Int. Journal of Math. Analysis, Vol. 3, 2009, no. 38, 1881 - 1892

On the Convergence of a New Two-Step Iteration in the Class of Quasi-Contractive Operators

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Abstract

In the class of quasi-contractive operators satisfying Zamfirescu's conditions, the most used fixed point iterative methods, that is, the Krasnoselskij, Mann and Ishikawa iterations, are all known to be convergent to the unique fixed point. In this paper, we consider a new two-step iterative scheme for approximating fixed points of quasi-contractive operators and we show that Krasnoselskij, Mann, Ishikawa and new two-step iterations are equivalent for quasi-contractive operators satisfying Zamfirescu's conditions.

Mathematics Subject Classifications: 47H10

Keywords: Krasnoselskij iteration, Mann iteration, Ishikawa iteration, two-step iteration, quasi-contractive operators

1. Introduction and Preliminaries

In the last three decades many papers have been published on the iterative approximation of fixed points for certain classes of operators, using the Krasnoselskij, Mann and Ishikawa iteration methods, see [2]. These papers were motivated by the fact that, under weaker contractive type conditions, the Picard iteration (or the method of successive approximations), need not converge to the fixed point of the operator in question. However, there exist large classes of operators, as for example that of quasi-contractive type operators introduced in [2, 3, 5, 6], for which not only the Picard iteration, but also the Ishikawa iterations can be used to approximate the fixed points.

Let K be a nonempty convex subset of a normed space E and $T : K \to K$ be a mapping. $F(T) = \{x^* \in K : Tx^* = x^*\}$ is the set of fixed point of T. For

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$u_0 \in K$, the sequence $\{u_n\}_{n=0}^{\infty}$ defined by

$$u_{n+1} = (1-\alpha_n)\, u_n + \alpha_n T u_n,\ \ n = 0, 1, 2, \ldots \tag{1}$$

where $\{\alpha_n\} \subset (0,1)$, is called the Mann iteration process. For $v_0 \in K$, the sequence $\{v_n\}_{n=0}^{\infty}$ defined by

$$\begin{aligned} v_{n+1} &= (1-\alpha_n)\, v_n + \alpha_n T w_n, \\ w_n &= (1-\beta_n)\, v_n + \beta_n T v_n,\ \ n = 0, 1, 2, \ldots \end{aligned} \tag{2}$$

where $\{\alpha_n\}\,,\ \{\beta_n\} \subset (0,1)$, is called the Ishikawa iteration process.

Recently, Thianwan [15] introduced a new two-step iteration as follows:

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)\, y_n + \alpha_n T y_n, \\ y_n &= (1-\beta_n)\, x_n + \beta_n T x_n,\ \ n = 0, 1, 2, \ldots \end{aligned} \tag{3}$$

where $\{\alpha_n\}\,,\ \{\beta_n\} \subset (0,1)$.

In 1972, Zamfirescu [16] obtained the following a very interesting fixed point theorem.

Theorem 1.1. *Let (X, d) be a complete metric space and $T : X \to X$ a mapping for which there exists the real numbers a, b and c satisfying $a \in (0,1)$; $b, c \in (0, \frac{1}{2})$ such that for each pair $x,\ y \in X$, at least one of the following conditions holds:*

$$\begin{aligned} &(\mathrm{z}_1) \quad d(Tx, Ty) \le a d(x, y)\,, \\ &(\mathrm{z}_2) \quad d(Tx, Ty) \le b\,[d(x, Tx) + d(y, Ty)]\,, \\ &(\mathrm{z}_3) \quad d(Tx, Ty) \le c\,[d(x, Ty) + d(y, Tx)]\,. \end{aligned}$$

Then T has a unique fixed point x^* and the Picard iteration $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_{n+1} = T x_n,\ \ n = 0, 1, 2, \ldots \tag{4}$$

converges to x^* for any arbitrary but fixed $x_0 \in X$.

An operator T satisfying the contractive conditions (z_1), (z_2) and (z_3) in the above theorem is called Zamfirescu operator (alternatively, we shall say that T satisfies condition Z).

Remark 1.2. *For $\alpha_n = \lambda$ (constant), the iteration (1) reduces to the so-called Krasnoselskij iteration while for $\alpha_n \equiv 1$ we obtain the Picard iteration (4), or the method of successive approximations, as it is commonly known. Obviously, for $\beta_n \equiv 0$ the Ishikawa iteration (2) and (3) reduces to (1).*

Berinde [1] introduced a new class of operators on an arbitrary Banach space E satisfying

$$\|Tx - Ty\| \leq \delta \|x - y\| + 2\delta \|Tx - x\| \tag{5}$$

for any $x, y \in E$, $0 \leq \delta < 1$. He proved that this class wider that the class of Zamfiresu operators and used the Ishikawa iteration process to approximate fixed points of this class of operators in an arbitrary Banach space given in the form of following theorem:

Theorem 1.3. *[1] Let K be a nonempty closed convex subset of an arbitrary Banach space E and $T : K \to K$ be an operator satisfying* (5). *Let $\{v_n\}_{n=o}^{\infty}$be defined through the iterative process* (2) *and $v_0 \in K$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of positive numbers in* $[0,1]$ *with $\{\alpha_n\}$ satisfying $\sum_{n=o}^{\infty} \alpha_n = \infty$. Then $\{v_n\}_{n=o}^{\infty}$ converges strongly to the fixed point of T.*

The purpose of this paper to approximate the fixed points of the class of mappings defined in (5) by using the iteration process (3) and to show the convergence of the two-step iteration equivalent to the convergence of the Krasnoselskij, Mann and Ishikawa iterations when applied to a map which satisfies condition Z.

In order to prove our main results, we will need the following lemma.

Lemma 1.4. [2] *Let $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$ be sequences of nonnegative numbers and $0 \leq q < 1$, so that $a_{n+1} \leq qa_n + b_n$, for all $n \geq 0$*

(i) If $\lim_{n\to\infty} b_n = 0$, then $\lim_{n\to\infty} a_n = 0$,

(ii) If $\sum_{n=0}^{\infty} b_n < \infty$, then $\sum_{n=0}^{\infty} a_n < \infty$.

2. Main results

Firstly, we shall prove strong convergence of the iteration process (3) to a fixed point of quasi-contractive operator.

Theorem 2.1. *Let E be an arbitrary Banach space, K a closed convex subset of E, and $T : K \to K$ an operator satisfying condition Z. Let $\{x_n\}_{n=0}^{\infty}$ be the two-step iteration defined by (3) and $x_0 \in K$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of positive number in* $[0,1]$ *satisfying*

$$\sum_{n=1}^{\infty} \alpha_n = \infty.$$

Then $\{x_n\}_{n=0}^{\infty}$ converges strongly to the fixed point of T.

Proof. From Theorem 1.1, we know that T has a unique fixed point in K, say x^*. Since T satisfies condition Z, at least one of the conditions from (z_1), (z_2) and (z_3) is satisfied. If (z_2) holds, then

$$\begin{aligned} \|Tx - Ty\| &\le b\left(\|x - Tx\| + \|y - Ty\|\right) \\ &\le b\left(\|x - Tx\| + \left(\|y - x\| + \|x - Tx\| + \|Tx - Ty\|\right)\right), \end{aligned}$$

this is implies that

$$(1 - b)\|Tx - Ty\| \le b\|x - y\| + 2b\|x - Tx\|.$$

From $0 \le b < 1$, we obtain

$$\|Tx - Ty\| \le \frac{b}{1 - b}\|x - y\| + \frac{2b}{1 - b}\|x - Tx\|.$$

We suppose that (z_3) holds, then

$$\begin{aligned} \|Tx - Ty\| &\le c\left(\|x - Ty\| + \|y - Tx\|\right) \\ &\le c\left(\|x - Tx\| + \|Tx - Ty\| + \|x - y\| + \|x - Tx\|\right), \end{aligned}$$

thus, from this inequality it follows that

$$(1 - c)\|Tx - Ty\| \le c\|x - y\| + 2c\|x - Tx\|$$

that is

$$\|Tx - Ty\| \le \frac{c}{(1 - c)}\|x - y\| + \frac{2c}{(1 - c)}\|x - Tx\|.$$

We assume that

$$\delta := \max\left\{a, \frac{b}{1 - b}, \frac{c}{1 - c}\right\}, \tag{6}$$

then $0 \le \delta < 1$. Finally, we get

$$\|Tx - Ty\| \le \delta\|x - y\| + 2\delta\|x - Tx\|,\ \forall x, y \in K. \tag{7}$$

Now, let $\{x_n\}_{n=0}^{\infty}$ be the two-step iteration defined by (3) and $x_0 \in K$ arbitrary. Then

$$\begin{aligned} \|x_{n+1} - x^*\| &= \|(1 - \alpha_n)y_n + \alpha_n Ty_n - (1 - \alpha_n + \alpha_n)x^*\| \\ &\le (1 - \alpha_n)\|y_n - x^*\| + \alpha_n\|Ty_n - x^*\|. \end{aligned} \tag{8}$$

Using (7) and (8) with $x := x^*$ and $y := y_n$, we get $\|Ty_n - x^*\| \leq \delta \|y_n - x^*\|$. where δ is given by (6). From (1) we have

$$\begin{aligned} \|y_n - x^*\| &= \|(1-\beta_n)\, x_n + \beta_n T x_n - (1-\beta_n+\beta_n)\, x^*\| \\ &\leq (1-\beta_n)\|x_n - x^*\| + \beta_n \|Tx_n - x^*\| . \end{aligned} \tag{9}$$

Again by (7), this time with $x := x^*$ and $y := x_n$, we find that $\|Tx_n - x^*\| \leq \delta \|x_n - x^*\|$ and hence, combining (8) and (9) we get

$$\begin{aligned} \|x_{n+1} - x^*\| &\leq (1-\alpha_n+\alpha_n\delta)\|y_n - x^*\| \\ &\leq (1-\alpha_n+\alpha_n\delta)(1-\beta_n+\beta_n\delta)\|x_n - x^*\| \end{aligned}$$

which, by the following inequality

$$(1-\alpha_n+\alpha_n\delta)(1-\beta_n+\beta_n\delta) \leq (1-\alpha_n(1-\delta)),$$

implies that

$$\|x_{n+1} - x^*\| \leq (1-\alpha_n(1-\delta))\|x_n - x^*\|,\ n = 0,1,2,.... \tag{10}$$

On repeating this process, we get

$$\|x_{n+1} - x^*\| \leq \prod_{k=0}^{n} (1-(1-\delta)\,\alpha_k)\|x_0 - x^*\|,\ n = 0,1,2,.... \tag{11}$$

Since $0 \leq \delta < 1,\ \alpha_k,\ \beta_n \in [0,1]$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$, then

$$\lim_{n\to\infty} \prod_{k=0}^{n} (1-(1-\delta)\,\alpha_k) = 0,$$

it follows from (11) that $\lim_{n\to\infty} \|x_{n+1} - x^*\| = 0$. Therefore $\{x_n\}_{n=0}^{\infty}$ converges strongly to the fixed point of T.

It is known, see Rhoades [7], that (z_1), (z_2) and (z_3) are independent conditions. In [9], the following conjecture was given: "if the Mann iteration converges, then so does the Ishikawa iteration." In a series of papers [8-12], the authors have given a positive answer to this conjecture, showing, for example, the equivalence between Mann and Ishikawa iterations for strongly and uniformly pseudocontractive maps.

Very recently, Şoltuz [14] had studied that the Picard, Mann and Ishikawa iterations are equivalent when applied to Zamfirescu operators and got the following result.

Theorem 2.2. [14] *Let E be a normed space, K a nonempty, convex, closed subset of E and $T : K \to K$ an operator satisfying condition Z (Zamfirescu* *operator). If the initial point is the same for all iterations, then the following are equivalent:*

(i) *the Mann iteration (1) converges to* x^**;*

(ii) *the Ishikawa iteration (2) converges to* x^**;*

(iii) *the Picard iteration (3) converges to* x^**.*

Now, we shall prove the equivalence between two-step iteration (3) and Mann iteration (1) for Zamfirescu operators. Let $F(T)$ denote the fixed point set with respect to K for the map T. Also, we suppose that $x^* \in F(T)$.

Theorem 2.3. *Let E be a normed space, K a nonempty, convex, closed subset of E and $T : K \to K$ a Zamfirescu operator. Suppose that T has a fixed point $x^* \in K$. If $u_0 = x_0 \in K$, then the following are true: if the Mann iteration (1) converges to x^*, then the two-step iteration (3) converges to x^*. Conversely, if the two-step iteration (3) converges to x^*, then the Mann iteration (1) converges to x^*, provided that $\alpha_n \geq A > 0$, $\forall n \in N$.*

Proof. Firstly, we shall prove the implication $(i) \Rightarrow (ii)$. Using (1), (3) and (7) with $x := u_n,\ y := y_n$, we have

$$\begin{aligned} \|u_{n+1} - x_{n+1}\| &\leq (1-\alpha_n)\|u_n - y_n\| + \alpha_n \|Tu_n - Ty_n\| \qquad (12) \\ &\leq (1-\alpha_n)\|u_n - y_n\| + \alpha_n\delta\|u_n - y_n\| + 2\alpha_n\delta\|u_n - Tu_n\| \\ &= (1-\alpha_n(1-\delta))\|u_n - y_n\| + 2\alpha_n\delta\|u_n - Tu_n\|. \end{aligned}$$

and with $x := u_n,\ y := x_n$, we get

$$\begin{aligned} \|u_n - y_n\| &\leq (1-\beta_n)\|u_n - x_n\| + \beta_n\|u_n - Tx_n\| \qquad (13) \\ &\leq (1-\beta_n)\|u_n - x_n\| + \beta_n\|u_n - Tu_n\| + \beta_n\delta\|u_n - x_n\| \\ &\quad +2\beta_n\delta\|u_n - Tu_n\| \\ &\leq \|u_n - x_n\| + (\beta_n + 2\beta_n\delta)\|u_n - Tu_n\|. \end{aligned}$$

Combining (12), (13) and using (7) and $\alpha_n \geq A > 0, \forall n \in \mathbb{N}$, we obtain

$$\begin{aligned} \|u_{n+1} - x_{n+1}\| &\leq (1 - A(1-\delta))\|u_n - x_n\| \\ &\quad + [(1 - A(1-\delta))(\beta_n + 2\beta_n\delta) + 2\alpha_n\delta]\|u_n - Tu_n\|. \end{aligned}$$

Denote

$$\begin{aligned} a_n : &= \|u_n - x_n\| \\ q : &= (1 - A(1-\delta)) \\ b_n : &= [(1 - A(1-\delta))(\beta_n + 2\beta_n\delta) + 2\alpha_n\delta]\|u_n - Tu_n\|. \end{aligned}$$

Let $x^* \in F(T)$. Since $\lim_{n\to\infty}\|u_n - x^*\| = 0$, utilizing (7) we get

$$\begin{aligned} 0 \leq \|u_n - Tu_n\| &\leq \|u_n - x^*\| + \|x^* - Tu_n\| \\ &\leq \|u_n - x^*\| + \delta\|x^* - u_n\| + 2\delta\|x^* - Tx^*\| \\ &= (1+\delta)\|u_n - x^*\| \to 0, \quad n \to \infty, \end{aligned}$$

which implies that $\lim_{n\to\infty} \|u_n - Tu_n\| = 0$. It follows from Lemma 1.4 that $\lim_{n\to\infty} \|u_n - x_n\| = 0$. In addition,

$$0 \leq \|x_n - x^*\| \leq \|x_n - u_n\| + \|u_n - x^*\|$$

and this implies that $\lim_{n\to\infty} x_n = x^*$.

Next, we shall prove $(ii) \Rightarrow (i)$, that is, if two-step iteration converges, then Mann iteration does converge. Applying (1), (3) and (7) with $x := y_n,\ y := u_n$, we have

$$\begin{aligned} \|x_{n+1} - u_{n+1}\| &\leq (1-\alpha_n)\|y_n - u_n\| + \alpha_n \|Ty_n - Tu_n\| \qquad (14) \\ &\leq (1-\alpha_n)\|y_n - u_n\| + \alpha_n\delta\|y_n - u_n\| + 2\alpha_n\delta\|y_n - Ty_n\| \end{aligned}$$

and

$$\begin{aligned} \|y_n - u_n\| &\leq (1-\beta_n)\|x_n - u_n\| + \beta_n\|Tx_n - u_n\| \qquad (15) \\ &\leq (1-\beta_n)\|x_n - u_n\| + \beta_n\|Tx_n - x_n\| + \beta_n\|x_n - u_n\| \\ &= \|x_n - u_n\| + \beta_n\|Tx_n - x_n\|. \end{aligned}$$

From (1) and (7) with $x := x_n, y := y_n$, we have also

$$\begin{aligned} \|y_n - Ty_n\| &\leq \|y_n - x_n\| + \|x_n - Tx_n\| + \|Tx_n - Ty_n\| \qquad (16) \\ &\leq (1+\delta)\left[(1-\beta_n)\|x_n - x_n\| + \beta_n\|Tx_n - x_n\|\right] \\ &\quad + (1+2\delta)\|x_n - Tx_n\| \\ &= (\beta_n(1+\delta) + (1+2\delta))\|x_n - Tx_n\|. \end{aligned}$$

Combining (14), (15) and (16) we obtain

$$\begin{aligned} \|x_{n+1} - u_{n+1}\| &\leq (1-\alpha_n(1-\delta))\left[\|x_n - u_n\| + \beta_n\|Tx_n - x_n\|\right] \\ &\quad + 2\alpha_n\delta\|y_n - Ty_n\| \\ &\leq (1-A(1-\delta))\|x_n - u_n\| + [\beta_n(1-\alpha_n(1-\delta)) \\ &\quad + 2\alpha_n\delta(\beta_n(1+\delta) + (1+2\delta))]\|Tx_n - x_n\|. \end{aligned}$$

Denote

$$\begin{aligned} a_n :&= \|x_n - u_n\| \\ q :&= (1-A(1-\delta)) \\ b_n :&= [\beta_n(1-\alpha_n(1-\delta)) + 2\alpha_n\delta(\beta_n(1+\delta) + (1+2\delta))]\|Tx_n - x_n\|. \end{aligned}$$

Since $\lim_{n\to\infty} \|x_n - x^*\| = 0$, T satisfies condition Z, and $x^* \in F(T)$, from (7) we obtain that

$$\begin{aligned} 0 \leq \|x_n - Tx_n\| &\leq \|x_n - x^*\| + \|x^* - Tx_n\| \\ &\leq \|x_n - x^*\| + \delta\|x^* - x_n\| + 2\delta\|x^* - Tx^*\| \\ &= (1+\delta)\|x_n - x^*\| \to 0, \quad n \to \infty. \end{aligned}$$

Therefore, $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$. From Lemma 1.4, we have $\lim_{n\to\infty}\|x_n - u_n\| = 0$. Hence, $\|x^* - u_n\| \le \|x_n - u_n\| + \|x_n - x^*\| \to 0$ as $n \to \infty$. This completes the proof.

Theorem 2.3 lead to the following Corollary:

Corollary 2.4. *Let E be a normed space, K a nonempty, convex, closed subset of E and $T : K \to K$ an operator satisfying condition Z (Zamfirescu operator). If the initial point is the same for all iterations, $\alpha_n \ge A > 0$, $\forall n \in N$, then the following are equivalent:*

(i) *the two-step iteration (3) converges to x^*;*

(ii) *the Mann iteration (1) converges to x^*;*

(iii) *the Ishikawa iteration (2) converges to x^*;*

(iv) *the Picard iteration (4) converges to x^*.*

3. The equivalence between T-stabilities

We shall now discuss the stability of iteration processes. Let E be a normed space, and T a selfmap of K. Let x_0 be a point of E, and assume that $x_{n+1} = f(T, x_n)$ is an iteration. We suppose that $\{x_n\}$ converges to a fixed point x^* of T. Let $\{y_n\}$ be an arbitrary sequence in E, and set $\varepsilon_n = \|y_{n+1} - f(T, y_n)\|$ for all $n = 0, 1, 2, \ldots$.

Remark 3.1. [4] *In practice, such a sequence $\{y_n\}$ could arise in the following way. Let choose the initial $x_0 \in E$. Set $x_{n+1} = f(T, x_n)$. Let $y_0 = x_0$. Now $x_1 = f(T, x_0)$. Because of rounding or discretization in the function T, a new value y_1 approximately equal to x_1 might be obtained instead of the true value of $f(T, x_0)$. Then the approximate y_2, the value $f(T, y_1)$ is computed to yields y_2, an approximation of $f(T, y_1)$. This computation is continued to obtain $\{y_n\}$ an approximate sequence of $\{x_n\}$.*

Definition 3.2. [4] *If $((\lim_{n\to\infty}\varepsilon_n = 0) \Rightarrow (\lim_{n\to\infty} y_n = x^*))$, then the iteration procedure $x_{n+1} = f(T, x_n)$ is said to be T-stable with respect to T.*

We suppose that $x^* \in F(T)$. Let $\{x_n\}$ be the two-step iteration and $\{u_n\}$ be the Mann iteration, $x_0 = u_0 \in K$. And, let the sequences $\{\alpha_n\} \subset (0,1), \{\beta_n\} \subset [0,1)$ satisfy

$$\lim_{n\to\infty}\alpha_n = \lim_{n\to\infty}\beta_n = 0, \ \ \sum_{n=1}^{\infty}\alpha_n = \infty. \tag{17}$$

The following nonnegative sequences are well-defined for all $n = 0, 1, 2, \ldots$.

$$\varepsilon_n := \|x_{n+1} - (1-\alpha_n)\, y_n - \alpha_n T y_n\|, \tag{18}$$

$$\delta_n := \|u_{n+1} - (1-\alpha_n)\, u_n - \alpha_n T u_n\|. \tag{19}$$

Definition 3.3. *(i) If* $\lim_{n\to\infty} \varepsilon_n = 0$ *implies that* $\lim_{n\to\infty} x_n = x^*$*, then the two-step iteration (3), is said to be* T*-stable.*

(ii) If $\lim_{n\to\infty} \delta_n = 0$ *implies that* $\lim_{n\to\infty} u_n = x^*$*, then the Mann iteration (1), is said to be* T*-stable.*

Remark 3.4. *Let* E *be a normed space and* $T : E \to E$ *a map. The following are equivalent:*

(i) for all $\{\alpha_n\} \subset (0,1)$, $\{\beta_n\} \subset [0,1)$ *satisfying (17), the two-step iteration is* T*-stable,*

(I) for all $\{\alpha_n\} \subset (0,1)$, $\{\beta_n\} \subset [0,1)$ *satisfying (17),*

$$\lim_{n\to\infty} \varepsilon_n = \lim_{n\to\infty} \|x_{n+1} - (1-\alpha_n)\, y_n - \alpha_n T y_n\| = 0 \;\Rightarrow \lim_{n\to\infty} x_n = x^*.$$

Remark 3.5. *[13] Let* E *be a normed space and* $T : E \to E$ *a map. The following are equivalent:*

(i) for all $\{\alpha_n\} \subset (0,1)$ *satisfying (17), the Mann iteration is* T*-stable,*

(II) for all $\{\alpha_n\} \subset (0,1)$ *satisfying (17),*

$$\lim_{n\to\infty} \delta_n = \lim_{n\to\infty} \|u_{n+1} - (1-\alpha_n)\, u_n - \alpha_n T u_n\| = 0 \;\Rightarrow \lim_{n\to\infty} u_n = x^*.$$

Theorem 3.6. *Let* E *be a normed linear space and* $T : E \to E$ *a operator satisfying condition Z (Zamfirescu operator). Let* x^* *be the fixed point of* T*. Then the following are equivalent:*

(i) for all $\{\alpha_n\} \subset (0,1)$, $\{\beta_n\} \subset [0,1)$ *satisfying (17), the two-step iteration is* T*-stable,*

(ii) for all $\{\alpha_n\} \subset (0,1)$ *satisfying (17), the Mann iteration is* T*-stable.*

Proof. From (18) and (19) assure that $(i) \Leftrightarrow (ii)$ is equivalent to $(I) \Leftrightarrow (II)$. So, we shall prove that $(I) \Rightarrow (II)$. That is, if two-step iteration is T-stable, then Mann iteration is T-stable. Using (6) and (19) with $x := u_n, y := y_n$, we have

$$\begin{aligned}
& \|u_{n+1} - (1-\alpha_n)\, u_n - \alpha_n T u_n\| \\
\le\; & \|u_{n+1} - (1-\alpha_n)\, u_n - \alpha_n T y_n\| + \|\alpha_n T y_n - \alpha_n T u_n\| \\
\le\; & \|u_{n+1} - (1-\alpha_n)\, y_n - \alpha_n T y_n\| + (1-\alpha_n+\alpha_n\delta)\,(1-\beta_n)\,\|u_n - u_n\| \\
& + \beta_n \|T u_n - u_n\|] + 2\alpha_n\delta \|u_n - T u_n\| \\
\le\; & \|u_{n+1} - (1-\alpha_n)\, y_n - \alpha_n T y_n\| \\
+\; & (\beta_n (1-\alpha_n+\alpha_n\delta) + 2\alpha_n\delta)\, \|u_n - T u_n\| \to 0, \quad n \to \infty.
\end{aligned}$$

Since condition (I) assures that $\lim_{n\to\infty} \|u_{n+1} - (1-\alpha_n)\, y_n - \alpha_n T y_n\| = 0 \;\Rightarrow \lim_{n\to\infty} u_n = x^*$, then we have $\lim_{n\to\infty} \|u_{n+1} - (1-\alpha_n)\, u_n - \alpha_n T u_n\| = 0$. Thus $\lim_{n\to\infty} u_n = x^*$.

Now, we shall prove $(II) \Rightarrow (I)$. Applying (6) and (18) with $x := x_n, y := y_n$, we have

$$\begin{aligned}
& \left\| x_{n+1} - (1-\alpha_n)\, y_n - \alpha_n T y_n \right\| \\
\leq\ & \left\| x_{n+1} - (1-\alpha_n)\, y_n - \alpha_n T x_n \right\| + \left\| \alpha_n T x_n - \alpha_n T y_n \right\| \\
\leq\ & \left\| x_{n+1} - (1-\alpha_n)\, x_n - \alpha_n T x_n \right\| + (1-\alpha_n+\alpha_n\delta) \left\| y_n - x_n \right\| \\
& +2\alpha_n\delta \left\| x_n - T x_n \right\| \\
\leq\ & \left\| x_{n+1} - (1-\alpha_n)\, x_n - \alpha_n T x_n \right\| \\
+\ & \left(\beta_n (1-\alpha_n+\alpha_n\delta) + 2\alpha_n\delta\right) \left\| T x_n - x_n \right\| \to 0, \quad n \to \infty.
\end{aligned}$$

Condition (II) assures that $\lim_{n\to\infty} \left\| x_{n+1} - (1-\alpha_n)\, x_n - \alpha_n T x_n \right\| = 0 \quad \Rightarrow$ $\lim_{n\to\infty} x_n = x^*$. Hence $\lim_{n\to\infty} \left\| x_{n+1} - (1-\alpha_n)\, y_n - \alpha_n T y_n \right\| = 0$, we have shown that $\lim_{n\to\infty} x_n = x^*$. This completes the proof.

Set in (1) and (3), $T := T^n$, to obtain the modified two-step and modified Mann iterations. Therefore, we obtain that the equivalence between $T-$stabilities of this modified iterations as follows:

Corollary 3.7. *Let E be a normed linear space and $T : E \to E$ a operator satisfying condition Z (Zamfirescu operator). Let x^* be the fixed point of T. Then the following are equivalent:*

(i) for all $\{\alpha_n\} \subset (0,1)$, $\{\beta_n\} \subset [0,1)$ satisfying (17), the modified two-step iteration is T-stable,

(ii) for all $\{\alpha_n\} \subset (0,1)$ satisfying (17), the modified Mann iteration is T-stable.

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Received: April, 2009