International Mathematical Forum, 5, 2010, no. 20, 957 - 966

A Note on Lightlike Hypersurfaces

Dhruwa Narain and Sachin Kumar Srivastava

Department of Mathematics and Statistics
D.D.U. Gorakhpur University, Gorakhpur-273009, India
Sachin_srivastava31@yahoo.com

Abstract

In this paper, we have studied the properties of Lightlike hypersurfaces of indefinite Sasakian manifold which are tangent to the structure vector field ξ.

Mathematics Subject Classification: 53C15, 53C25, 53C50

Keywords: Lightlike hypersurfaces, Indefinite Sasakian manifold, Screen distribution

1. Introduction

The general theory of lightlike (degenerate) submanifolds of semi- Riemannian (or Riemannian) manifolds is one of the interesting topic of Differential Geometry. Many authors have studied lightlike hypersurfaces (or submanifolds) of semi-Riemannian manifolds [5], [6] and others.

The aim of this note is to study the properties of lightlike hypersurfaces of indefinite Sasakian manifold which are tangent to the structure vector field.

2. Preliminaries

A $(2n+1)-$dimensional semi-Riemannian manifold (M, g) is said to be an indefinite Sasakian manifold if it admits an almost contact structure (ϕ, ξ, η), i.e. ϕ is a tensor field of type (1, 1) of rank $2n$, ξ is a vector field, η is a $1-$form, satisfying

(2.1) $$\phi^2 = -I + \eta \otimes \xi$$

where I denotes the identity transformation.

(2.2) $$\eta(\xi) = 1$$

(2.3)(a) $$\eta\, o\, \phi = 0 \qquad (b) \qquad \phi\xi = 0$$

$$g(\phi X,\, \phi Y) = g(X,\, Y) - \eta(X)\,\eta(Y) \tag{2.4}$$

$$g(\xi, X) = \eta(X) \tag{2.5}$$

$$(\nabla_X \eta)\, Y = g(\phi X,\, Y) \tag{2.6}$$

$$(\nabla_X \phi)\, Y = g(X,\, Y)\xi - \eta(Y)X \tag{2.7}$$

$$\nabla_X \xi = -\,\phi X, \qquad \forall X,\, Y \in \Gamma(TM) \tag{2.8}$$

where ∇ is the Levi-Civita connection for a semi-Riemannian metric g.

If we define $'F(X,Y) = g(\phi X, Y)$, then in addition to above relation we find

$$'F(X,Y) + {}'F(Y,X) = 0 \tag{2.9}$$

$$'F(X,\phi Y) = {}'F(Y,\phi X) \tag{2.10}$$

$$'F(\phi X,\phi Y) = {}'F(X,Y) \tag{2.11}$$

A plane section ϕ in T_pM is called a $\phi-$section if it is spanned by X and ϕX, where X is a unit tangent vector field orthogonal to ξ. The sectional curvature of a $\phi-$section π is called a *$\phi-$sectional curvature*. A Sasakian manifold M with constant $\phi-$ sectional curvature, c is said to be a *Sasakian space form* and is denoted by $M(c)$. The curvature tensor R of a Sasakian space form $M(c)$ is given by [7]

$$\begin{aligned} R(X,Y)\,Z = {} & \frac{c+3}{4}(g(Y,Z)X - g(X,Z)Y) + \frac{c-1}{4}(\eta(X)\eta(Z)Y \\ & -\eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi + {}'F(Y,Z)\phi X \\ & -{}'F(X,Z)\phi Y - 2'F(X,Y)\phi Z), \qquad \forall X,Y,Z \in \Gamma(TM) \end{aligned} \tag{2.12}$$

Let (M,g) be a $(2n+1)-$dimensional semi-Riemanniann manifold with index s, $0 < s < 2n+1$ and let $(\tilde{M},\, \tilde{g})$ be a hypersurface of M, with $\tilde{g} = g_{|M}$, $\tilde{M}$ is a lightlike hypersurface of M if $\tilde{g}$ is of constant rank $(2n-1)$ and the normal bundle $T\tilde{M}^{\perp}$ is a distribution of rank 1 on $\tilde{M}$ [5]. A complementary bundle of $T\tilde{M}^{\perp}$ in $T\tilde{M}$ is a rank $(2n-1)$ non-degenerate distribution over M. It is called *screen distribution* and is often denoted by $S(T\tilde{M})$. A lightlike hypersurface endowed with a specific screen distribution is denoted by the triple $(\tilde{M}, g, S(T\tilde{M}))$.

The following characterisation is prove in [5].

Proposition (2.1). *Let $(\tilde{M},\tilde{g})$ of an $(2n+1)-$dimensional semi-Riemannian manifold (M,g). Then the following assertions are equivalent :*

(i) *$\tilde{M}$ is a lightlike hypersurface of M.*

(ii) *g has a constant rank $2n-1$ on $\tilde{M}$.*

(iii) *$T\tilde{M}^{\perp} = \bigcup_{x\in M} T_x\tilde{M}^{\perp}$ is a distribution on $\tilde{M}$.*

As $T\tilde{M}^{\perp}$ lies in the tangent bundle, the following result has an important role in studying the geometry of a lightlike hypersurface.

Proposition (2.2)([5]). *Let $(\tilde{M}, \tilde{g}, S(T\tilde{M}))$ of a lightlike hypersurface of M. Then there exists a unique vector subbundle $tr(T\tilde{M})$ of rank 1 over $\tilde{M}$ such that for any non-zero section E of $T\tilde{M}^{\perp}$ on a coordinate neighbourhood $\mathcal{U} \subset \tilde{M}$, there exists a unique section N of $tr(T\tilde{M})$ on $\mathcal{U}$ satisfying*

$$\tilde{g}(N, E) = 1 \tag{2.13}$$

and

$$\tilde{g}(N, N) = \tilde{g}(N, W) = 0, \qquad \forall W \in \Gamma(S(T\tilde{M})\big|_{\mathcal{U}}).$$

Remark. Throughout the paper, all manifolds are supposed to be paracompact and smooth. We denote $\Gamma(E)$ the smooth sections of the vector bundle E. Also by $\perp$ and $\oplus$ we denote the orthogonal and non-orthogonal direct sum of two vector bundles. By proposition 2.2 we may write down the following decompositions [5] :

$$T\tilde{M} = S(T\tilde{M}) \perp (T\tilde{M}^{\perp}) \tag{2.15}$$

$$TM\big|_{\tilde{M}} = T\tilde{M} \oplus tr(T\tilde{M}) \tag{2.16}$$

and

$$TM\big|_{\tilde{M}} = S(T\tilde{M}) \perp (T\tilde{M}^{\perp} \oplus tr(T\tilde{M})). \tag{2.17}$$

Let ∇ be the Levi-Civita connection on (M, g), then by using (2.17), we have Gauss and Weingarten formulae in the form

$$\nabla_X Y = \tilde{\nabla}_X Y + h(X, Y), \qquad \forall X, Y \in \Gamma(T\tilde{M}) \tag{2.18}$$

and

$$\nabla_X V = -A_V X + \tilde{\nabla}^{\perp}_X V, \qquad \forall X, Y \in \Gamma(T\tilde{M}) \qquad \forall V \in \Gamma(tr(T\tilde{M})) \tag{2.19}$$

where $\tilde{\nabla}_X Y$, $A_V X \in \Gamma(T\tilde{M})$ and $h(X, Y)$, $\tilde{\nabla}^{\perp}_X V \in \Gamma(tr(T\tilde{M}))$.

$\tilde{M}$ is a symmetric linear connection on $\tilde{M}$ called an induced linear connection, $\tilde{\nabla}^{\perp}$ is a linear connection on the vector bundle $tr(T\tilde{M})$. h is a $\Gamma(tr(T\tilde{M}))-$ valued symmetric bilinear form and A_V is the shape operator of $\tilde{M}$ concerning V.

Equivalently, consider a normalizing pair $\{E, N\}$ as in Proposition 2.2. Then (2.18) and (2.19) take the form

$$\nabla_X Y = \tilde{\nabla}_X Y + B(X,Y)N, \qquad \forall X, Y \in \Gamma(T\tilde{M}\big|_{\mathcal{U}}) \tag{2.20}$$

and

$$\nabla_X N = -\,A_N X + \tau(X)N, \qquad \forall X \in \Gamma(T\tilde{M}\big|_{\mathcal{U}}) \tag{2.21}$$

It is important to mention that the second fundamental form B is independent of choice of screen distribution, infact, from (2.20) and (2.21), we obtain

$$B(X,Y) = \tilde{g}(\nabla_X Y, E), \tag{2.22}$$

and

$$\tau(X) = \tilde{g}(\tilde{\nabla}^{\perp}_X N, E). \tag{2.23}$$

Let P be the projection morphism of $S(T\tilde{M})$ with respect to the orthogonal decomposition of $T\tilde{M}$. We have

$$\nabla_X PY = \nabla^*_X PY + C(X, PY) \tag{2.24}$$

and

$$\nabla_X E = -\,A^*_E X - \tau(X)E. \tag{2.25}$$

where $\nabla^*_X PY$ and $A^*_E X$ belongs to $\Gamma(S(T\tilde{M})).C$, A^*_E and ∇^* are called the local second fundamental form, the local shape operator and the induced connection on $S(T\tilde{M})$. The induced linear connection $\tilde{\nabla}$ is not a metric connection and we have

$$(\tilde{\nabla}_X \tilde{g})(Y,Z) = B(X,Y)\alpha(Z) + B(X,Z)\alpha(Y), \qquad \forall X, Y \in \Gamma(T\tilde{M}\big|_{\mathcal{U}}), \tag{2.26}$$

where α is a differential 1 form locally defined on $\tilde{M}$ by $\alpha(.) = g(N, .)$. Also, we have the following identities

$$\tilde{g}(A^*_E X, PY) = B(X, PY), \tag{2.27}$$

$$\tilde{g}(A^*_E X, N) = 0, \tag{2.28}$$

and

$$B(X,\ E) = 0. \tag{2.29}$$

3. Curvature Tensor

Let R and $\tilde{R}$ be the curvature tensors of M and $\tilde{M}$. Since

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \tag{3.1}$$

$$\tilde{R}(X,Y)Z = \tilde{\nabla}_X \tilde{\nabla}_Y Z - \tilde{\nabla}_Y \tilde{\nabla}_X Z - \tilde{\nabla}_{[X,Y]} Z \tag{3.2}$$

where $\tilde{\nabla}$ is induced connection on the hypersurface $\tilde{M}$. From (3.2) by use of (2.20) and (2.21), we have

$$R(X,Y)Z = \tilde{R}(X,Y)Z + B(X,Z)A_N Y - B(Y,Z)A_N X + \{(\nabla_X B)(Y,Z)$$
$$-(\nabla_Y B)(X,Z)(X,Z) + \tau(X)B(Y,Z) - \tau(Y)B(X,Z)\}N.$$

This leads to the following theorem :

Theorem (3.1). *On the lightlike hypersurface $\tilde{M}$ of indefinite Sasakian manifold M. The curvature tensor of M and $\tilde{M}$ are related as*

$$R(X,Y)Z = \tilde{R}(X,Y)Z + B(X,Z)A_N Y - B(Y,Z)A_N X + \tag{3.3}$$
$$\{(\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) + \tau(X)B(Y,Z) - \tau(Y)B(X,Z)\}N,$$

$$(\nabla_X B)(Y,Z) = X.B(Y,Z) - B(\nabla_X Y, Z) - B(Y, \nabla_X Z). \tag{3.4}$$

Since $\{E,N\}$ is normalising pair. Therefore from theorem (3.1) we can state the following corollary :

Corollary (3.1.1). *On the lightlike hypersurface $\tilde{M}$ of indefinite Sasakian manifold M, we have*

$$'R(X,Y,Z,E) = (\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) + \tau(X)B(Y,Z) \tag{3.5}$$
$$-\tau(Y)B(X,Z) \qquad \forall X,Y,Z \in \Gamma(T\tilde{M}\big|_{\mathcal{U}})$$

where,

$$'R(X,Y,Z,E) = g(R(X,Y)Z,E) \tag{3.6}$$

From (3.5) and by using (2.12) we have the following theorem :

Theorem (3.2). *On the lightlike hypersurface $\tilde{M}$ of indefinite Sasakian manifold, with $\xi \in (T\tilde{M})$, we have*

$$(\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) = \tau(Y)B(X,Z) - \tau(X)B(Y,Z)$$

$$+\frac{c-1}{4}\{'F(Y,Z)u(X) - 'F(X,Z)u(Y) - 2'F(X,Y)u(Z)\}.$$

4. Lightlike Hypersurfaces of Indefinite Sasakian Manifolds :

Let (M, ϕ, ξ, η, g) be an indefinite Sasakian manifold and $(\tilde{M}, \tilde{g})$ be its lightlike hypersurface, tangent to the structure vector field $\xi(\xi \in T\tilde{M})$. If E is a local section of $T\tilde{M}^{\perp}$, then $\tilde{g}(\phi E, E) = 0$ and ϕE is tangent to $\tilde{M}$. Thus $\phi(T\tilde{M}^{\perp})$ is a distribution of rank 1 such that $\phi(T\tilde{M}^{\perp}) \cap T\tilde{M}^{\perp} = \{0\}$. This enables us to choose a screen distribution $S(T\tilde{M})$ such that it contains $\phi(T\tilde{M}^{\perp})$ as vector subbundle.

We consider a local section N of $tr(T\tilde{M})$. Since $\tilde{g}(\phi N, E) = -\tilde{g}(N, \phi E) = 0$, we deduce that ϕN is also tangent to $\tilde{M}$ and belongs to $S(T\tilde{M})$. On the other hand, since $\tilde{g}(\phi N, N) = 0$, we see that the components of ϕN with respect to E vanishes. Thus $\phi N \in \Gamma(S(T\tilde{M}))$.

From (2.4), we have $\tilde{g}(\phi N, \phi E) = 1$. Therefore $\phi(T\tilde{M}^{\perp}) \oplus \phi(tr(T\tilde{M}))$ (direct sum but not orthogonal) is nondegenerate vector subbundle of $S(T\tilde{M})$ of rank 2.

It is known [8] that if $\tilde{M}$ is tangent to the structure vector field ξ, then ξ belongs to $S(T\tilde{M})$. Using this, and since

$$\tilde{g}(\phi E, \xi) = \tilde{g}(\phi N, \xi) = 0$$

there exists a nondegenerate distribution D_0 of rank $(2n-4)$ on $\tilde{M}$ such that

$$S(T\tilde{M}) = \{\phi(T\tilde{M}^{\perp}) \oplus \phi(tr(T\tilde{M}))\} \perp D_0 \perp <\xi> \tag{4.1}$$

where $<\xi> = \text{span}\ \{\xi\}$. The distribution D_0 is an invariant under ϕ, that is

$$\phi(D_0) = D_0.$$

Moreover, from (2.15) and (4.1), we obtain the decomposition

$$T\tilde{M} = \{\phi(T\tilde{M}^{\perp}) \oplus (tr(T\tilde{M}))\} \perp D_0 \perp <\xi> \perp T\tilde{M}^{\perp}. \tag{4.2}$$

Now, we consider the distributions on M, $D := T\tilde{M}^{\perp} \perp \phi(T\tilde{M}^{\perp}) \perp D_0$, $D' := \phi(tr(T\tilde{M}))$. Then D is invariant under ϕ and

$$T\tilde{M} = D \oplus D' \perp <\xi>\ . \tag{4.3}$$

Now, we consider the local lightlike vector fields $U := -\phi N$, $V := -\phi E$. Then, from (4.3) any $X \in \Gamma(T\tilde{M})$ is written as

$$X = RX + QX + \eta(X)\xi, \tag{4.4}$$

$$QX = u(X)U, \tag{4.5}$$

where R and Q are the projection morphisms of $T\tilde{M}$ into D and D', respectively, and u is a differential 1−form locally defined on $\tilde{M}$ by $u(\cdot) := g(V, \cdot)$.

Applying ϕ to X and using (2.3) (note that $\phi^2 N = -N$), we obtain

$$\phi X = \tilde{\phi} X + u(X)N, \qquad \text{where } \tilde{\phi} \text{ is a tensor field of type } (1,1) \tag{4.6}$$

defined on $\tilde{M}$ by $\tilde{\phi} X := \phi RX$ and we also have

$$\tilde{\phi}^2 X = -X + \eta(X)\xi + u(X)U, \qquad \forall X \in \Gamma(T\tilde{M}) \tag{4.7}$$

Now applying $\tilde{\phi}$ to $\tilde{\phi}^2 X$ and since $\phi U = 0$, we obtain

$$\tilde{\phi}^3 + \tilde{\phi} = 0. \tag{4.8}$$

This shows that $\tilde{\phi}$ is on $f-$structure [7] of constant rank, we also have

$$\tilde{g}(\tilde{\phi} X, \tilde{\phi} Y) = g(X, Y) - \eta(X)\eta(Y) - u(Y)v(X) - u(X)v(Y) \tag{4.9}$$

where v is a locally defined on $\tilde{M}$ by $v(\cdot) = g(U, \cdot)$.

From the straight forward calculations we can state the following theorems :

Theorem (4.1). *Let $\tilde{M}$ be a lightlike hypersurface of an indefinite Sasakian manifold M with $\xi \in T\tilde{M}$. Then, we have*

$$\tilde{\nabla}_X \xi = -\tilde{\phi} X \tag{4.10}$$

and

$$B(X, \xi) = -u(X). \tag{4.11}$$

Theorem (4.2). *Let $\tilde{M}$ be a lightlike hypersurface of an indefinite Sasakian manifold M with $\xi \in T\tilde{M}$. Then, we have*

$$(\tilde{\nabla}_X u)Y = -B(X, \phi Y) - u(Y)\,\tau(X) \tag{4.12}$$

$$(4.13)\qquad (\tilde{\nabla}_X \tilde{\phi})Y = g(X,Y)\,\xi - \eta(Y)X - B(X,Y)U + u(Y)\,A_N X.$$

Replacing Y by U in (4.12) and (4.13), we have

$$A_N X = -\tilde{\phi}(\tilde{\nabla}_X U) - \tilde{g}(X,U)\xi + B(X,U)U$$

and

$$\tau(X) = u(\tilde{\nabla}_X U) = -\,(\tilde{\nabla}_X u)(U).$$

From above we can state the following theorem :

Theorem (4.3). *Let $\tilde{M}$ be a lightlike hypersurface of an indefinite Sasakian manifold. Then for any $X \in \Gamma(T\tilde{M})$, we have*

$$(4.14)\qquad A_N X = -\,\tilde{\phi}(\nabla_X U) - v(X)\xi + B(X,U)U$$

$$(4.15)\qquad \tau(X) = u(\tilde{\nabla}_X U) = -\,(\tilde{\nabla}_X u)\,U$$

$$(4.16)\qquad A_N X = \eta(A_N X)\xi + u(A_N X)U, \quad \tau(X) = 0,$$

is parallel i.e. $\tilde{\nabla}_X U = 0$. since $u(Y) = 0, \forall Y \in \Gamma(D)$. Then (4.13) reduced to the equation

$$(\tilde{\nabla}_X \tilde{\phi})(Y) = \tilde{g}(X,Y)\xi - \eta(Y)X - B(X,Y)U.$$

If $B(X,Y) = 0$, i.e. $\tilde{M}$ is totally geodesic then

$$(\tilde{\nabla}_X \tilde{\phi})(Y) = \tilde{g}(X,Y)\xi - \eta(Y)X.$$

This leads to the following theorem :

Theorem (4.4). *A lightlike hypersurface $\tilde{M}$ of an indefinite Sasakian manifold M is totally geodesic i.e. $B = 0$, if and only if*

$$(4.17)\qquad (\tilde{\nabla}_X \tilde{\phi})(Y) = (\nabla_X \phi)(Y).$$

5. Covariant almost analytic vector fields :

An arbitrary $1-$form u is said to be covariant almost analytic if

$$(5.1)\qquad u((\tilde{\nabla}_X \tilde{\phi})(Y) - (\tilde{\nabla}_Y \tilde{\phi})(X)) = (\tilde{\nabla}_{\tilde{\phi}X} u)(Y) - (\tilde{\nabla}_X u)(\tilde{\phi}Y)$$

By use of (4.12), (4.13) the equation (5.1) reduces to

$$(5.2)\qquad B(\tilde{\phi}X, \tilde{\phi}Y) + B(X,Y) + u(Y)B(X,U) = u(X)\{\eta(Y) + u(A_N Y)\}$$

$$-u(Y)\{u(\tilde{\nabla}_{\tilde{\phi}X}U)+\eta(X)+u(A_NX)\}$$

This leads to the following theorem :

Theorem (5.1). *On the lightlike hypersurface $\tilde{M}$ of indefinite Sasakian manifold, if $1-form$ u is covariant almost analytic vector field, then we have* $\forall \;\; X,Y\in\Gamma(T\tilde{M})$.

$$B(\tilde{\phi}X,\tilde{\phi}Y)+B(X,Y)+u(Y)B(X,U)=u(X)\{\eta(Y)+u(A_NY)\}$$

$$-u(Y)\{u(\tilde{\nabla}_{\tilde{\phi}X}u)+\eta(X)+u(A_NX)\}.$$

If $1-$form u is killing i.e. $(\tilde{\nabla}_Xu)(Y)+(\tilde{\nabla}_Yu)(X)=0$, then by use of (4.13), we can state the following theorem :

Theorem (5.2). *On the lightlike hypersurface $\tilde{M}$ of indefinite Sasakian manifold, if $1-$form u is killing, then we have*

$$B(X,\tilde{\phi}Y)+B(Y,\tilde{\phi}X)=-u(Y)\tau(X)-u(X)\tau(Y). \tag{5.3}$$

Using (5.2) and (5.3) we have the following theorem :

Theorem (5.3). *On the lightlike hypersurface $\tilde{M}$ of indefinite Sasakian manifold M, if $1-form$ u is covariant almost analytic and killing both then, we have*

$$2B(X,Y)=u(X)\{B(Y,U)+\eta(Y)+u(A_NY)\} \tag{5.4}$$

$$-u(Y)\{B(X,U)+2\eta(X)+u(A_NX)\}$$

Acknowledgement :

The second author is thankful to University Grants Commission, New Delhi for the support during this work.

References

[1] C. Atindogbe and B. B. Lionel, A note on conformal connections on lightlike hypersurfaces, Conformal Geometry and Dynamics (AMS), 11 (2007), 1 - 11.

[2] C. Calin, Contribution to geometry of CR-submanifold, Ph. D. Thesis, University of Iasi, Romania, 1998.

[3] D. E. Blair, Riemannian Geometry of contact and Symplectic Manifolds, Progress in Mathematics, Birkhauser Boston, Inc., Boston M. A., 2002.

[4] K. L. Duggal, On scalar curvature in lightlike geometry, J. Geom. Phys., 57 (2007), 473-481.

[5] K. L. Duggal and A. Bejancu, Lightlike submanifolds of semi-Riemannian manifolds and its applications, Kluwer Academic Publishers, Dordrecht, 1996.

[6] K. L. Duggal and A. Giménez, Lightlike hypersurfaces of Lorentzian manifolds with distinguished screen, J. Geom. Phys., 55 (2005), 107-122.

[7] K. Yano and M. Kon, Structures on manifolds, World Scientific, Singapore, 1984.

[8] T. H. Kang, S. D. Jung, B. H. Kim, H. K. Pak and J. S. Pak, Lightlike hypersurfaces of indefinite Sasakian manifolds, Indian J. pure appl. Math., 34(9) (2003), 1369-1380.

Received: October, 2009