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Double Integral Representation of Sums

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Abstract. We consider sums involving the product binomial coefficient and polynomial terms and develop some double integral identities. In particular cases it is possible to express the sums in closed form, recover some known results and produce new identities.

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1. Introduction

In this paper we will develop integral identities for sums of the form

$$\sum_{n=0}^{\infty} \frac{t^n \binom{n+m-1}{n}}{(an+j+1)^{k+1} \binom{an+j}{bn}}.$$

We recover and extend some results published by Batir [1] and other authors. There has recently been renewed interest in the study of series involving binomial coefficients and a number of authors have obtained either closed form representation or integral representation for some particular cases of these series. The interested reader is referred to [1, 2, 3, 4, 5, 6, 7, 8] and references therein. The following Lemma and Theorem are the main results presented in this paper.

2. The main results

The following Lemma deals with the integral representation of $\frac{1}{(k+\alpha)^j}$.

Lemma 1. *Let $k+\alpha \geq 0,\ k,\alpha \in \mathbb{R}$ and $j \geq 0$ then,*

$$\frac{1}{(k+\alpha)^j} = \begin{cases} \frac{1}{(j-1)!}\int\limits_0^\infty y^{j-1} e^{-y(k+\alpha)} dy, & \text{for } j \geq 1 \\ 1, & \text{for } j = 0 \end{cases}. \tag{2.1}$$

Proof. The proof follows easily by the application of integration by parts. □

We now state the following theorem.

Theorem 1. *Let a be a positive real number, $|t| \leq 1$, $j \geq 0, k \geq 0$, and $m \geq 1$, then*

$$S(a,j,k,m,t) = \sum_{n=0}^{\infty} \frac{t^n \binom{n+m-1}{n}}{(an+j+1)^{k+1} \begin{pmatrix} an+j \\ j \end{pmatrix}} \tag{2.2}$$

$$= \begin{cases} \frac{1}{(k-1)!} \int\limits_{y=0}^{\infty} \int\limits_{x=0}^{1} \frac{(1-x)^j \ y^{k-1} \ e^{-y(j+1)}}{(1-tx^a e^{-ay})^m} dx dy, & \text{for } k \geq 1 \\ \int\limits_0^1 \frac{(1-x)^j}{(1-tx^a)^m} dx, & \text{for } k = 0 \end{cases}.$$

Proof. Consider

$$\sum_{n=0}^{\infty} \frac{t^n \binom{n+m-1}{n}}{(an+j+1)^{k+1} \begin{pmatrix} an+j \\ j \end{pmatrix}} = \sum_{n=0}^{\infty} \frac{t^n \binom{n+m-1}{n} \Gamma(j+1)\Gamma(an+1)}{(an+j+1)^k \ \Gamma(an+j+2)}$$

$$= \sum_{n=0}^{\infty} \frac{t^n \binom{n+m-1}{n}}{(an+j+1)^k} B(an+1, j+1) \tag{2.3}$$

where $B(\alpha,\beta) = \int\limits_0^1 x^{\alpha-1}(1-x)^{\beta-1}\ dx$, is the classical Beta function and $\Gamma(\alpha) = \int\limits_0^\infty u^{\alpha-1} e^{-u} du$ is the Gamma function. From (2.3) we have

$$\sum_{n=0}^{\infty} \frac{t^n \binom{n+m-1}{n}}{(an+j+1)^k} B(an+1, j+1) = \sum_{n=0}^{\infty} \frac{t^n \binom{n+m-1}{n}}{(an+j+1)^k} \int\limits_0^1 x^{an}(1-x)^j dx$$

$$= \sum_{n=0}^{\infty} \frac{t^n \binom{n+m-1}{n}}{(k-1)!} \int\limits_0^\infty y^{k-1} e^{-y(j+1)} e^{-any} dy \int\limits_0^1 x^{an}(1-x)^j dx$$

upon utilising Lemma 1. Now changing the order of integration and summation we have,

$$\frac{1}{(k-1)!}\int_0^{\infty}\int_0^{1}(1-x)^j\,y^{k-1}e^{-y(j+1)}\sum_{n=0}^{\infty}\binom{n+m-1}{n}\left(tx^a e^{-ay}\right)^n dxdy,$$

$$=\frac{1}{(k-1)!}\int_{y=0}^{\infty}\int_{x=0}^{1}\frac{(1-x)^j\;y^{k-1}\;e^{-y(j+1)}}{(1-tx^a e^{-ay})^m}dxdy,\ \text{ for } k\geq 1.$$

The case of $k=0$ follows in a similar way so that

$$\begin{aligned}S(a,j,0,m,t) &= \sum_{n=0}^{\infty}\frac{t^n\ \binom{n+m-1}{n}}{(an+j+1)\begin{pmatrix}an+j\\ j\end{pmatrix}}\\ &= \int_0^1\frac{(1-x)^j}{(1-tx^a)^m}dx.\end{aligned}$$

□

An alternative representation of (2.2) is given in the following corollary.

Corollary 1. *Let the conditions of Theorem 1 hold then :*

$$\sum_{n=0}^{\infty}\frac{t^n\ \binom{n+m-1}{n}}{(an+j+1)^{k+1}\begin{pmatrix}an+j\\ j\end{pmatrix}}$$

$$(2.4)\qquad =\frac{amt}{k!}\int_{y=0}^{\infty}\int_{x=0}^{1}\frac{(1-x)^j\;x^{a-1}\;y^k\;e^{-y(a+j+1)}}{(1-tx^a e^{-ay})^{m+1}}dxdy,\ \textit{for } k\geq 0.$$

Proof. From Theorem 1

$$\sum_{n=0}^{\infty}\frac{t^n\binom{n+m-1}{n}}{(an+j+1)^{k+1}\binom{an+j}{j}}=a\sum_{n=0}^{\infty}\frac{t^n\binom{n+m-1}{n}\Gamma(j+1)\,n\Gamma(an)}{(an+j+1)^{k+1}\ \Gamma(an+j+1)}$$

$$= a\sum_{n=0}^{\infty}\frac{nt^n\binom{n+m-1}{n}}{(an+j+1)^{k+1}}B(an,j+1)$$

$$= a\sum_{n=0}^{\infty}\frac{nt^n\binom{n+m-1}{n}}{(an+j+1)^{k+1}}\int_0^1 x^{an-1}(1-x)^j\,dx$$

$$= \frac{a}{k!}\int_0^{\infty}\int_0^1\frac{(1-x)^j\,y^k e^{-y(j+1)}}{x}\sum_{n=0}^{\infty}n\binom{n+m-1}{n}\left(tx^a e^{-ay}\right)^n dxdy$$

$$= \frac{a}{k!}\int_0^{\infty}\int_0^1\frac{(1-x)^j\,y^k e^{-y(j+1)}mtx^a e^{-ay}}{x\left(1-tx^a e^{-ay}\right)^{m+1}}dxdy \text{ for } k\geq 0,$$

and rearranging the integrand we obtain (2.4). □

Remark 1. *For $a=1, t=1,\ j=p,\ p$ a positive integer, $m=p+3,$ and k a positive integer*

$$S(1,p,k,p+3,1)=\sum_{n=0}^{\infty}\frac{\binom{n+p+2}{n}}{(n+p+1)^{k+1}\binom{n+p}{p}}$$

$$= \frac{1}{(k-1)!}\int_{y=0}^{\infty}\int_{x=0}^{1}\frac{(1-x)^p\ y^{k-1}\ e^{-y(p+1)}}{(1-xe^{-y})^{p+3}}dxdy$$

$$= \frac{1}{(p+2)(p+1)}\left[\zeta(k)+\zeta(k-1)-H_p^{(k)}-H_p^{(k-1)}\right],$$

where $\zeta(z)$ is the Zeta function and $H_p^{(k)}=\sum_{i=0}^{p}\frac{1}{i^k}$ are the generalised Harmonic numbers of order k.

The next Theorem deals with a more general version of the above Theorem.

Theorem 2. *Let a and b be a positive real number with $(a-b)\geq 0,\ j\geq 0, k\geq 0,\ t\in\mathbb{R}$ and $m\geq 1$. For $\left|\frac{tb^b(a-b)^{a-b}}{a^a}\right|\leq 1$ then*

$$S(a,b,j,k,m,t)=\sum_{n=0}^{\infty}\frac{t^n\binom{n+m-1}{n}}{(an+j+1)^{k+1}\binom{an+j}{bn}}$$

$$(2.5)\qquad = \begin{cases} \frac{1}{(k-1)!} \int\limits_{y=0}^{\infty} \int\limits_{x=0}^{1} \frac{(1-x)^j \ y^{k-1} \ e^{-y(j+1)}}{\left(1-tx^b(1-x)^{a-b}e^{-ay}\right)^m} dxdy, \ \textit{for } k \geq 1 \\ \int\limits_0^1 \frac{(1-x)^j}{\left(1-tx^b(1-x)^{a-b}\right)^m} dx, \qquad \textit{for } k = 0 \end{cases}.$$

Proof. Consider

$$\sum_{n=0}^{\infty} \frac{t^n \ \binom{n+m-1}{n}}{(an+j+1)^{k+1} \begin{pmatrix} an+j \\ bn \end{pmatrix}} = \sum_{n=0}^{\infty} \frac{t^n \ \binom{n+m-1}{n} \ \Gamma\left((a-b)\,n+j+1\right)\Gamma\left(bn+1\right)}{(an+j+1)^k \ \ \Gamma\left(an+j+2\right)}$$

$$= \sum_{n=0}^{\infty} \frac{t^n \ \binom{n+m-1}{n}}{(an+j+1)^k} B\left(bn+1,(a-b)\,n+j+1\right).$$

Now

$$\sum_{n=0}^{\infty} \frac{t^n \ \binom{n+m-1}{n} B\left(bn+1,(a-b)\,n+j+1\right)}{(an+j+1)^k} = \sum_{n=0}^{\infty} \frac{t^n \ \binom{n+m-1}{n}}{(an+j+1)^k} \int\limits_0^1 x^{bn} \left(1-x\right)^{(a-b)n+j} dx$$

$$= \sum_{n=0}^{\infty} \frac{t^n \ \binom{n+m-1}{n}}{(k-1)!} \int\limits_0^{\infty} y^{k-1} e^{-y(j+1)} e^{-any} dy \int\limits_0^1 x^{bn} \left(1-x\right)^{(a-b)n+j} dx$$

upon utilising Lemma 1. By an allowable change of the order of integration and summation we have,

$$\frac{1}{(k-1)!} \int\limits_0^{\infty} \int\limits_0^1 (1-x)^j \, y^{k-1} e^{-y(j+1)} \sum_{n=0}^{\infty} \binom{n+m-1}{n} \left(tx^b \left(1-x\right)^{(a-b)} e^{-ay}\right)^n dxdy,$$

$$= \frac{1}{(k-1)!} \int\limits_{y=0}^{\infty} \int\limits_{x=0}^{1} \frac{(1-x)^j \ \ y^{k-1} \ e^{-y(j+1)}}{\left(1-tx^b\left(1-x\right)^{(a-b)} e^{-ay}\right)^m} dxdy, \ \text{for } k \geq 1.$$

Similarly for $k = 0$,

$$S\left(a,b,j,0,m,t\right) = \sum_{n=0}^{\infty} \frac{t^n \ \binom{n+m-1}{n}}{(an+j+1) \begin{pmatrix} an+j \\ bn \end{pmatrix}}$$

$$= \int\limits_0^1 \frac{(1-x)^j}{\left(1-tx^b\left(1-x\right)^{a-b}\right)^m} dx.$$

□

Remark 2. *For the special case $a=2, b=1,\ j=0, k\geq 1, m\geq 1$ and $|t|<4$ we have that*

$$\begin{aligned} S(2,1,0,k,m,t) &= \sum_{n=0}^{\infty} \frac{t^n \binom{n+m-1}{n}}{(2n+1)^{k+1}\binom{2n}{n}} \\ &= \frac{1}{(k-1)!}\int_{y=0}^{\infty}\int_{x=0}^{1} \frac{y^{k-1}\ e^{-y}}{\left(1-tx(1-x)e^{-2y}\right)^m}dxdy \\ &= \int_0^1 {}_{k+1}F_k\left[\left.\begin{array}{c} \frac{1}{2},\frac{1}{2},\frac{1}{2},\ldots,\frac{1}{2},m \\ \frac{3}{2},\frac{3}{2},\frac{3}{2},\ldots,\frac{3}{2}\end{array}\right| \left(x-x^2\right)t\right]dx \\ &= {}_{k+2}F_{k+1}\left[\left.\begin{array}{c} \frac{1}{2},\frac{1}{2},\frac{1}{2},\ldots,\frac{1}{2},m \\ \frac{3}{2},\frac{3}{2},\frac{3}{2},\ldots,\frac{3}{2}\end{array}\right| \frac{t}{4}\right] \end{aligned}$$

where the generalised hypergeometric representation ${}_pF_q\ [\cdot,\cdot]$, is defined as

$${}_pF_q\left[\left.\begin{array}{c} a_1,a_2,...,a_p \\ b_1,b_2,...,b_q\end{array}\right| t\right] = \sum_{n=0}^{\infty}\frac{(a_1)_n\ (a_2)_n\ ...\ (a_p)_n\ \ t^n}{(b_1)_n\ (b_2)_n\ ...\ (b_q)_n\ \ n!}$$

and $(w)_\alpha = w(w+1)(w+2)\ldots(w+\alpha-1) = \frac{\Gamma(w+\alpha)}{\Gamma(w)}$ is Pochhammer's symbol.

For $k=1,\ t=1$ and $m=1$ we recover Batir's [1] result

$$\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2\binom{2n}{n}} = \frac{8}{3}G - \frac{\pi}{3}\ln\left(2+\sqrt{3}\right) = \int_{y=0}^{\infty}\int_{x=0}^{1}\frac{e^{-y}}{\left(1-x(1-x)e^{-2y}\right)}dxdy.$$

where G is known as Catalan's constant.

For $k=0$,

$$S(2,1,0,0,m,t) = \sum_{n=0}^{\infty}\frac{t^n\binom{n+m-1}{n}}{(2n+1)\binom{2n}{n}} = \int_0^1 \frac{dx}{\left(1-tx(1-x)\right)^m} = {}_2F_1\left[\left.\begin{array}{c} 1,m \\ \frac{3}{2}\end{array}\right| \frac{t}{4}\right],$$

in particular for $m=5$ and $t=-\frac{5}{4}$

$$\begin{aligned} \sum_{n=0}^{\infty}\frac{\left(-\frac{5}{4}\right)^n\binom{n+4}{n}}{(2n+1)\binom{2n}{n}} &= \int_0^1\frac{dx}{\left(1+\frac{5}{4}x(1-x)\right)^5} = {}_2F_1\left[\left.\begin{array}{c} 1,5 \\ \frac{3}{2}\end{array}\right| -\frac{5}{16}\right] \\ &= \frac{2}{583443}\left\{102879 + 2048\sqrt{105}\ln\left(\frac{13+\sqrt{105}}{8}\right)\right\}. \end{aligned}$$

3. Conclusion

We have applied the method of integral representation for binomial sums, in some cases expressed them in closed form, and have extended some published results.

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