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$(*)$-Generalized Projective Modules and Lifting Modules

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Abstract

In this paper we introduce the $(*)$-generalized projective modules as a proper generalization of projective modules. We also investigate the "When is direct sum of two lifting modules lifting?". Let $M = M_1 \oplus M_2$ be an amply supplemented module and for every decomposition of $M = M_i \oplus M_j$, M_i is $(*)$-M_j-generalized projective. Then M is lifting if and only if M_1 and M_2 are lifting. We also prove that for any amply supplemented module $M = M_1 \oplus M_2$ such that M_1 and M_2 are indecomposable lifting modules, if M_1 is $(*)$-M_2-generalized projective and every supplement submodule N of M such that $M = N + M_1$ is a direct summand of M, then M is lifting.

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1 Introduction

Throughout this paper R be a ring with identity and let all modules be unitary right R-modules. Projective modules, injective modules and generalizations of these modules have been studied extensively in recent years by many authors (see [2]-[14]).

Let M be a module and $S \leq M$ is called *small* in M (notation $S \ll M$) if $M \neq S + T$ for any proper submodule T of M. M is called *hollow* if every proper submodule of M is small in M. Let N be a submodule of M. A submodule K of M is called *supplement* of N in M provided that K is minimal with respect to the property $M = N + K$, equivalently, $M = N + K$ and $N \cap K \ll K$. A submodule K of M is called *supplement* in M provided there exists a submodule N of M such that K is supplement of N in M. M is called *amply supplemented* if for all submodule N and L of M with $M = N + L$, N contains a supplement of L in M. By [14, 41.7], every factor module of amply supplemented module is also amply supplemented. M is called a *lifting* module if for every submodule A of M there exists a direct summand B of M such that $A/B \ll M/B$ equivalently, M is amply supplemented and every supplement submodule of M is direct summand.

Let M be a module and $N \leq A \leq M$. If $A/N \ll M/N$, then N is called *co-essential* submodule of A in M. A submodule A is called *coclosed* if A has no proper co-essential submodule. In [6, Lemma 1.7], if M is amply supplemented and $A \leq M$, then there exists a submodule B of A such that B is a coessential submodule of A and B is coclosed submodule of M. If M is amply supplemented, coclosed submodule and supplement submodule of M coincide each other by [10, Chapter1, Proposition 2.1].

Finite direct sum of lifting modules need not be lifting. For example $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/8\mathbb{Z}$ are lifting but $M = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z}$ is not lifting. Hence (finite) direct sums of lifting modules are considered some by authors (see [4]-[13]). Projective modules and generalization of projective modules play an important role in the study of direct sums of lifting modules to be lifting. In this paper we consider this problem.

In [12], Mohamed and Müller defined dual-ojective modules as a generalization of projective modules. A is *B-dual-ojective* if, for any homomorphism $\varphi : A \to X$ and any epimorphism $\pi : B \to X$, there exist decompositions $A = A_1 \oplus A_2$, $B = B_1 \oplus B_2$, a homomorphism $\varphi_1 : A_1 \to B_1$ and an epimorphism $\varphi_2 : B_2 \to A_2$ such that $\pi\varphi_1 = \varphi \mid_{A_1}$ and $\varphi\varphi_2 = \pi \mid_{B_2}$. For example $\mathbb{Z}/p\mathbb{Z}$ is $\mathbb{Z}/p^2\mathbb{Z}$-dual-ojective but is not $\mathbb{Z}/p^2\mathbb{Z}$-projective. Note that in [7], dual-ojective modules are called *generalized projective* modules and investigate the finite direct sum of lifting modules. Also in [13], dual-ojective modules are called *cojective* modules. In this paper we use the term generalized projective modules for these modules.

We consider the $(*)$-generalized projective modules as a proper generalization of generalized projective modules and derived some consequences and properties (see Example 2.7).

2 $(*)$-Generalized Projective Modules

Let M_1 and M_2 be two R-modules. M_1 is called $(*)$-M_2-*generalized projective* modules if for any submodule A of M_2 and any homomorphism $\varphi = M_1 \to M_2/A$ with $Ker\varphi \ll M_1$ then there exists decomposition $M_1 = M_1' \oplus M_1''$, $M_2 = M_2' \oplus M_2''$ and homomorphisms $\varphi_1 = M_1' \to M_2'$, $\varphi_2 = M_2'' \to M_1''$ such that φ_2 is onto, $\pi\varphi_1 = \varphi \mid_{M_1'}$ and $\varphi\varphi_2 = \pi \mid_{M_2''}$.

Firstly we give the characterization of $(*)$-generalized projective modules as follows:

Theorem 2.1 *Let M_1 and M_2 be R-modules and $M = M_1 \oplus M_2$. Then the following are equivalent.*
(1) *M_1 is a $(*)$-M_2-generalized projective module.*
(2) *For every submodule A of M with $M = A + M_2$ and $A \cap M_1 \ll M_1$, there exists a decomposition $M = B \oplus M_1' \oplus M_2' = B + M_2$ such that $B \leq A$, $M_1' \leq M_1$ and $M_2' \leq M_2$.*

Proof. (1) $\Rightarrow$ (2) : Assume M_1 is $(*)$-M_2-generalized projective, let $M = A+M_2$ and $A \cap M_1 \ll M_1$ for any submodule A of M. Let $\eta : M \to M/A$ be the natural epimorphism. Define $\pi = \eta \mid_{M_2}$: $M_2 \to M_2/A \cap M_2 \cong (A + M_2)/A = M/A$ and $\varphi = \eta \mid_{M_1}$: $M_1 \to M/A$ Clearly $Ker\varphi = M_1 \cap A \ll M_1$. By hypothesis, there exist decompositions $M_1 = M_1' \oplus M_1''$ and $M_2 = M_2' \oplus M_2''$ and homomorphisms $\varphi_1 : M_1' \to M_2'$ and $\varphi_2 = M_2'' \to M_1''$ such that φ_2 is onto, $\pi\varphi_1 = \varphi \mid_{M_1'}$ and $\varphi\varphi_2 = \pi \mid_{M_2''}$. Define $A_1 = \{x - \varphi_1(x) \mid x \in M_1'\}$ and $A_2 = \{x - \varphi_2(x) \mid x \in M_2''\}$. Clearly $M = A_1 \oplus A_2 \oplus M_2' \oplus M_1''$. We say $B = A_1 \oplus A_2$. Now we will show that $B \leq A$ and $M = B + M_2$. Take an element $m_1' - \varphi_1(m_1') \in A_1$. $\eta(m_1' - \varphi_1(m_1')) = \eta(m_1') - \eta\varphi_1(m_1') = \varphi(m_1') - \pi\varphi_1(m_1') = \pi\varphi_1(m_1') - \pi\varphi_1(m_1') = 0$, since $\eta \mid_{M_1} = \varphi$, $\eta \mid_{M_2} = \pi$ and $\pi\varphi_1 = \varphi \mid_{M_1'}$. Hence $A_1 \leq A$. By the same proof we have $A_2 \leq A$. Therefore $B \leq A$. It is not hard to see that $A_1 + M_2' = M_1' + M_2'$ and $A_2 + M_2'' = M_1'' + M_2''$. Thus $M = B + M_2$.
(2) $\Rightarrow$ (1) : Let $\pi : M_2 \to M_2/A$ be the natural epimorphism for any submodule A of M_2 and $\varphi : M_1 \to M_2/A$ any homomorphism with $Ker\varphi \ll M_1$. Define homomorphism $\eta : M \to M_2/A$ with $\eta(m_1 + m_2) = \varphi(m_1) + \pi(m_2)$. Say $K = Ker\eta$. Now we will show that $K + M_2 = M$ and $Ker\varphi = K \cap M_1$. Take an element $m = m_1 + m_2 \in M$, where $m_1 \in M_1$ and $m_2 \in M_2$. Say $\varphi(m_1) = m_2' + A = \pi(m_2')$. Also $m_1 + m_2 = m_1 + m_2 + m_2' - m_2'$. Clearly $m_1 - m_2' \in K$. Hence $M = K + M_2$. Now take an element $a \in K \cap M_1$. $\eta(a) = 0 + \varphi(a) = 0$ implies $a \in Ker\varphi$. Also $Ker\varphi$ is a submodule of M_1 and K. Hence $Ker\varphi = K \cap M_1$. By assumption there exist decomposition $M = B \oplus M_1' \oplus M_2' = B + M_2$ such that $B \leq K$, $M_1' \leq M_1$ and $M_2' \leq M_2$. We have $M_1 = M_1 \cap M = M_1 \cap (B \oplus M_1' \oplus M_2') = M_1' \oplus M_1''$ and $M_2 = M_2 \cap M = M_2 \cap [B \oplus M_1' \oplus M_2'] = M_2' \oplus M_2''$ where $M_1'' = M_1 \cap (B \oplus M_2')$ and $M_2'' = M_2 \cap (B \oplus M_1')$.

Now we will define the homomorphism $\varphi_1 : M_1'' \to M_2'$ and $\varphi_2 : M_2'' \to M_1'$. Let $\alpha : M \to M_2'$ be the projection $B \oplus M_1'$ and $\beta : M \to M_1'$ be the projection $B \oplus M_2'$. Say $\varphi_1 = \alpha \mid_{M_1''}: M_1'' \to M_2'$ with $Ker\varphi_1 = M_1'' \cap (B \oplus M_1')$ and $\varphi_2 : \beta \mid_{M_2''}: M_2'' \to M_1'$ with $Ker\varphi_2 = M_2'' \cap (B \oplus M_2')$. Take an element $m_1' \in M_1'$. Since $M = B + M_2$ there exist $b \in B$ and $m_2 \in M_2$ such that $m_1' = b + m_2$. $m_2 = m_1' - b \in M_2 \cap (B \oplus M_1') = M_2''$. Hence $m_2 = m_2''$ where $m_2'' \in M_2''$. Therefore $\varphi_2(m_2) = \varphi_2(m_2'') = \varphi_2(m_1' - b) = m_1'$. Hence φ_2 is onto. Take an element $m_1'' \in M_1''$. $\varphi(m_1'') = \eta(m_1'') = \eta(b + m_2') = \eta(m_2') = \pi(m_2') = \pi\varphi_1(m_1'')$, where $m_1'' = b + m_2'$ such that $b \in B$ and $m_2' \in M_2'$. Hence $\pi\varphi_1 = \varphi \mid_{M_1''}$. Take an element $m_2'' \in M_2''$. There exist $b \in B$ and $m_1' \in M_1'$ such that $m_2'' = b + m_1'$. $\pi(m_2'') = \eta(m_2'') = \eta(b + m_1') = \eta(m_1') = \varphi(m_1') = \varphi\varphi_2(m_2'')$. Therefore $\varphi\varphi_2 = \pi \mid_{M_2''}$. This completes the proof. □

Lemma 2.2 *Let P is $(*)$-N-generalized projective and K is direct summand of N. Then P is $(*)$-K-generalized projective.*

Proof. Let $M = P \oplus K$ and A is a submodule of M. Assume that $M = A + K$ and $A \cap P \ll P$. $N = K \oplus T$ for some submodule T of N. Then $P \oplus N = M \oplus T = A + N$. And also $A \cap P \ll P$. By assumption, there exists a decomposition $P \oplus N = A' \oplus P' \oplus N' = A' + N$, where $A' \leq A$, $P' \leq P$ and $N' \leq N$. Then $M = M \cap (P \oplus N) = M \cap (A' \oplus P' \oplus N') = A' \oplus P' \oplus (M \cap N') = A' + (M \cap N)$. Also $M \cap N = K$ and $M \cap N' \leq K$. Hence P is $(*)$-K-generalized projective. □

A module M is called the *finite internal exchange property* if, for any finite direct sum decomposition $M = M_1 \oplus \ldots \oplus M_n$ and any direct summand X of M, there exist $M_i' \leq M_i$ $(i : 1, \ldots, n)$ such that $M = X \oplus M_1' \oplus \ldots \oplus M_n'$. It is easy to see that the finite internal exchange property is inherited by direct summands.

Lemma 2.3 *Let M be a module with the finite internal exchange property and let N be a direct summand of M. If M is $(*)$-P-generalized projective, then N is $(*)$-P-generalized projective for any module P.*

Proof. By similar proof of the [7, Proposition 2.2]. □

Proposition 2.4 *Let $M = M_1 \oplus M_2$ be a module such that M_2 is lifting. Assume that M_1 is $(*)$-M_2-generalized projective. If P is a supplement of M_2 in M and $P \cap M_1 \ll M_1$, then there exists a decomposition $M = P \oplus M_1' \oplus M_2'$ such that $M_1' \leq M_1$ and $M_2' \leq M_2$.*

Proof. Since M_2 is lifting, there exists a decomposition $M_2 = P^* \oplus P^{**}$ such that $(P \cap M_2)/P^* \ll M/P^*$. Clearly $M = P + P^{**}$. We say $N = M_1 \oplus P^{**}$, $P' = P \cap N$. So $M = N \oplus P^*$. $P' \oplus P^* = (P \cap N) \oplus P^* = P \cap (N \oplus P^*) = P$. Since $M = P + P^{**}$, $N = N \cap M = N \cap (P + P^{**}) = P^{**} + N \cap P = P^{**} + P'$. By Lemma 2.2, M_1 is $(*)$-P^{**}-generalized projective. And also $P' \cap M_1 \ll M_1$ since $P \cap M_1 \ll M_1$. By Theorem 2.1, there exists a decomposition $N = P'' \oplus M_1' \oplus P^{***} = P'' + P^{**}$ such that $P'' \leq P'$, $M_1' \leq M_1$ and $P^{***} \leq P^{**}$. Hence $M = N \oplus P^* = P'' \oplus M_1' \oplus P^{***} + P^*$. Since $N = P'' + P^{**}$ and $M = N + P^*$, we have $P'' + M_2 = M$. Since P is supplement of M_2 in M, $P'' = P$. Thus $M = P \oplus M_1' \oplus P^{***} \oplus P^* = P \oplus M_1' \oplus M_2'$ with $P^{***} \oplus P^* = M_2' \leq M_2$. □

Lemma 2.5 *Let M_1 and M_2 be modules and $M = M_1 \oplus M_2$. If M is amply supplemented module, then the following conditions are equivalent.*
(1) *M_1 is $(*)$-M_2-generalized projective.*
(2) *For every supplement submodule A of M with $M = A + M_2$ and $A \cap M_1 \ll M_1$, there exists a decomposition $M = B \oplus M_1' \oplus M_2' = B + M_2$ such that $B \leq A$ and $M_1' \leq M_1$ and $M_2' \leq M_2$.*

Proof. (1) $\Rightarrow$ (2) : Clear by Theorem 2.1.
(2) $\Rightarrow$ (1) : Let A be submodule of M, $M = A + M_2$ and $A \cap M_1 \ll M_1$. Since M is amply supplemented, A contains a supplement K of M_2. Hence $M = K + M_2$ and $K \cap M_2 \ll K \leq A$. By [11, Lemma 4.2], $K \cap M_1 \ll M_1$. By assumption there exists a decomposition $M = B \oplus M_1' \oplus M_2'$ such that $B \leq K \leq A$ and $M_1' \leq M_1$ and $M_2' \leq M_2$. Hence M_1 is $(*)$-M_2-generalized projective by Theorem 2.1. □

Corollary 2.6 *Let $M = M_1 \oplus M_2$ be an amply supplemented module. Then the following conditions are equivalent:*
(1) *M_1 is $(*)$-M_2-generalized projective.*
(2) *For every supplement P of M_2 in M with $P \cap M_1 \ll M_1$, there exists a decomposition $M = P \oplus M_1' \oplus M_2'$ such that $M_1' \leq M_1$ and $M_2' \leq M_2$.*

Proof. (1) $\Rightarrow$ (2) : Let P be a supplement of M_2 in M and $P \cap M_1 \ll M_1$. By Lemma 2.5 ,there exists a decomposition $M = B \oplus M_1' \oplus M_2' = B + M_2$ such that $B \leq P$ and $M_1' \leq M_1$ and $M_2' \leq M_2$. Since P is supplement of M_2 in M, $P = B$. Hence $M = P \oplus M_1' \oplus M_2'$.
(2) $\Rightarrow$ (1) : Let A be submodule of M, $M = A + M_2$ and $P \cap M_1 \ll M_1$. Since M is amply supplemented, A contains a supplement P of M_2 in M. i.e. $M = P + M_2$ and $P \cap M_2 \ll P \leq A$. By assumption, there exists a decomposition $M = P \oplus M_1' \oplus M_2'$ such that $M_1' \leq M_1$ and $M_2' \leq M_2$. By Theorem 2.1, M_1 is $(*)$-M_2-generalized projective. □

Clearly $(*)$-generalized projective module is a generalized projective module. But the converse is not true in general. After Corollary 2.6, we will give an example which is $(*)$-generalized projective but not generalized projective.

Example 2.7 *([11, Lemma A.5]) Let R be an incomplete rank one discrete valuation ring, with quotient field K. Then K is $(*)$-K-generalized projective but fails to be K-generalized projective.*

Proof. By [12, Definition 2.1 and Example 3.6] and Corollary 2.6, K is $(*)$-K-generalized projective. By [12, Example 3.6], K is not K-generalized projective. □

Proposition 2.8 *Let M_1 be a lifting module with the finite internal exchange property and let M_2 be any module. If M_1 is $(*)$-M_2-generalized projective, then M_1 is M_2-generalized projective.*

Proof. Let $\varphi : M_1 \to X$ be any homomorphism and $\pi : M_2 \to X$ be any epimorphism. Consider the submodule $Ker\varphi$ of M_1. Since M_1 is lifting, there exists a decomposition $M_1 = M_1' \oplus M_1''$ such that $M_1' \leq Ker\varphi$ and $Ker\varphi \cap M_1'' \ll M_1''$. Say $\varphi_2 = \varphi \mid_{M_1''}: M_1'' \to X$. Then $Ker\varphi_2 \ll M_1''$. By Lemma 2.3, M_1'' is $(*)$-M_2-generalized projective. Then there exist decompositions $M_1'' = A_1 \oplus A_2$, $M_2 = M_2' \oplus M_2''$ and homomorphisms $f_1 : A_1 \to M_2'$ and $f_2 : M_2'' \to A_2$ such that f_2 is onto, $\pi f_1 = \varphi_2 \mid_{A_1}$ and $\varphi_2 f_2 = \pi \mid_{M_2''}$. Define homomorphism $h_1 : M_1' \oplus A_1 \to M_2'$, where $h(m_1' + a_1) = f_1(a_1)$. Let $m_1' \in M_1'$ and $a_1 \in A_1$. $\pi h_1(m_1' + a_1) = \pi f_1(a_1) = \varphi_2(a_1) = \varphi(m_1' + a_2')$, since $M_1' \leq Ker\varphi$ and $\pi f_1 = \varphi_2 \mid_{A_1}$. Hence $\pi h_1 = \varphi \mid_{M_1' \oplus A_1}$. Let $m_2'' \in M_2''$ and $f_2(m_2'') = a_2$, where $a_2 \in A_2$. $\varphi f_2(m_2'') = \varphi(a_2) = \varphi_2(a_2) = \varphi_2 f_2(m_2'') = \pi(m_2'')$, since $\varphi_2 = \varphi \mid_{M_1''}$ and $\varphi_2 f_2 = \pi \mid_{M_2''}$. Hence $\varphi f_2 = \pi \mid_{M_2''}$. Therefore M_1 is M_2-generalized projective. □

3 Direct Sums of Lifting Modules

Theorem 3.1 *$M = M_1 \oplus M_2$ be an amply supplemented module. Assume that M_1 and M_2 are lifting and for every decomposition of $M = M_i \oplus M_j$, M_i is $(*)$-M_j-generalized projective. Then M is lifting.*

Proof. Let P be a supplement submodule of M. Since $M = M_1 + M_2 + P$ and M is amply supplemented, there exists a supplement M_2' of $P + M_1$ such that $M_2' \leq M_2$. i.e. $M = P + M_1 + M_2'$ and $(P + M_1) \cap M_2' \ll M_2' \leq M_2$. Since M_2 is lifting, there exists a direct summand H of M_2 such that $M_2'/H \ll M_2/H$. Then $M = (P + M_1) + H$. By the minimality of M_2', $M_2' = H$. Therefore $M_2' \leq_d$

M_2. Then $M_2 = M_2' \oplus M_2''$ for some submodule M_2'' of M_2. Again same process, we have a decomposition of $M_1 = M_1' \oplus M_1''$ such that $M = P + M_1' + M_2'$ and $(P+M_2')\cap M_1' \ll M_1' \ll M_1$. $P\cap(M_1'+M_2') \leq [(P+M_1')\cap M_2']+[(P+M_2')\cap M_1']$ implies that $P \cap (M_1' + M_2') \ll M_1' + M_2'$. Hence $M_1' + M_2'$ is supplement of P in M. Also P is a supplement submodule of M, then $P \cap (M_1' + M_2') \ll P$. Therefore $(M_1' + P) \cap M_2' \leq [(M_1' + M_2') \cap P] + [(P + M_2') \cap M_1']$ implies that $(M_1' + P) \cap M_2' \ll M_1' + P$. Hence $P + M_1'$ is a supplement of M_2' in M. $M = M_1 \oplus M_2 = M_1 \oplus M_2' \oplus M_2'' = A \oplus M_2'$, where $A = M_1 \oplus M_2''$. By assumption, A is $(*)$-M_2'-generalized projective. By Corollary 2.6 , $M = (P + M_1') \oplus A' \oplus B$ such that $A' \leq A$, $B \leq M_2'$. We also have $(P + M_1') \cap (A' \oplus B \oplus P) = P$. Then $M/P = (P + M_1')/P \oplus (A' \oplus B \oplus P)/P$. Hence $(A' \oplus B \oplus P)/P$ is a supplement submodule of M/P. By [6, Lemma 1.4] $A' \oplus B \oplus P$ is supplement submodule of M. Since M is amply supplemented, there exists a submodule K of M_1 such that $M = P \oplus A' \oplus B + K$ and $(P \oplus A' \oplus B) \cap K \ll K$. Since $P \oplus A' \oplus B$ is supplement submodule of M, $P \oplus A' \oplus B$ is also supplement of K in M. Since M_1 is lifting, there exists $K' \leq_d M_1$ such that $K/K' \ll M_1/K'$. Then $M = K' \oplus K'' = P \oplus A' \oplus B + K'$ for some submodule K'' of M. Since K is supplement of $(P \oplus A' \oplus B)$ in M, $K = K'$. Hence $M = K \oplus K''$. By assumption K'' is $(*)$-K-generalized projective module. Since $P \oplus A' \oplus B$ is supplement of K in M and Corollary 2.6, $M = P \oplus A' \oplus B \oplus K^* \oplus K^{**}$ such that $K^* \leq K$ and $K^{**} \leq K''$. Hence M is lifting. □

Corollary 3.2 *Let $M = M_1 \oplus M_2$ be an amply supplemented module and for every decomposition of $M = M_i \oplus M_j$, M_i is $(*)$-M_j-generalized projective. Then M is lifting if and only if M_1 and M_2 are lifting.*

Proof. It is clear by Theorem 3.1. □

Lemma 3.3 *Let M_1 and M_2 be indecomposable lifting modules and let $M = M_1 \oplus M_2$ be an amply supplemented module. If M_1 is $(*)$-M_2-generalized projective then every supplement submodule N of M such that $M = N + M_2$ and $N \cap M_1 \ll M_1$ is a direct summand.*

Proof. Let N be a supplement submodule of M such that $M = N + M_2$ and $N \cap M_1 \ll M_1$. By assumption, there exists a decomposition $M = N' \oplus M_1' \oplus M_2' = N' + M_2$ such that $N' \leq N$ and $M_1' \leq M_1$ and $M_2' \leq M_2$. Since M_1 and M_2 are indecomposable, $M = N' \oplus M_1$ or $M = N' \oplus M_2$. In both cases M/N' is lifting. Since N is supplement submodule of M, N/N' is a supplement submodule of M/N'. Thus N/N' is a direct summand of M/N'. Then N is a direct summand of M. □

In [13], Orhan and Keskin Tütüncü defined the (small, pseudo-)generalized projective modules via the class of $\mathcal{B}(M,X)$ as follows and investigate the $\mathcal{B}(M,X)$-lifting modules.

$$\mathcal{B}(M,X)=\{A\leq M \mid \exists Y\leq X, \exists f\in Hom(M,X/Y), Kerf/A\ll M/A\}$$

M_1 is *(small, pseudo)-$\mathcal{B}(M_2,X)$-generalized projective* if for any submodule A of M_2 with $A\in\mathcal{B}(M_2,X)$, any homomorphism $\varphi: M_1\to M_2/A$ (with $Im\varphi\ll M_2/A$, $Im\varphi=M_2/A$ respectively), there exist decompositions $M_1=M_1'\oplus M_1''$, $M_2=M_2'\oplus M_2''$, a homomorphism $\varphi_1: M_1'\to M_2'$ and an epimorphism $\varphi_2: M_2''\to M_1''$ such that $\pi\varphi_1=\varphi\mid_{M_1'}$ and $\varphi\varphi_2=\pi\mid_{M_2''}$, where $\pi: M_2\to M_2/A$.
In [13], they also defined M_1 is called *almost-$\mathcal{B}(M_2,X)$-projective* as a generalization of almost projective modules if for every epimorphism $f: M_2\to M_2/A$ with $A\in\mathcal{B}(M_2,X)$ and any homomorphism $g: M_1\to M_2/A$, either there exists $h: M_1\to M_2$ with $fh=g$ or there exists a nonzero direct summand N of M_2 and $\bar{h}=: N\to M_1$ with $g\bar{h}=f\mid_N$.
Also in the set of $\mathcal{B}(M,X)$ if we take $X=M$, then $\mathcal{B}(M,X)$ coincides with the set of all submodules of M and if $M=M_1\oplus M_2$ then $\mathcal{B}(M_2,X)$ coincides with the set of all submodules of M_2. Hence we get the following corollaries.

Corollary 3.4 *Let M_1 be a simple module and M_2 be an indecomposable module. Then M_1 is M_2-generalized projective if and only if M_1 is almost-M_2-projective.*

Proof. By [13, Theorem 4.15]. □

Remark: Clearly for a hollow module M_1 and any module M_2, if M_1 is $(*)$-M_2-generalized projective module, then M_1 is M_2-generalized projective.

Corollary 3.5 *Let M_1 be a simple module and M_2 be an indecomposable module. Then M_1 is $(*)$-M_2-generalized projective if and only if M_1 is almost-M_2-projective.*

Theorem 3.6 *Let M_1 and M_2 be indecomposable lifting modules and let $M=M_1\oplus M_2$ be an amply supplemented module. If one of the following conditions hold, then M is lifting.*
(1) *M_1 is small-M_2-generalized projective and every supplement submodule N of M such that $M=N+M_1$ is a direct summand.*
(2) *M_1 is small-M_2-generalized projective, M_2 is small -M_1-generalized projective and M_1 pseudo-M_2-generalized projective.*
(3) *M_1 is small-M_2-generalized projective, M_2 is small-M_1-generalized projective and M_2 is pseudo-M_1-generalized projective.*
(4) *M_1 is $(*)$-M_2-generalized projective and every supplement submodule N of M such that $M=N+M_1$ is a direct summand.*

(5) *M_2 is $(*)$-M_1-generalized projective and M_1 is small -M_2-generalized projective.*
(6) *M_1 simple and small-M_2-generalized projective.*
(7) *M_1 is simple and almost-M_2-projective.*
(8) *M_1 is simple and $(*)$-M_2-generalized projective.*
(9) *M_2 is M_1-generalized projective and M_1 is small-M_2-generalized projective.*
(10) *M_1 is M_2-generalized projective and every supplement submodule N of M such that $M = N + M_1$ is a direct summand.*

Proof. (1)-(2)-(3)-(6)-(7)-(9) follow from [13, Theorem 4.16].
(4) Let N be a supplement submodule of M. Assume that $N + M_1 \neq M$. Since M/N is amply supplemented, there exists a supplement submodule K/N of M/N such that $(N + M_1)/K \ll M/K$. Also by [6, Lemma 1.4], K is supplement submodule of M. $M/K = (N + M_1 + M_2)/K = (N + M_1)/K + (K + M_2)/K$ implies that $M = K + M_2$.
Case 1: Assume that $K \cap M_1 = M_1$. Clearly K/M_1 is supplement submodule of $M/M_1 \cong M_2$. By assumption K/M_1 is direct summand of M/M_1. Hence K is a direct summand of M.
Case 2: Assume that $K \cap M_1 \neq M_1$. By [11, Corollary 4.9], M_1 is also hollow. Then $K \cap M_1 \ll M_1$. By Lemma 3.3, K is a direct summand of M.
In both two cases K is a direct summand of M. Hence $M = K \oplus K'$ for some submodule K' of M. We have $M/N = K/N \oplus (K' \oplus N)/N$. Clearly $(K' \oplus N)/N$ is coclosed submodule of M/N. By [6, Lemma 1.4], $K' \oplus N$ is also coclosed submodule of M. Clearly $M = M_1 + K' \oplus N$. By assumption $N \oplus K'$ is a direct summand of M. Then the desired result follows.
(5) follows from Remark and [13, Theorem 4.16].
(8) follows by Corollary 3.5.
(10) follows from Remark and (4). □

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