (ii), we have $(x,y) \in E(s)$ or $(y,x) \in E(s)$ and there exists $(u,v) \in H^*_{(y,x)}(s)$ such that $(v,u) \in E(s)$. If $(x,y) \in E(s)$, then $(h(x),h(y)) \in E$. Suppose that $(y,x) \in E(s)$, since $(x,y) \notin E(s)$, hence there exists $(u,v) \in H^*_{(y,x)}(s)$ such that

$(v,u)\in E(s)$. We have $(u,v)\in H^n_{(y,x)}(s)$ for some n with $n\geq 1$. Let $(u,v)\in H^1_{(u',v')}(s)$ where $(u',v')\in H^{n-1}_{(y,x)}(s)$. We have $(u',v'),(u',v),(v,u'),(v',v)\in E(s)$. Hence $(h(u'),h(v')),(h(u'),h(v)),(h(v),h(u')),(h(v'),h(v))\in E$.

Since $G\in\mathcal{K}_6$, by Table 1, we have $(h(v'),h(u'))\in E$ for each case. Let $(u',v')\in H^1_{(u'',v'')}(s)$ where $(u'',v'')\in H^{n-2}_{(y,x)}(s)$. Consider in the similar way, we have $(h(v''),h(u''))\in E$. Continue this process we get $(h(x),h(y))\in E$. Therefore h is a homomorphism from $G(t)$ into G. In the same way, we can prove that if h is a homomorphism from $G(t)$ into G, then it is a homomorphism from $G(s)$ into G. Hence by Proposition 3.3, we get $s\approx t\in Id\mathcal{K}_6$. □

Theorem 3.7. *Let s and t be non-trivial terms and difference from variables. Then $s\approx t\in Id\mathcal{K}_7$ if and only if*

(i) for any $x\in V(s)$, $(x,x)\in E(s)$ if and only if $(x,x)\in E(t)$,

(ii) for any $x,y\in V(s)$ with $x\neq y$, $(x,y)\in E(s)$ or $(y,x)\in E(s)$ and there exists $(u,v)\in I^*_{(y,x)}(s)$ such that $(v,u)\in E(s)$ if and only if $(x,y)\in E(t)$ or $(y,x)\in E(t)$ and there exists $(u',v')\in I^*_{(y,x)}(t)$ such that $(v',u')\in E(t)$.

Proof. The proof is similar to the proof of Theorem 3.6. □

Theorem 3.8. *Let s and t be non-trivial terms and difference from variables. Then $s\approx t\in Id\mathcal{K}_8$ if and only if*

(i) for any $x\in V(s)$, $(x,x)\in E(s)$ if and only if $(x,x)\in E(t)$,

(ii) for any $x,y\in V(s)$ with $x\neq y$, $(x,y)\in E(s)$ or $(y,x)\in E(s)$ and there exists $(u,v)\in J^*_{(y,x)}(s)$ such that $(v,u)\in E(s)$ if and only if $(x,y)\in E(t)$ or $(y,x)\in E(t)$ and there exists $(u',v')\in J^*_{(y,x)}(t)$ such that $(v',u')\in E(t)$.

Proof. Suppose that there exists $x\in V(s)$ such that $(x,x)\in E(s)$ but $(x,x)\notin E(t)$. Consider the graph $G=(V,E)$ such that $V=\{0,1\}$, $E=\{(0,1),(1,0),(1,1)\}$. By Table 1, we see that $G\in\mathcal{K}_8$. Let $h:V(s)\longrightarrow V$ such that $h(x)=0$, $h(y)=1$ for all other $y\in V(s)$. We see that $h(s)=\infty$, $h(t)=h(L(t))$. Hence $s\approx t\notin Id\mathcal{K}_8$. Suppose that there exist $x,y\in V(s)$ with $x\neq y$, $(x,y)\in E(s)$ or $(y,x)\in E(s)$ and there exists $(u,v)\in J^*_{(y,x)}(s)$ such that $(v,u)\in E(s)$ but $(x,y)\notin E(t)$ and $(y,x)\notin E(t)$ or there exists no $(u',v')\in J^*_{(y,x)}(t)$ such that $(v',u')\in E(t)$. If $(x,y)\notin E(t)$ and $(y,x)\notin E(t)$, then consider the graph $G=(V,E)$ such that $V=\{0,1,2\}$, $E=\{(0,0),(0,1),(1,0),(1,1),(0,2),(2,0),(2,2)\}$. By Table 1, we see that $G\in\mathcal{K}_8$. Let $h:V(s)\longrightarrow V$ such that $h(x)=1$, $h(y)=2$ and $h(z)=0$ for all other $z\in V(s)$. We see that $h(s)=\infty$, $h(t)=h(L(t))$. Hence $s\approx t\notin Id\mathcal{K}_8$. Suppose that $(y,x)\in E$ and there exists no $(u',v')\in J^*_{(y,x)}(t)$ such that $(v',u')\in E(t)$. We see that the subgraph of $G(t)$ induced by J where J is the set of end vertices of edges in $J^*_{(y,x)}(t)$ belong to $\mathcal{K}_8$. Consider the graph $G=(V,E)$ which obtains from $G(t)$ by adding minimum edges to $G(t)$ until $G\in\mathcal{K}_8$. Let $h:V(s)\longrightarrow V$ such that $h(x)=x$ for all $x\in V(s)$. We have that $h(s)=\infty$, $h(t)=h(L(t))$. Hence $s\approx t\notin Id\mathcal{K}_8$.

Conversely, suppose that s and t are non-trivial terms satisfying (i) and (ii). Let $G = (V, E)$ be a graph in $\mathcal{K}_8$ and let $h : V(s) \longrightarrow V$ be a function. Suppose that h is a homomorphism from $G(s)$ into G and let $(x, y) \in E(t)$. If $x = y$, then by (i), we have $(x, x) \in E(s)$. Hence $(h(x), h(x)) \in E$. If $x \neq y$, then by (ii), we have $(x, y) \in E(s)$ or $(y, x) \in E(s)$ and there exists $(u, v) \in J^*_{(y,x)}(s)$ such that $(v, u) \in E(s)$. If $(x, y) \in E(s)$, then $(h(x), h(y)) \in E$. Suppose that $(y, x) \in E(s)$, since $(x, y) \notin E(s)$, hence there exists $(u, v) \in J^*_{(y,x)}(s)$ such that $(v, u) \in E(s)$. We have $(u, v) \in J^n_{(y,x)}(s)$ for some n with $n \geq 1$. Let $(u, v) \in J^1_{(u',v')}(s)$ where $(u', v') \in J^{n-1}_{(y,x)}(s)$. We have $(u', v'), (u', u') \in E(s)$ or $(u', v'), (u', v), (v, u'), (v, v') \in E(s)$ or $(u', v'), (u', u), (u, v'), (v', u) \in E(s)$ or $(u', v'), (u', v), (v, u'), (v', v) \in E(s)$. Hence $(h(u'), h(v')), (h(u'), h(u')) \in E$ or $(h(u'), h(v')), (h(u'), h(v)), (h(v), h($
$u')), (h(v), h(v')) \in E$ or $(h(u'), h(v')), (h(u'), h(u)), (h(u), h(v')), (h(v'), h(u))$
$\in E$ or $(h(u'), h(v')), (h(u'), h(v)), (h(v), h(u'), (h(v'), h(v)) \in E$.

Since $G \in \mathcal{K}_8$, by Table 1, we have $(h(v'), h(u')) \in E$ for each case. Let $(u', v') \in J^1_{(u'',v'')}(s)$ where $(u'', v'') \in J^{n-2}_{(y,x)}(s)$. Consider in the similar way, we have $(h(v''), h(u'')) \in E$. Continue this process we get $(h(x), h(y)) \in E$. Therefore h is a homomorphism from $G(t)$ into G. In the same way, we can prove that if h is a homomorphism from $G(t)$ into G, then it is a homomorphism from $G(s)$ into G. Hence by Proposition 3.3, we get $s \approx t \in Id\mathcal{K}_8$. $\square$

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