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Asymptotic Formulas for the Number of Negative Eigenvalues of a Differential Operator with Operator Coefficent

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Abstract

In this work, we find some asymptotic formulas for the number $N(\varepsilon)$ of eigenvalues smaller than $-\varepsilon \quad (\varepsilon > 0)$ of a self adjoint operator L which is formed by differential expression

$$(Ly)(x) = -(p(x)y'(x))' - Q(x)y(x)$$

and with the boundary condition

$$y(0) = 0$$

as $\varepsilon \to 0$,in the space $L_2(0, \infty; H)$

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1 INTRODUCTION

Let H be a separable Hilbert space with infinite dimension and $Q(x)$ be a self adjoint operator from H to H for all x in $[0, \infty)$. Moreover, we consider $Q(x)$ as a continuous operator function in the interval $[0, \infty)$ with respect to the norm on the space $B(H)$. We suppose that there are positive constants c_1, c_2 such that

$$c_1 \leq p(x) \leq c_2.$$

Let us denote the set of all functions $y = y(x) \in L_2(0, \infty; H)$ which satisfy the following conditions by $D(L)$;

a) $y(x)$ and $y'(x)$ are absolutely continuous with respect to the norm on the space H in every finite interval $[0, a]$.

b) $l(y) = -(p(x)y'(x))' - Q(x)y(x) \in L_2(0, \infty;\ H\ )$

c) $y(0) = 0$.

Let us consider that self adjoint operator $L : D(L) \longrightarrow L_2(0, \infty;\ H)$ defined by

$$(Ly)(x) = -(p(x)y'(x))' - Q(x)y(x).$$

The operator L is formed by differential expression

$$l(y) = -(p(x)y'(x))' - Q(x)y(x) \tag{1.1}$$

and the boundary condition

$$y(0) = 0.$$

In this work, we have find some asymptotic formulas for the number $N(\varepsilon)$ of eigenvalues smaller than $-\varepsilon \quad (\varepsilon > 0)$ of the operator L as $\varepsilon \to 0$. In the work [6], asymptotic behavior of the number of negative eigenvalues of a scalar differential operator with high order has been investigated. In the work [1] and work [5], the asymptotic formulas for the number of negative eigenvalues of differential operators with unbounded operator coefficient have been found.

2 SOME INEQUALITIES FOR THE NUMBER OF NEGATIVE EIGENVALUES

In this part, we have find some inequalities for the number of negative eigenvalues of the operator L .

Let us suppose that following conditions are satisfied:

1) $Q(x) : H \longrightarrow H$ is an absolutely continuous, self adjoint and positive operator, for all $x \in [0, \infty)$.

2) $Q(x)$ is monotonous decreasing.

3) $Q(x)$ is continuous with respect to the norm on the space $B(H)$ and

$$\lim_{x \to \infty} \|Q(x)\| = 0$$

.

4) There are positive constants c_1 and c_2 such that $c_1 \le p(x) \le c_2$.

5) The function $p(x)$ has continuous and bounded derivative.

6) The function $p(x)$ is not decreasing.

If the conditions 1), 3), 4) and 5) are satisfied the operator L is bounded from below and negative part of the spectrum is discrete.

Let $\alpha_1(x) \geq \alpha_2(x) \geq \cdots \geq \alpha_j(x) \geq \cdots$ be eigenvalues of the operator $Q(x) : H \longrightarrow H$. Since $Q(x) > 0$ for every $x \in [0, \infty)$, we have $\alpha_j(x) > 0 \quad (j = 1, 2, \cdots)$. Furthermore, since

$$\alpha_1(x) = \sup_{\|f\|=1} (Q(x)f, f)$$

[2] and

$$\|Q(x)\| = \sup_{\|f\|=1} \Big|(Q(x)f, f)\Big| = \sup_{\|f\|=1} (Q(x)f, f),$$

[4], we have $\alpha_1(x) = \|Q(x)\|$. Since $Q(x)$ is continuous, the function $\alpha_1(x)$ is continuous in the interval $[0, \infty)$. Since $Q(x)$ is monotonous decreasing, it can be shown that the functions $\alpha_1(x), \alpha_2(x), \cdots, \alpha_j(x), \cdots$ are monotonous decreasing. On the other hand

$$\lim_{x \to \infty} \alpha_1(x) = 0$$

so, the imagine set of the function α_1 is the interval $(0, \alpha_1(0)]$. Therefore, the function α_1 has an inverse function which is continuous in $(0, \alpha_1(0)]$. Let us denote this inverse function by ψ_1. Let us consider following operators where $\varepsilon \in (0, \alpha_1(0))$:

1) Let L_1 and L_2 be operators which are formed by expression (1.1) and with the boundary conditions

$$y(0) = y(\psi_1(\varepsilon)) = 0$$

$$y'(0) = y'(\psi_1(\varepsilon)) = 0$$

respectively, in the space $L_2(0, \psi_1(\varepsilon); H)$.

2) Let $L_{(1)i}$ and $L_{(2)i}$ be operators which are formed by expression (1.1) and with the boundary conditions

$$y(x_{i-1}) = y(x_i) = 0$$

$$y'(x_{i-1}) = y'(x_i) = 0$$

respectively, in the space $L_2(x_{i-1}, x_i; H)$.
Divide the interval $[0, \psi_1(\varepsilon)]$ into the intervals at the length

$$\delta = \frac{\psi_1(\varepsilon)}{[\mid \psi_1^k(\varepsilon) \mid] + 1}. \tag{2.1}$$

Here k is a constant which belongs to the interval $(0,1)$ and ε is any positive number which satisfies the inequality $\psi_1^k(\varepsilon) \geq 2$. Let the partition points of the interval $[0, \psi_1(\varepsilon)]$ be

$$0 = x_0 < x_1 < x_2 < \cdots < x_M = \psi_1(\varepsilon).$$

Let us take

$$\psi_j(\varepsilon) = \sup\{x \in [0,\infty) : \alpha_j(x) \geq \varepsilon\}. \tag{2.2}$$

Let $N(\varepsilon)$, $N_1(\varepsilon)$, $N_2(\varepsilon)$, $n_{1(i)}$ and $n_{2(i)}$ be number of the eigenvalues smaller than $-\varepsilon$ $(\varepsilon > 0)$ of the operators L, L_1, L_2, $L_{1(i)}$ and $L_{2(i)}$ respectively. When the conditions 1)-6) are satisfied, it can be shown that

$$N_1(\varepsilon) \leq N(\varepsilon) \leq N_2(\varepsilon), \tag{2.3}$$

$$n_{(1)i} > \sum_{\substack{j \\ \alpha_j(x_i) > \varepsilon}} \Big[\frac{1}{\pi} \int_{x_i}^{\varphi_{i,j}(\varepsilon)} \sqrt{\frac{\alpha_j(x) - \varepsilon}{p(x)}} dx - 1\Big], \tag{2.4}$$

$$n_{(2)i} < \sum_{\substack{j \\ \alpha_j(x_{i-1}) > \varepsilon}} \Big[\frac{1}{\pi} \int_{x_{i-2}}^{x_{i-1}} \sqrt{\frac{\alpha_j(x) - \varepsilon}{p(x)}} dx + 1\Big]. \tag{2.5}$$

Here, $\varphi_{i,j}(\varepsilon) = min\{x_{i+1}, \psi_j(\varepsilon)\}$ $\quad (i = 1, 2, \cdots, M-1)$.

Theorem 2.1 *If the conditions 1)-6) are satisfied then we have*

$$N(\varepsilon) > \frac{1}{\pi} \sum_{j=1}^{l_\varepsilon} \int_0^{\psi_j(\varepsilon)} \sqrt{\frac{\alpha_j(x) - \varepsilon}{p(x)}} dx - const.l_\varepsilon \int_0^{\delta} \sqrt{\alpha_1(x)} dx - const.l_\varepsilon \psi_1^k(\varepsilon)$$

for small values of ε.

Here,

$$l_\varepsilon = \sum_{\alpha_j(0) \geq \varepsilon} 1.$$

Proof: From the variation principles of R. Courant [3], we have

$$N_1(\varepsilon) \geq \sum_{i=1}^{M} n_{(1)i}$$

By using this inequality and the inequality (2.4), we obtain

$$N_1(\varepsilon) \geq \sum_{i=1}^{M-1} n_{(1)i} > \sum_{i=1}^{M-1} \sum_{\substack{j \\ \alpha_j(x_i)>\varepsilon}} \Big[\frac{1}{\pi} \int\limits_{x_i}^{\varphi_{i,j}(\varepsilon)} \sqrt{\frac{\alpha_j(x)-\varepsilon}{p(x)}} dx - 1\Big]. \tag{2.6}$$

Here, the sum

$$\sum_{i=1}^{M-1} \sum_{\substack{j \\ \alpha_j(x_i)>\varepsilon}} \Big[\frac{1}{\pi} \int\limits_{x_i}^{\varphi_{i,j}(\varepsilon)} \sqrt{\frac{\alpha_j(x)-\varepsilon}{p(x)}} dx - 1\Big] \tag{2.7}$$

is the sum of expressions

$$\frac{1}{\pi} \int\limits_{x_i}^{\varphi_{i,j}(\varepsilon)} \sqrt{\frac{\alpha_j(x)-\varepsilon}{p(x)}} dx - 1 \qquad (i, j = 1, 2, ...)$$

for natural numbers $i \geq 1$ and $j \geq 1$ satisfying the condition $\alpha_j(x_i) > \varepsilon$. If we consider monotonous decreasing functions $\alpha_j(x)$ $(j = 1, 2, ...)$, we can write

$$\sum_{i=1}^{M-1} \sum_{\substack{j \\ \alpha_j(x_i)>\varepsilon}} \Big[\frac{1}{\pi} \int\limits_{x_i}^{\varphi_{i,j}(\varepsilon)} \sqrt{\frac{\alpha_j(x)-\varepsilon}{p(x)}} dx - 1\Big] = \sum_{\substack{j \\ \alpha_j(x_1)>\varepsilon}} \sum_{\substack{i\geq 1 \\ \alpha_j(x_i)>\varepsilon}} \Big[\frac{1}{\pi} \int\limits_{x_i}^{\varphi_{i,j}(\varepsilon)} \sqrt{\frac{\alpha_j(x)-\varepsilon}{p(x)}} dx - 1\Big] \tag{2.8}$$

for the sum (2.7). By using the expression

$$\varphi_{i,j}(\varepsilon) = min\{x_{i+1}, \psi_j(\varepsilon)\} \qquad (i = 1, 2, ..., M-1)$$

we obtain

$$\sum_{\substack{j \\ \alpha_j(x_1)>\varepsilon}} \sum_{\substack{i\geq 1 \\ \alpha_j(x_i)>\varepsilon}} \int\limits_{x_i}^{\varphi_{i,j}(\varepsilon)} \sqrt{\frac{\alpha_j(x)-\varepsilon}{p(x)}} dx = \sum_{\substack{j \\ \alpha_j(x_1)>\varepsilon}} \Big[\int\limits_{x_1}^{x_2} \sqrt{\frac{\alpha_j(x)-\varepsilon}{p(x)}} dx$$

$$+ \int\limits_{x_2}^{x_3} \sqrt{\frac{\alpha_j(x)-\varepsilon}{p(x)}} dx + \cdots + \int\limits_{x_{i_0}}^{\psi_j(\varepsilon)} \sqrt{\frac{\alpha_j(x)-\varepsilon}{p(x)}} dx\Big]$$

$$= \sum_{\substack{j \\ \alpha_j(x_1)>\varepsilon}} \int_{x_1}^{\psi_j(\varepsilon)} \sqrt{\frac{\alpha_j(x)-\varepsilon}{p(x)}}dx. \tag{2.9}$$

Here, i_0 is a natural number satisfying the condition

$$x_{i_0} < \psi_j(\varepsilon) \le x_{i_0+1}.$$

From (2.2), we obtain

$$\psi_j(\varepsilon) \ge x_1$$

for $j \in \mathbf{N}$ satisfying the condition $\alpha_j(x_1) > \varepsilon$. It can be easily shown that $\alpha_j(x_0) > \varepsilon$ for every $x_0 \in [0, \psi_j(\varepsilon))$. Therefore, we have

$$\alpha_j(x_1) > \varepsilon$$

for $j \in \mathbf{N}$ satisfying the condition $\psi_j(\varepsilon) \ge x_1$. Hence, from (2.9), we find

$$\sum_{\substack{j \\ \alpha_j(x_1)>\varepsilon}} \sum_{\substack{i\ge 1 \\ \alpha_j(x_i)>\varepsilon}} \int_{x_i}^{\varphi_{i,j}(\varepsilon)} \sqrt{\frac{\alpha_j(x)-\varepsilon}{p(x)}}dx = \sum_{\psi_j(\varepsilon)\ge x_1} \int_{x_1}^{\psi_j(\varepsilon)} \sqrt{\frac{\alpha_j(x)-\varepsilon}{p(x)}}dx$$

$$= \sum_{\psi_j(\varepsilon)\ge x_1} \int_{0}^{\psi_j(\varepsilon)} \sqrt{\frac{\alpha_j(x)-\varepsilon}{p(x)}}dx - \sum_{\psi_j(\varepsilon)\ge x_1} \int_{0}^{x_1} \sqrt{\frac{\alpha_j(x)-\varepsilon}{p(x)}}dx$$

$$= \sum_{j=1}^{l_\varepsilon} \int_{0}^{\psi_j(\varepsilon)} \sqrt{\frac{\alpha_j(x)-\varepsilon}{p(x)}}dx - \sum_{\psi_j(\varepsilon)<x_1} \int_{0}^{\psi_j(\varepsilon)} \sqrt{\frac{\alpha_j(x)-\varepsilon}{p(x)}}dx$$

$$- \sum_{\psi_j(\varepsilon)\ge x_1} \int_{0}^{x_1} \sqrt{\frac{\alpha_j(x)-\varepsilon}{p(x)}}dx$$

$$= \sum_{j=1}^{l_\varepsilon} \int_{0}^{\psi_j(\varepsilon)} \sqrt{\frac{\alpha_j(x)-\varepsilon}{p(x)}}dx - \sum_{j} \int_{0}^{\varphi_{0,j}(\varepsilon)} \sqrt{\frac{\alpha_j(x)-\varepsilon}{p(x)}}dx$$

$$\geq \sum_{j=1}^{l_\varepsilon} \int_0^{\psi_j(\varepsilon)} \sqrt{\frac{\alpha_j(x)-\varepsilon}{p(x)}}dx - \sum_{\alpha_j(0)\geq\varepsilon} \int_0^{x_1} \sqrt{\frac{\alpha_1(x)-\varepsilon}{p(x)}}dx$$

$$\geq \sum_{j=1}^{l_\varepsilon} \int_0^{\psi_j(\varepsilon)} \sqrt{\frac{\alpha_j(x)-\varepsilon}{p(x)}}dx - const \int_0^{\delta} \sqrt{\alpha_1(x)-\varepsilon}dx \sum_{\alpha_j(0)\geq\varepsilon} 1$$

$$= \sum_{j=1}^{l_\varepsilon} \int_0^{\psi_j(\varepsilon)} \sqrt{\frac{\alpha_j(x)-\varepsilon}{p(x)}}dx - const l_\varepsilon \int_0^{\delta} \sqrt{\alpha_1(x)-\varepsilon}dx. \tag{2.10}$$

We have

$$\sum_{\substack{i\geq 1\\ \alpha_j(x_i)>\varepsilon}} 1 = \max\{i : \alpha_j(x_i) > \varepsilon\}$$

for a fixed number $j \geq 1$ satisfying the condition $\alpha_j(x_1) > \varepsilon$.
Let us take $\max\{i : \alpha_j(x_i) > \varepsilon\} = m(j,\varepsilon)$. Since $\alpha_j(x_{m(j,\varepsilon)}) > \varepsilon$ from (2.2), we obtain

$$x_{m(j,\varepsilon)} \leq \psi_j(\varepsilon).$$

On the other hand, since

$$m(j,\varepsilon) = \frac{x_{m(j,\varepsilon)}}{\delta}$$

we have

$$m(j,\varepsilon) \leq \frac{\psi_j(\varepsilon)}{\delta}$$

or

$$\sum_{\substack{i\geq 1\\ \alpha_j(x_i)>\varepsilon}} 1 \leq \frac{\psi_j(\varepsilon)}{\delta}.$$

By using the last inequality, we find

$$\sum_{\substack{j\\ \alpha_j(x_1)>\varepsilon}} \sum_{\substack{i\geq 1\\ \alpha_j(x_i)>\varepsilon}} 1 < \sum_{\substack{j\\ \alpha_j(x_1)>\varepsilon}} \frac{\psi_j(\varepsilon)}{\delta} < \sum_{\alpha_j(0)>\varepsilon} \frac{\psi_j(\varepsilon)}{\delta}$$

$$< \frac{\psi_1(\varepsilon)}{\delta} \sum_{\alpha_j(0)>\varepsilon} 1 = \delta^{-1}\psi_1(\varepsilon) l_\varepsilon. \tag{2.11}$$

From (2.6), (2.8), (2.10) and (2.11), we obtain

$$N_1(\varepsilon) > \frac{1}{\pi}\sum_{j=1}^{l_\varepsilon} \int_0^{\psi_j(\varepsilon)} \sqrt{\frac{\alpha_j(x)-\varepsilon}{p(x)}}dx \; - \; const.l_\varepsilon \int_0^{\delta} \sqrt{\alpha_1(x)-\varepsilon}dx \; - \; \delta^{-1}\psi_1(\varepsilon) l_\varepsilon. \tag{2.12}$$

From the relations (2.1), (2.3) and (2.12), we find

$$N(\varepsilon) > \frac{1}{\pi}\sum_{j=1}^{l_\varepsilon} \int_0^{\psi_j(\varepsilon)} \sqrt{\frac{\alpha_j(x)-\varepsilon}{p(x)}}dx \; - \; const.l_\varepsilon \int_0^{\delta} \sqrt{\alpha_1(x)}dx \; - \; const.l_\varepsilon \psi_1^k(\varepsilon).\square$$

Theorem 2.2 *If the conditions 1)-6) are satisfied then we have*

$$N(\varepsilon) < n_{(2)1} + \pi^{-1}\sum_{j=1}^{l_\varepsilon} \int_0^{\psi_j(\varepsilon)} \sqrt{\frac{\alpha_j(x)-\varepsilon}{p(x)}}dx + l_\varepsilon \delta^{-1}\psi_1(\varepsilon)$$

for small values of ε .

Proof: From the variation principles of R. Courant [3], we have

$$N_2(\varepsilon) \le \sum_{i=1}^{M} n_{(2)i}. \tag{2.13}$$

From (2.13) and (2.5), we obtain

$$N_2(\varepsilon) < n_{(2)1} + \sum_{i=2}^{M} \sum_{\substack{j \\ \alpha_j(x_{i-1})>\varepsilon}} \left[\frac{1}{\pi}\int_{x_{i-2}}^{x_{i-1}} \sqrt{\frac{\alpha_j(x)-\varepsilon}{p(x)}}dx + 1\right]. \tag{2.14}$$

The sum

$$\sum_{i=2}^{M} \sum_{\substack{j \\ \alpha_j(x_{i-1})>\varepsilon}} \Big[\frac{1}{\pi}\int_{x_{i-2}}^{x_{i-1}} \sqrt{\frac{\alpha_j(x)-\varepsilon}{p(x)}}dx+1\Big] \tag{2.15}$$

is the sum of the expressions

$$\frac{1}{\pi}\int_{x_{i-2}}^{x_{i-1}} \sqrt{\frac{\alpha_j(x)-\varepsilon}{p(x)}}dx+1 \qquad (i=2,3,...;\ j=1,2,...)$$

for the natural numbers $i \geq 2$ and $j \geq 1$ satisfying the condition

$$\alpha_j(x_{i-1}) > \varepsilon.$$

If we consider the functions $\alpha_j(x) \quad (j=1,2,...)$ monotonous decreasing, then we obtain

$$\sum_{i=2}^{M} \sum_{\substack{j \\ \alpha_j(x_{i-1})>\varepsilon}} \Big[\frac{1}{\pi}\int_{x_{i-2}}^{x_{i-1}} \sqrt{\frac{\alpha_j(x)-\varepsilon}{p(x)}}dx+1\Big] = \sum_{\substack{j \\ \alpha_j(x_1)>\varepsilon}} \sum_{\substack{i\geq 2 \\ \alpha_j(x_{i-1})>\varepsilon}} \Big[\frac{1}{\pi}\int_{x_{i-2}}^{x_{i-1}} \sqrt{\frac{\alpha_j(x)-\varepsilon}{p(x)}}dx+1\Big]$$

$$= \sum_{\substack{j \\ \alpha_j(x_1)>\varepsilon}} \Big[\frac{1}{\pi}\int_{0}^{x_1} \sqrt{\frac{\alpha_j(x)-\varepsilon}{p(x)}}dx+\frac{1}{\pi}\int_{x_1}^{x_2} \sqrt{\frac{\alpha_j(x)-\varepsilon}{p(x)}}dx+\cdots+$$

$$+\frac{1}{\pi}\int_{x_{i_0-1}}^{x_{i_0}} \sqrt{\frac{\alpha_j(x)-\varepsilon}{p(x)}}dx+i_0\Big] \tag{2.16}$$

for the sum (2.15). Here, i_0 is a natural number satisfying the conditions

$$\alpha_j(x_{i_0}) > \varepsilon, \quad \alpha_j(x_{i_0+1}) \leq \varepsilon.$$

From (2.2), we find

$$x_{i_0} \leq \psi_j(\varepsilon).$$

On the other hand, if we consider

$$i_0 = \frac{x_{i_0}}{\delta},$$

from (2.14) and (2.16), we obtain

$$N_2(\varepsilon) \;<\; n_{(2)1} + \sum_{\substack{j \\ \alpha_j(x_1)>\varepsilon}} \Big[\frac{1}{\pi}\int_0^{\psi_j(\varepsilon)} \sqrt{\frac{\alpha_j(x)-\varepsilon}{p(x)}}dx + \frac{\psi_j(\varepsilon)}{\delta}\Big]$$

$$\le n_{(2)1} + \sum_{j=1}^{l_\varepsilon}\Big[\frac{1}{\pi}\int_0^{\psi_j(\varepsilon)} \sqrt{\frac{\alpha_j(x)-\varepsilon}{p(x)}}dx + \frac{\psi_j(\varepsilon)}{\delta}\Big]. \qquad (2.17)$$

If we consider $\psi_j(\varepsilon) \le \psi_1(\varepsilon) \quad (j = 1, 2, ..., l_\varepsilon)$, from (2.3) and (2.17), we obtain

$$N(\varepsilon) < n_{(2)1} + \frac{1}{\pi}\sum_{j=1}^{l_\varepsilon}\int_0^{\psi_j(\varepsilon)} \sqrt{\frac{\alpha_j(x)-\varepsilon}{p(x)}}dx + l_\varepsilon\frac{\psi_1(\varepsilon)}{\delta}. \qquad (2.18)$$

3 THE ASYMPTOTIC FORMULAS FOR THE NUMBER OF NEGATIVE EIGENVALUES OF THE OPERATOR L

In this section, we will find some asymptotic formulas for the number $N(\varepsilon)$ of eigenvalues smaller than $-\varepsilon \quad (\varepsilon > 0)$ of the self adjoint operator L, as $\varepsilon \to 0$.

Let us denote the functions of the form $ln_0 x = x$, $ln_n x = ln(ln_{n-1}x)$ by $ln_n x \quad (n = 0, 1, 2, ...)$ and suppose that the function $\alpha_1(x) = \|Q(x)\|$ satisfies the following condition:

7) There are a number $\xi > 0$ and a natural number n so that the function $\alpha_1(x) - (ln_n x)^{-\xi}$ is neither negative nor monotonous increasing in an interval $[a, \infty) \quad (a > 0)$.

Theorem 3.1 *If the conditions 1)-7) are satisfied and the series*

$$\sum_{j=1}^{\infty}(\alpha_j(0))^m$$

is convergent for a fixed number $m \in (0, \infty)$, then the asymptotic formula

$$N(\varepsilon) = \pi^{-1}\Big[1 + O(e^{-\varepsilon^{-\beta}})\Big]\sum_j \int_{\alpha_j(x)\ge\varepsilon} \sqrt{\frac{\alpha_j(x)-\varepsilon}{p(x)}}dx$$

is satisfied as $\varepsilon \to 0$, where β is a positive constant.

Proof: By using the theorem 2.2, it is shown that the inequality

$$N(\varepsilon) < \pi^{-1} \sum_{j=1}^{l_\varepsilon} \int_0^{\psi_j(\varepsilon)} \sqrt{\frac{\alpha_j(x) - \varepsilon}{p(x)}} dx + const l_\varepsilon \int_0^{\delta} \sqrt{\alpha_1(x)} dx + const l_\varepsilon \psi_1^k(\varepsilon) \tag{3.1}$$

is satisfied. By the theorem 2.1 and the inequality (3.1), we obtain

$$\left| N(\varepsilon) - \pi^{-1} \sum_{j=1}^{l_\varepsilon} \int_0^{\psi_j(\varepsilon)} \sqrt{\frac{\alpha_j(x) - \varepsilon}{p(x)}} dx \right| < const \left[l_\varepsilon \delta + l_\varepsilon \psi_1^k(\varepsilon) \right]$$

for small positive values of ε $(\varepsilon > 0)$.
Here, if we take $k = \frac{1}{2}$ and consider the formula (2.1) then we find

$$\left| N(\varepsilon) - \pi^{-1} \sum_{j=1}^{l_\varepsilon} \int_0^{\psi_j(\varepsilon)} \sqrt{\frac{\alpha_j(x) - \varepsilon}{p(x)}} dx \right| < const l_\varepsilon \sqrt{\psi_1(\varepsilon)}. \tag{3.2}$$

Let us limit from above the sum on the left side of the inequality (3.2).
Let us take

$$f(\varepsilon) = \psi_1(\varepsilon)[ln \psi_1(\varepsilon)]^{-1}.$$

By using the operator function $Q(x)$ which satisfies the condition 2) and the function $p(x)$ which satisfies the condition 4) and 6), we obtain

$$\begin{aligned} \sum_{j=1}^{l_\varepsilon} \int_0^{\psi_j(\varepsilon)} \sqrt{\frac{\alpha_j(x) - \varepsilon}{p(x)}} dx &> \int_0^{\psi_1(\varepsilon)} \sqrt{\frac{\alpha_1(x) - \varepsilon}{p(x)}} dx \\ &> \int_{\frac{f(\varepsilon)}{2}}^{f(\varepsilon)} \sqrt{\frac{\alpha_1(x) - \varepsilon}{p(x)}} dx \\ &> \sqrt{c_2^{-1}} \int_{\frac{f(\varepsilon)}{2}}^{f(\varepsilon)} \sqrt{\alpha_1(x) - \varepsilon} dx \\ &> \frac{f(\varepsilon)}{2\sqrt{c_2}} \sqrt{\alpha_1(f(\varepsilon) - \varepsilon)}. \end{aligned} \tag{3.3}$$

On the other hand, if the conditions 1), 2), 3) and 7) have been satisfied, it can be shown that

$$\alpha_1\Big(\frac{\psi_1(\varepsilon)}{ln\psi_1(\varepsilon)}\Big)-\varepsilon>(ln\psi_1(\varepsilon))^{-(\xi+1)(n+1)} \tag{3.4}$$

for small positive values of ε. From (3.3) and (3.4), we find

$$\begin{aligned}\sum_{j=1}^{l_\varepsilon}\int_0^{\psi_j(\varepsilon)}\sqrt{\frac{\alpha_j(x)-\varepsilon}{p(x)}}dx &> \frac{\psi_1(\varepsilon)}{2\sqrt{c_2}ln\psi_1(\varepsilon)}(ln\psi_1(\varepsilon))^{-\frac{1}{2}(\xi+1)(n+1)}\\ &> c_{11}\psi_1^{\frac{3}{4}}(\varepsilon).\end{aligned} \tag{3.5}$$

From (3.2) and (3.3), we find

$$\left|\frac{N(\varepsilon)}{\pi^{-1}\sum\limits_{j=1}^{l_\varepsilon}\int\limits_0^{\psi_j(\varepsilon)}\sqrt{\frac{\alpha_j(x)-\varepsilon}{p(x)}}dx}-1\right|<c_{11}l_\varepsilon\psi_1^{-\frac{1}{4}}(\varepsilon). \tag{3.6}$$

Furthermore, since the series

$$\sum_{j=1}^{\infty}(\alpha_j(0))^m$$

is convergent, we have

$$const\geq\sum_{j=1}^{\infty}(\alpha_j(0))^m\geq\sum_{\alpha_j(0)\geq\varepsilon}(\alpha_j(0))^m\geq\sum_{\alpha_j(0)\geq\varepsilon}(\varepsilon)^m=\varepsilon^m l_\varepsilon.$$

Here,

$$l_\varepsilon=const\varepsilon^{-m}. \tag{3.7}$$

Since the function $\alpha_1(x)$ satisfies the condition 7), we have

$$\varepsilon=\alpha_1(\psi_1(\varepsilon))\geq(ln_n\psi_1(\varepsilon))^{-\xi}$$

for small positive values of ε. From here, we obtain

$$\psi_1(\varepsilon)>e^{\varepsilon-\frac{1}{\xi}}. \tag{3.8}$$

From (3.6), (3.7) and (3.8), we find

$$\left|\frac{N(\varepsilon)}{\pi^{-1}\sum\limits_{j=1}^{l_\varepsilon}\int\limits_0^{\psi_j(\varepsilon)}\sqrt{\frac{\alpha_j(x)-\varepsilon}{p(x)}}dx}-1\right|<c_{13}\varepsilon^{-m}e^{-\frac{1}{4}\varepsilon^{-\frac{1}{4}}}<e^{-\varepsilon^{-\beta}}.$$

From this inequality, we obtain the asymptotic formula

$$N(\varepsilon) = \pi^{-1}\Big[1 + O(e^{-\varepsilon^{-\beta}})\Big] \sum_j \int_{\alpha_j(x)\geq\varepsilon} \sqrt{\frac{\alpha_j(x)-\varepsilon}{p(x)}}dx$$

as $\varepsilon \to 0$. □

First of all, we suppose that the function $\alpha_1(x)$ satisfies following condition:

8) For every $\eta > 0$

$$\lim_{x\to\infty} \alpha_1(x)x^{k_0-\eta} = \lim_{x\to\infty} [\alpha_1(x)x^{k_0+\eta}]^{-1} = 0$$

where k_0 is a constant which belongs to the interval $(0, 2)$.

By using the theorem 2.1 and the inequality (3.1), the following theorem can be proved.

Theorem 3.2 *We suppose that the conditions 1)-6) and 8) are satisfied. In addition, if the series*

$$\sum_{j=1}^{\infty} (\alpha_j(0))^m$$

is convergent for a fixed number m *which satisfies the condition*

$$0 < m < \frac{(2-k_0)^2}{8k_0 - 2{k_0}^2},$$

then the asymptotic formula

$$N(\varepsilon) = \pi^{-1}\Big[1 + O(\varepsilon^{t_0})\Big] \sum_j \int_{\alpha_j(x)\geq\varepsilon} \sqrt{\frac{\alpha_j(x)-\varepsilon}{p(x)}}dx$$

is satisfied as $\varepsilon \to 0$ *, where* t_0 *is a positive constant.*

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