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Strong Convergence Theorems for Fixed Point Problems, Equilibrium Problems and Applications

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Abstract

In this paper, we introduce an iterative scheme by hybrid methods for finding a common element of the set of fixed points of nonexpansive mappings, the set of solution of an equilibrium problems and the set of solution of a variational inequality problems for a monotone, k-Lipschitz continuous mappings in a Hilbert space. We show that the iterative sequence converges strongly to a common element of the three sets. Our results extend and improve the recent results of Takahashi, Takeuchi and Kubota [W. Takahashi, Y. Takeuchi, R. Kubota, Strong Convergence Theorems by Hybrid Methods for Families of Nonexpansive Mappings in Hilbert Spaces, J. Math. Anal. Appl., 341(2008), 276-286] and many others.

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1 Introduction

Let H be a real Hilbert space with norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$ and let C be a nonempty closed convex subset of H. Let P_C be the metric projection of H onto C. A mapping $S : C \to C$ is said to be *nonexpansive* if $\|Sx - Sy\| \leq$

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$\|x-y\|$, for all $x, y \in C$. We denote by $F(S)$ the set of fixed points of S. Let F be a bifunction of $C \times C$ into $\mathbf{R}$, where $\mathbf{R}$ is the set of real numbers. The equilibrium problem for $F : C \times C \to \mathbf{R}$ is to find $x \in C$ such that

$$F(x, y) \geq 0, \quad \forall y \in C. \tag{1}$$

The set of such solutions of (1)is denoted by $EP(F)$. Given a mapping $T : C \to H$, let $F(x, y) = \langle Tx, y-x \rangle$ for all $x, y \in C$. Then $z \in EP(F)$ if and only if $\langle Tz, y-z \rangle \geq 0$ for all $y \in C$, i.e., z is a solution of the variational inequality problem. Numerous problems in physics, optimization and economics reduce to find a solution of (1). Some methods have been proposed to solve the equilibrium problem (see [1, 4, 14, 19]). In 2005, Combettes and Hirstoaga [2] introduced an iterative scheme of finding the best approximation to the initial data when $EP(F)$ is nonempty and they also proved a strong convergence theorem. If C is bounded nonempty closed convex and S is a nonexpansive mapping of C into itself, then $F(S)$ is nonempty (see [7]).

Definition 1.1 *Let $A : C \to H$ be a mapping. Then A is called*

1. *monotone if*
$$\langle Ax - Ay, x - y \rangle \geq 0, \quad \forall x, y \in C;$$

2. *β-strongly monotone if there exists a positive real number β such that*
$$\langle Ax - Ay, x - y \rangle \geq \beta \|x - y\|^2, \quad \forall x, y \in C;$$

3. *ξ-Lipschitz continuous if there exists a positive real number ξ such that*
$$\|Ax - Ay\| \leq \xi \|x - y\|, \quad \forall x, y \in C;$$

4. *α-inverse-strongly monotone [1, 9] if there exists a positive real number α such that*
$$\langle Ax - Ay, x - y \rangle \geq \alpha \|Ax - Ay\|^2, \quad \forall x, y \in C.$$

Remark 1.2 *It is obvious that any α-inverse-strongly monotone mapping A is monotone and Lipschitz continuous.*

The classical *variational inequality problem* is to find $u \in C$ such that $\langle v - u, Au \rangle \geq 0$ for all $v \in C$. The set of solutions of this variational inequality problem is denoted by $VI(C, A)$. It is well known that if A is a strongly monotone and Lipschitz continuous mapping on C, then the variational inequality problem has a unique solution. How to actually find a solution of the variational inequality problem is one of the most important topics in the

study of the variational inequality problem. The variational inequality has been extensively studied in the literature. See, e.g., [22, 23] and the references therein.

In 1953, Mann [13] introduced the iteration as follows: a sequence $\{x_n\}$ defined by

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) S x_n, \tag{2}$$

where the initial guess element $x_0 \in C$ is arbitrary and $\{\alpha_n\}$ is a real sequence in $[0, 1]$. Mann iteration has been extensively investigated for nonexpansive mappings. One of the fundamental convergence results is proved by Reich [18]. In an infinite-dimensional Hilbert space, the Mann iteration can conclude only weak convergence [5]. Attempts to modify the Mann iteration method (2) so that strong convergence is guaranteed have recently been made.

In 1976, Korpelevich [8] introduced the following so-called extragradient method:

$$\begin{cases} x_0 = x \in C, \\ y_n = P_C(x_n - \lambda A x_n), \\ x_{n+1} = P_C(x_n - \lambda A y_n) \end{cases} \tag{3}$$

for all $n \geq 0$, where $\lambda \in (0, \frac{1}{k}), C$ is a nonempty closed convex subset of $\mathbf{R}^n$ and A is a monotone and k-Lipschitz continuous mapping of C in to $\mathbf{R}^n$. He proved that if $VI(C, A)$ is nonempty, then the sequences $\{x_n\}$ and $\{\bar{x}_n\}$, generated by (3), converge to the same point $z \in VI(C, A)$.

In 2006, Zeng and Yao[24] proved the following strong convergence theorem:

$$\begin{cases} x_0 = x \in C, \\ y_n = P_C(x_n - \lambda_n A x_n), \\ x_{n+1} = \alpha_n x_0 + (1 - \alpha_n) S P_C(x_n - \lambda_n A y_n), \ \ \forall n \geq 0, \end{cases} \tag{4}$$

where $\{\lambda_n\}$ and $\{\alpha_n\}$ satisfy the following conditions:

(i) $\lambda_n k \subset (0, 1 - \delta)$ for some $\delta \in (0, 1)$;

(ii) $\alpha_n \subset (0, 1), \sum_{n=1}^{\infty} \alpha_n = \infty, \lim_{n\to\infty} \alpha_n = 0$.

They proved that the sequence $\{x_n\}$ and $\{y_n\}$ converges strongly to the same point $P_{F(S)\cap VI(C,A)} x_0$ provided that $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$.

Later, motivated by the idea of Korpelevich's extragradient method [8], Lu-Chuan Cenga, and Jen-Chih Yao [3] and Liwei Li and W. Song [12] studies a hybrid of the extragradient method of monotone operator and maximal monotone operators, Nadezhkina and Takahashi [15] who introduced the following iterative scheme for finding an element of $F(S) \cap VI(C, A)$ and proved strong convergence theorem. Recently, Takahashi et al. [21] proved the following strong convergence theorem by the new hybrid projection method.

Theorem 1.3 ([21, Theorem 4.1]) *Let C be a nonempty closed convex subset of a real Hilbert space H. Let T be a nonexpansive mappings from C into itself such that $F(T) \neq \emptyset$ and let $x_0 \in H$. For $C_1 = C$, and $u_1 = P_{C_1}x_0$, and let*

$$\begin{cases} y_n = \alpha_n u_n + (1-\alpha_n)Tu_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|u_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \quad n \in \mathbf{N} \end{cases} \tag{5}$$

where $0 \leq \alpha_n < a < 1$. Then $\{u_n\}$ converges strongly to $z_0 = P_{F(T)}x_0$.

On the other hand, Takahashi and Takahashi [19] introduced an iterative scheme by the viscosity approximation method for finding a common element of the set of the solution (1) and the set of fixed points of a nonexpansive mapping in a real Hilbert space. Let $S : C \to C$ be a nonexpansive mapping. Starting with an arbitrary initial $x_1 \in C$, define sequences $\{x_n\}$ and $\{u_n\}$ recursively by

$$\begin{cases} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \ \ \forall y \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)Su_n, \forall n \geq 1. \end{cases} \tag{6}$$

They proved that under certain appropriate conditions imposed on $\{\alpha_n\}$ and $\{r_n\}$, the sequences $\{x_n\}$ and $\{u_n\}$ converge strongly to $z \in F(S) \cap EP(F)$, where $z = P_{F(S)\cap EP(F)}f(z)$.

In this paper, motivated by Zeng and Yao [24], Takahashi et al. [21] and Takahashi and Takahashi [19], we will introduce a new hybrid extragradient method (3.1) which is mixed with the iterative schemes considered in [24], [21] and [19] for finding a common element of the set of fixed points of a nonexpansive mapping, the set of solutions of an equilibrium problem and the set of solutions of the variational inequality for k-Lipschitz continuous mappings in a Hilbert space. Then, the strong convergence theorem is proved under some parameter controlling conditions. Our results extend and improved that the corresponding ones announced by Kumam [10, 11], Takahashi and Takahashi [19], Takahashi et al. [21] and Zeng and Yao [24].

2 Preliminary

Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. We denote weak convergence and strong convergence by notations $\rightharpoonup$ and $\to$, respectively. In a real Hilbert space H, it is well known that

$$\|\lambda x - (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2,$$

for all $x, y \in H$ and $\lambda \in [0,1]$. Recall that the (nearest point) projection P_C from H onto C assigns to each $x \in H$ the unique point in $P_C \in C$ satisfying the property

$$\|x - P_C x\| = \min_{y \in C} \|x - y\|.$$

The following characterizes the projection P_C. It is well known that P_C is a nonexpansive mapping of H onto C and satisfies

$$\langle x - y, P_C x - P_C y \rangle \geq \|P_C x - P_C y\|^2, \tag{7}$$

for every $x, y \in H$. Moreover, $P_C x$ is characterized by the following properties: $P_C x \in C$ and

$$\langle x - P_C x, y - P_C x \rangle \leq 0, \tag{8}$$

$$\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2, \tag{9}$$

for all $x \in H, y \in C$. It is easy to see that the following is true:

$$u \in VI(C, A) \Leftrightarrow u = P_C(u - \lambda A u), \lambda > 0. \tag{10}$$

Let A be a monotone, k-Lipschitz continuous mapping of C into H and let $N_C v$ be the normal cone to C at $v \in C$; i.e.,

$$N_C v := \{w \in H : \langle v - u, w \rangle \geq 0, \ \forall u \in C\}.$$

Define

$$Tv = \begin{cases} Av + N_C v, & v \in C, \\ \emptyset, & v \notin C. \end{cases}$$

Then, T is maximal monotone and $0 \in Tv$ if and only if $v \in VI(C, A)$ (see [16] for more details). It is also known that H satisfies the *Opial's condition* [17], that is, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

holds for every $y \in H$ with $y \neq x$. Hilbert space H, satisfies the *Kadec-Klee property* [6, 20], that is, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$ together imply $\|x_n - x\| \to 0$.

For solving the equilibrium problem for a bifunction $F : C \times C \to \mathbf{R}$, let us assume that F satisfies the following conditions:

(A1) $F(x, x) = 0$ for all $x \in C$;

(A2) F is monotone, i.e., $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$;

(A3) for each $x, y, z \in C$, $\lim_{t \downarrow 0} F(tz + (1-t)x, y) \leq F(x, y)$;

(A4) for each $x \in C, y \mapsto F(x,y)$ is convex and lower semicontinuous.

The following lemma appears implicitly in [1].

Lemma 2.1 [1] *Let C be a nonempty closed convex subset of H and let F be a bifunction of $C \times C$ into $\mathbf{R}$ satisfying (A1)-(A4). Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that*

$$F(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0 \text{ for all } y \in C.$$

The following lemma was also given in [2].

Lemma 2.2 [2] *Assume that $F : C \times C \to \mathbf{R}$ satisfies (A1)-(A4). For $r > 0$ and $x \in H$, define a mapping $T_r : H \to C$ as follows:*

$$T_r(x) = \left\{ z \in C : F(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C \right\}$$

for all $x \in H$. Then, the following hold:

1. *T_r is single-valued;*
2. *T_r is firmly nonexpansive, i.e. for any $x, y \in H$ $\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle$;*
3. *$F(T_r) = EP(F)$;*
4. *$EP(F)$ is closed and convex.*

3 Main Results

In this section, we prove a strong convergence theorem by hybrid extragradient method for a monotone, k-Lipschitz continuous mapping in a Hilbert space.

Theorem 3.1 *Let C be a nonempty closed convex subset of a real Hilbert space H, let F be a bifunction from $C \times C$ into $\mathbf{R}$ satisfying* (A1)-(A4)*, let A be a monotone and k-Lipschitz continuous mappings of C into H and let S be nonexpansive mappings from C into itself such that $F(S) \cap VI(C,A) \cap EP(F) \neq \emptyset$. Let $\{x_n\}$, $\{y_n\}$, $\{w_n\}$ and $\{u_n\}$ be sequences generated by $C_1 = C$, $x_1 = P_{C_1}x_0$, and let*

$$\begin{cases} u_n \in C \text{ such that } \quad F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ y_n = P_C(u_n - \lambda_n A u_n), \\ w_n = \alpha_n x_n + (1 - \alpha_n) S P_C(x_n - \lambda_n A y_n), \\ C_{n+1} = \{z \in C_n : \|w_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} x_0, \end{cases} \tag{11}$$

for all $n \in \mathbf{N}$, *where* $\{\lambda_n\} \subset [a,b]$ *for some* $a, b \in (0, \frac{1}{\xi})$, $\{\alpha_n\} \subset [c,d]$ *for some* $c, d \in (0,1)$ *and* $\{r_n\} \subset (0,\infty)$ *satisfies* $\liminf_{n\to\infty} r_n > 0$. *Then* $\{x_n\}$ *converges strongly to* $P_{F(S)\cap VI(C,A)\cap EP(F)}x_0$.

Proof. We show that $F(S) \cap VI(C,A) \cap EP(F) \subset C_n$. for all $n \in \mathbf{N}$. It is obvious that $F(S) \cap VI(C,A) \cap EP(F) \subset C = C_1$. Suppose that $F(S) \cap VI(C,A) \cap EP(F) \subset C_k$. for some $k \in \mathbf{N}$. Hence, for $v \in F(S) \cap VI(C,A) \cap EP(F) \subset C_k$. Let $\{T_{r_k}\}$ be a sequence of mappings defined as in Lemma 2.2. Then $v = P_C(v - \lambda_n Av) = T_{r_k}v$ and from $u_k = T_{r_k}x_k$, for any $k \in \mathbf{N}$, we note that

$$\|u_k - v\| = \|T_{r_k}x_k - T_{r_k}v\| \le \|x_k - v\|.$$

Put $v_k = P_C(u_k - \lambda_n Ak_n)$ for any $k \in \mathbf{N}$. From (9) and the monotonicity of A, we have

$$\begin{aligned}
\|v_k - v\|^2 &\le \|u_k - \lambda_n Ay_k - v\|^2 - \|u_k - \lambda_n Ay_k - v_k\|^2 \\
&= \|u_k - v\|^2 - \|u_k - v_k\|^2 + 2\lambda_k\langle Ay_k, v - v_k\rangle \\
&= \|u_k - v\|^2 - \|u_k - v_k\|^2 \\
&\quad +2\lambda_k(\langle Ay_k - Av, v - y_k\rangle + \langle Av, v - y_k\rangle + \langle Ay_k, y_k - v_k\rangle) \\
&\le \|u_k - v\|^2 - \|u_k - v_k\|^2 + 2\lambda_k\langle Ay_k, y_k - v_k\rangle \\
&= \|u_k - v\|^2 - \|u_k - y_k\|^2 - 2\langle u_k - y_k, y_k - v_k\rangle - \|y_k - v_k\|^2 \\
&\quad +2\lambda_k\langle Ay_k, y_k - v_k\rangle \\
&= \|u_k - v\|^2 - \|u_k - y_k\|^2 - \|y_k - v_k\|^2 \\
&\quad +2\langle u_k - \lambda_n Ay_k - y_k, v_k - y_k\rangle.
\end{aligned}$$

Moreover, from $y_k = P_C(u_k - \lambda_n Au_k)$ and (8), we have

$$\langle u_k - \lambda_k Au_k - y_k, v_k - y_k\rangle \le 0. \tag{12}$$

Since A is k-Lipschitz-continuous, it follows that

$$\begin{aligned}
\langle u_k - \lambda_k Ay_k - y_n, v_k - y_k\rangle &= \langle u_k - \lambda_n Au_k - y_k, v_k - y_k\rangle + \langle \lambda_n Au_k - \lambda_n Ay_k, v_k - y_k\rangle \\
&\le \langle \lambda_n Au_k - \lambda_n Ay_k, v_k - y_k\rangle \\
&\le \lambda_n\xi\|u_k - y_k\|\|v_k - y_k\|.
\end{aligned}$$

So, we have

$$\begin{aligned}
\|v_k - v\|^2 &\le \|u_k - v\|^2 - \|u_k - y_k\|^2 - \|y_k - v_k\|^2 \\
&\quad +2\lambda_k\xi\|u_k - y_k\|\|v_k - y_k\| \\
&\le \|u_k - v\|^2 - \|u_k - y_k\|^2 - \|y_k - v_k\|^2 \\
&\quad +\lambda_k^2\xi^2\|u_k - y_k\|^2 + \|v_k - y_k\|^2 \\
&= \|u_k - v\|^2 + (\lambda_k^2\xi^2 - 1)\|u_k - y_k\|^2 \qquad (13) \\
&\le \|u_k - v\|^2,
\end{aligned}$$

and hence

$$\|v_k - v\| \leq \|u_k - v\| \leq \|x_k - v\|.$$

Thus, from (13) we have

$$\begin{aligned}
\|w_k - v\|^2 &\leq \alpha_n\|x_k - v\|^2 + (1-\alpha_n)\|Sv_k - v\|^2 \\
&\leq \alpha_k\|x_k - v\|^2 + (1-\alpha_k)\|v_k - v\|^2 \\
&\leq \alpha_k\|x_k - v\|^2 + (1-\alpha_k)\{\|u_k - v\|^2 + (\lambda_k^2\xi^2 - 1)\|u_k - y_k\|^2\} \\
&\leq \alpha_k\|x_k - v\|^2 + (1-\alpha_k)\|x_k - v\|^2 + (1-\alpha_k)(\lambda_k^2\xi^2 - 1)\|u_k - y_k\|^2 \\
&= \|x_k - v\|^2 + (1-\alpha_k)(\lambda_k^2\xi^2 - 1)\|u_k - y_k\|^2 \leq \|x_k - v\|^2.
\end{aligned}$$

and hence $u \in C_{k+1}$. This implies that

$$F(S) \cap VI(C, A) \cap EP(F) \subset C_n, \quad \text{for all} \quad n \in \mathbf{N}. \tag{14}$$

Next, we show that C_n is closed and convex for all $n \in \mathbf{N}$. It is obvious that $C_1 = C$ is closed and convex. Suppose that C_k is closed and convex for some $k \in \mathbf{N}$. For $z \in C_k$, we know that $\|w_k - z\| \leq \|x_k - z\|$ is equivalent to

$$\|w_k - x_k\|^2 + 2\langle w_k - x_k, x_k - z\rangle \geq 0. \tag{15}$$

So, C_{k+1} is closed and convex. Then, for any $n \in \mathbf{N}$, C_n is closed and convex. This implies that $\{x_n\}$ and $\{u_n\}$ are well-defined. From $x_n = P_{C_n}x_0$, we have

$$\langle x_0 - x_n, x_n - y\rangle \geq 0$$

for each $y \in C_n$. Using $F(S) \cap VI(C, A) \cap EP(F) \subset C_n$, we also have

$$\langle x_0 - x_n, x_n - u\rangle \geq 0 \quad \text{for all} \quad u \in F(S) \cap VI(C, A) \cap EP(F) \quad \text{and} \quad n \in \mathbf{N}.$$

So, for $u \in F(S) \cap VI(C, A) \cap EP(F)$. we have

$$\begin{aligned}
0 &\leq \langle x_0 - x_n, x_n - u\rangle \\
&= \langle x_n - x_0, x_n - x_0 + x_0 - u\rangle \\
&= -\|x_0 - x_n\|^2 + \|x_0 - x_n\|\|x_0 - u\|.
\end{aligned}$$

This implies that

$$\|x_0 - x_n\| \leq \|x_n - u\| \quad \text{for all} \quad u \in F(S) \cap VI(C, A) \cap EP(F) \quad \text{and} \quad n \in \mathbf{N}.$$

From $x_n = P_{C_n}x_0$, and $x_{n+1} = P_{C_{n+1}}x_0 \in C_{n+1} \subset C_n$ we obtain

$$\langle x_0 - x_n, x_n - x_{n+1}\rangle \geq 0 \tag{16}$$

From (16), we have, for $n \in \mathbf{N}$,

$$\begin{aligned} 0 &\leq \langle x_0 - x_n, x_n - x_{n+1} \rangle \\ &= \langle x_0 - x_n, x_n - x_0 + x_0 - x_{n+1} \rangle \\ &= -\|x_0 - x_n\|^2 + \langle x_0 - x_n, x_n - x_{n+1} \rangle \\ &= -\|x_0 - x_n\|^2 + \|x_0 - x_n\| \|x_0 - x_{n+1}\|. \end{aligned}$$

and hence

$$\|x_0 - x_n\| \leq \|x_0 - x_{n+1}\|.$$

Thus the sequence $\{\|x_n - x_0\|\}$ is a bounded and nonincreasing sequence, so $\lim_{n\to\infty} \|x_n - x_0\|$ exists, that is there exists

$$c = \lim_{n\to\infty} \|x_n - x_0\|. \tag{17}$$

Next we can show that $\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0$. Indeed, from (16) we get

$$\begin{aligned} \|x_n - x_{n+1}\|^2 &= \|x_n - x_0 + x_0 - x_{n+1}\|^2 \\ &= \|x_n - x_0)\|^2 + 2\langle x_n - x_0, x_0 - x_{n+1} \rangle \\ &= -\|x_n - x_0)\|^2 + 2\langle x_n - x_0, x_0 - x_n + x_n - x_{n+1} \rangle + \|x_0 x_{n+1}\| \\ &\leq -\|x_n - x_0\|^2 + \|x_0 - x_{n+1}\|^2. \end{aligned}$$

From (17), we obtain

$$\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0. \tag{18}$$

Since $x_{n+1} \in C_{n+1} \subset C_n$, we have

$$\|x_n - w_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - w_n\| \leq 2\|x_n - x_{n+1}\|.$$

By (18), we obtain

$$\lim_{n\to\infty} \|x_n - w_n\| = 0. \tag{19}$$

From (27), we have

$$\|x_n - w_n\| = \|x_n - \alpha_n x_n - (1 - \alpha_n) S v_n\| = \|(1 - \alpha_n)(x_n - S v_n)\|,$$

it follows by (19) that

$$\lim_{n\to\infty} \|x_n - S v_n\| = 0.$$

Let $v \in F(S) \cap VI(C, A) \cap EP(F)$. Then, we obtain that

$$\begin{aligned} \|u_n - v\|^2 &= \|T_{r_n} x_n - T_{r_n} v\|^2 \leq \langle T_{r_n} x_n - T_{r_n} v, x_n - v \rangle = \langle u_n - v, x_n - v \rangle \\ &= \frac{1}{2}(\|u_n - v\|^2 + \|x_n - v\|^2 - \|x_n - u_n\|^2). \end{aligned}$$

Therefore, $\|u_n - v\|^2 \leq \|x_n - v\|^2 - \|x_n - u_n\|^2$. Thus, we can calculate

$$\begin{aligned}
\|w_n - v\|^2 &= \|\alpha_n(x_n - v) + (1-\alpha_n)(Sv_n - v)\|^2 \\
&\leq \alpha_n\|x_n - v\|^2 + (1-\alpha_n)\|v_n - v\|^2 \\
&\leq \alpha_n\|x_n - v\|^2 + (1-\alpha_n)\|u_n - v\|^2 \\
&\leq \alpha_n\|x_n - v\|^2 + (1-\alpha_n)\{\|x_n - v\|^2 - \|x_n - u_n\|^2\} \\
&= \|x_n - v\|^2 - (1-\alpha_n)\|x_n - u_n\|^2,
\end{aligned}$$

and hence

$$\|w_n - v\|^2 \leq \|x_n - v\|^2 - (1-\alpha_n)\|u_n - x_n\|^2.$$

Since $0 < c \leq \alpha_n \leq d < 1$, it follows that

$$\begin{aligned}
(1-d)\|x_n - u_n\|^2 &\leq (1-\alpha_n)\|x_n - u_n\|^2 \\
&= \|x_n - v\|^2 - \|w_n - v\|^2 \\
&= (\|x_n - v\| + \|w_n - v\|)(\|x_n - v\| - \|w_n - v\|) \\
&\leq \|x_n - w_n\|(\|x_n - v\| + \|w_n - v\|). \qquad (20)
\end{aligned}$$

From (19) and (20), we have

$$\lim_{n\to\infty} \|x_n - u_n\| = 0. \qquad (21)$$

Since $\liminf_{n\to\infty} r_n > 0$, we obtain

$$\lim_{n\to\infty} \|\frac{x_n - u_n}{r_n}\| = \lim_{n\to\infty} \frac{1}{r_n}\|x_n - u_n\| = 0. \qquad (22)$$

For $v \in F(S) \cap VI(C, A) \cap EP(F)$, from (10) we obtain

$$\|w_n - v\|^2 \leq \|x_n - v\|^2 + (1-\alpha_n)(\lambda_n^2 k^2 - 1)\|u_n - y_n\|^2.$$

Therefore, we have

$$\begin{aligned}
\|u_n - y_n\| &\leq \frac{1}{(1-\alpha_n)(1-\lambda_n^2 k^2)}(\|x_n - v\|^2 - \|w_n - v\|^2) \\
&= \frac{1}{(1-\alpha_n)(1-\lambda_n^2 k^2)}(\|x_n - v\| + \|w_n - x^*\|)(\|x_n - v\| - \|w_n - v\|) \\
&\leq \frac{1}{(1-\alpha_n)(1-\lambda_n^2 k^2)}\|x_n - w_n\|(\|x_n - v\| + \|w_n - v\|).
\end{aligned}$$

So, by (19) we obtain

$$\lim_{n\to\infty} \|u_n - y_n\| = 0. \qquad (23)$$

Since $\|x_n - y_n\| \leq \|x_n - u_n\| + \|u_n - y_n\|$, from (21) and (23) we also have

$$\lim_{n\to\infty} \|x_n - y_n\| = 0. \qquad (24)$$

From $\|v_n - y_n\| \leq \|u_n - y_n\|$ and (23), we obtain

$$\lim_{n\to\infty} \|v_n - y_n\| = 0, \tag{25}$$

we also obtain

$$\|Sv_n - v_n\| \leq \|Sv_n - x_n\| + \|x_n - y_n\| + \|y_n - v_n\|,$$

and hence

$$\lim_{n\to\infty}\|Sv_n - v_n\| = 0. \tag{26}$$

Since $\{v_n\}$ is bounded, there exists a subsequence $\{v_{n_i}\}$ of $\{v_n\}$ which converges weakly to z. Without loss of generality, we can assume that $v_{n_i} \rightharpoonup z$. Since $v_{n_i} \subset C$ and C is closed and convex, C is weakly closed and hence $z \in C$. From $\|Sv_n - v_n\| \to 0$, we obtain $Sv_{n_i} \rightharpoonup z$. We will show that $z \in EP(F)$. Using the same argument as in the proof in [11, Theorem 3.1 pp. 273] or [10, Theorem 3 pp. 1251], we can prove that $z \in EP(F)$. From Opial's condition, we obtain that $z \in F(S)$. Finally, we can show that $z \in VI(C, A)$. Define,

$$Tv = \begin{cases} Av + N_C v, & v \in C, \\ \emptyset, & v \notin C. \end{cases}$$

Then, T is maximal monotone. Let $(v, u) \in G(T)$. Since $u - Av \in N_C v$ and $v_n \in C$, we have $\langle v - v_n, u - Av\rangle \geq 0$. On the other hand, from $v_n = P_C(u_n - \lambda_n Ay_n)$, we have $\langle v - v_n, v_n - (u_n - \lambda_n Ay_n)\rangle \geq 0$, and hence, $\left\langle v - v_n, \frac{v_n - u_n}{\lambda_n} + Ay_n \right\rangle \geq 0$. Therefore, we have

$$\begin{aligned}
\langle v - v_{n_i}, u\rangle &\geq \langle v - v_{n_i}, Av\rangle \\
&\geq \langle v - v_{n_i}, Av\rangle - \left\langle v - v_{n_i}, \frac{v_{n_i} - u_{n_i}}{\lambda_{n_i}} + Ay_{n_i} \right\rangle \\
&= \left\langle v - v_{n_i}, Av - Ay_{n_i} - \frac{v_{n_i} - u_{n_i}}{\lambda_{n_i}} \right\rangle \\
&= \langle v - v_{n_i}, Av - Av_{n_i}\rangle + \langle v - v_{n_i}, Av_{n_i} - Ay_{n_i}\rangle - \left\langle v - v_{n_i}, \frac{v_{n_i} - u_{n_i}}{\lambda_{n_i}} \right\rangle \\
&\geq \langle v - v_{n_i}, Av_{n_i} - Ay_{n_i}\rangle - \left\langle v - v_{n_i}, \frac{v_{n_i} - u_{n_i}}{\lambda_{n_i}} \right\rangle,
\end{aligned}$$

Since $\lim_{n\to\infty} \|v_n - u_n\| = \lim_{n\to\infty} \|v_n - y_n\| = 0$, $u_{n_i} \rightharpoonup p$ and A is Lipschitz continuous, we obtain that $\lim_{n\to\infty} \|Av_n - Ay_n\| = 0$ and $v_{n_i} \rightharpoonup p$. From $\liminf_{n\to\infty} \lambda_n > 0$ and $\lim_{n\to\infty} \|v_n - u_n\| = 0$, we obtain

$$\langle v - z, u\rangle \geq 0.$$

Since T is maximal monotone, we have $z \in T^{-1}0$ and hence $z \in VI(C, A)$. Hence, we have $z \in F(S) \cap VI(C, A) \cap EP(F)$. Finally, we show that $x_n \to$

z, where $z = P_{F(S)\cap VI(C,A)EP(F)}x_0$. Since $x_n = P_{Q_n}x_0$ and $z \in F(S) \cap VI(C,A) \cap EP(F) \subset Q_n$, we have $\|x_n - x_0\| \leq \|z - x_0\|$. It follows from $z' = P_{F(S)\cap VI(C,A)\cap EP(F)}x_0$ and the lower semicontinuity of the norm that

$$\|z' - x_0\| \leq \|z - x_0\| \leq \liminf_{i\to\infty}\|x_{n_i} - x_0\| \leq \limsup_{i\to\infty}\|x_{n_i} - x_0\| \leq \|z' - x_0\|.$$

Thus, we obtain that $\lim_{k\to\infty}\|x_{n_i} - x_0\| = \|z - x_0\| = \|z' - x_0\|$. Using the Kadec-Klee property of H, we obtain that

$$\lim_{i\to\infty} x_{n_i} = z = z'.$$

Since $\{x_{n_i}\}$ is an arbitrary weakly convergent subsequence of $\{x_n\}$, we can conclude that $\{x_n\}$ converges strongly to z, where $z = P_{F(T)\cap VI(C,A)\cap EP(F)}x_0$.

Corollary 3.2 *Let C be a nonempty closed convex subset of a real Hilbert space H. Let A be monotone, k-Lipschitz continuous mapping of C into H and let S be a nonexpansive mappings from C into itself such that $F(S) \cap VI(C,A) \neq \emptyset$. Let $\{x_n\}$, $\{y_n\}$ and $\{w_n\}$ be sequences generated by $x_0 \in C$ and let*

$$\begin{cases} y_n = P_C(x_n - \lambda_n Ax_n), \\ w_n = \alpha_n x_n + (1-\alpha_n)SP_C(x_n - \lambda_n Ay_n), \\ C_{n+1} = \{z \in C_n : \|w_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \end{cases} \tag{27}$$

for all $n \in \mathbf{N}$, where $\{\lambda_n\} \subset [a,b]$ for some $a, b \in (0, \frac{1}{k})$, $\{\alpha_n\} \subset [c,d]$ for some $c, d \in (0,1)$. Then $\{x_n\}$ converges strongly to $P_{F(S)\cap VI(C,A)}x_0$.

Proof. Put $F(x,y) = 0$ for all $x, y \in C$ and $\{r_n\} = 1$ in Theorem 3.1. Thus, we have $u_n = x_n$. Then the sequence $\{x_n\}$ generated in Corallary 3.2 converges strongly to $P_{F(S)\cap VI(C,A)}x_0$.

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