Int. Journal of Math. Analysis, Vol. 3, 2009, no. 39, 1905 - 1915

On Approximation Properties of the Families of Flett and Generalized Flett Potentials

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Abstract

This paper aims to study the approximation properties of the Flett $(\mathcal{F}^{\alpha}f)$ and the generalized Flett potentials $(\mathcal{F}_{\nu}^{\alpha}f)$, which are represented by the Poisson integral $(\mathcal{P}_t)$ and the generalized Poisson integrals $(\mathcal{P}_t^{\nu})$ respectively, when the parameter $\alpha > 0$ tends to zero.

Mathematics Subject Classification: 41A35, 26A33

Keywords: Flett potentials, generalized translation, Lipschitz class, Poisson integral

1 Introduction and Preliminaries

The Flett potentials, $\mathcal{F}^{\alpha}f$ and the generalized Flett potentials, $\mathcal{F}_{\nu}^{\alpha}f$ are defined in terms of Fourier and Fourier- Bessel transforms by the following formulas:

$$F(\mathcal{F}^{\alpha}f)(x) = (1+|x|)^{-\alpha}F(f)(x),\ (x \in \mathbf{R}^n\ , \alpha > 0) \tag{1}$$

$$F_{\nu}(\mathcal{F}_{\nu}^{\alpha}f)(x) = (1+|x|)^{-\alpha}F_{\nu}(f)(x),\ (x \in \mathbf{R}_+^n\ , \alpha > 0). \tag{2}$$

These potentials are interpreted as a negative fractional powers of the operator $(E+(-\Delta)^{1/2})$ and $(E+(-\Delta_{\nu})^{1/2})$ respectively. Here, $\Delta = \sum\limits_{k=1}^{n} \frac{\partial^2}{\partial x_k^2}$ is the Laplacean, $\Delta_{\nu} = \sum\limits_{k=1}^{n-1} \frac{\partial^2}{\partial x_k^2} + \left(\frac{\partial^2}{\partial x_n^2} + \frac{2\nu}{x_n}\frac{\partial}{\partial x_n}\right)$ $(\nu > 0)$ is the singular Laplace-Bessel differential operator and E is the identity operator.

Note that, the Flett potentials are introduced by T.M. Flett in his fundamental paper [5, p.445-448] (see also [3, p.541-542]). One should also mention

the papers [3] by I.A. Aliev, S. Sezer and M. Eryigit and [15] by S. Sezer, devoted to inversion formulas for the Flett and generalized Flett potentials.

Approximation properties of the potentials such as Riesz, Bessel, generalized Riesz and generalized Bessel, have been studied by T. Kurokawa[9] and A.D. Gadjiev, A.Aral and I.A. Aliev [7].

The main aim of this paper is to inverstigate the approximation properties of the Flett and generalized Flett potentials as the parameter α goes to zero.

We start by giving some auxilary definitions, notations and lemmas.

Let $\mathbf{R}^n_+ = \{x : x = (x_1, \dots, x_{n-1}, x_n) \in \mathbf{R}^n,\ x_n > 0\}$.

$$L_p \equiv L_p(\mathbf{R}^n) = \left\{ f : \|f\|_p = \left(\int\limits_{\mathbf{R}^n} |f(x)|^p dx \right)^{1/p} < \infty \right\},$$

$$L_{p,\nu} \equiv L_{p,\nu}(\mathbf{R}^n_+) = \left\{ f : \|f\|_{p,\nu} = \left(\int\limits_{\mathbf{R}^n_+} |f(x)|^p x_n^{2\nu} dx \right)^{1/p} < \infty \right\},$$

where $1 \le p < \infty, \nu > 0, dx = dx_1 \cdots dx_n$.

$C_0 \equiv C_0(\mathbf{R}^n)$ is the class of all continuous functions on $\mathbf{R}^n$ vanishing at infinity.

In this paper, the letters c, c_0, c_1, $c_2, \dots$ are used for constants and if the function $\varphi(\alpha)$ is bounded as $\alpha \to 0^+$, we will write $\varphi(\alpha) = O(1)$ as $\alpha \to 0^+$.

The Fourier transform and Fourier-Bessel transform of a function f are defined by the following formulas:

$$(Ff)(\xi) = \int\limits_{\mathbf{R}^n} e^{i\,\xi\cdot x} f(x) dx, \qquad \xi \cdot x = \xi_1 x_1 + \cdots + \xi_n x_n,$$

$$(F_\nu f)(\xi) = \int\limits_{\mathbf{R}^n_+} f(x) e^{i\,\xi'\cdot x'} j_{\nu-\frac{1}{2}}(\xi_n x_n) x_n^{2\nu} dx,$$

where $x' = (x_1, \dots, x_{n-1})$, $\xi' \cdot x' = \xi_1 x_1 + \cdots + \xi_{n-1} x_{n-1}$ and $j_\lambda(\tau) = 2^\lambda \Gamma(\lambda + 1) J_\lambda(\tau)/\tau^\lambda$, $J_\lambda(\tau)$ is the Bessel function of the first kind.

The Fourier-Bessel harmonic analysis is adopted to the generalized convolution

$$(f \otimes g)(x) = \int\limits_{\mathbf{R}^n_+} f(y)(T^y g)(x) y_n^{2\nu} dy,\ x \in \mathbf{R}^n_+,$$

where T^y is the generalized translation operator, acting according to the law

$$(T^y g)(x) = \frac{\Gamma(\nu + 1/2)}{\Gamma(\nu)\Gamma(1/2)} \int\limits_0^\pi g(x' - y',\ \sqrt{x_n^2 - 2x_n y_n \cos\alpha + y_n^2}) \sin^{2\nu-1}\alpha d\alpha \tag{3}$$

(see, e.g., [1], [2], [6],[8], [10] and [17]). Actually we deal with the usual (Euclidean) translation in $x' = (x_1, \dots, x_{n-1})$ and the generalized translation with in x_n-variable.

It is well known that (see, [12])

$$\begin{aligned} &\|T^y f\|_{p,\nu} \le \|f\|_{p,\nu} \ , \forall y \in \mathbf{R}^n_+, \ 1 \le p < \infty, \\ &\|T^y f - f\|_{p,\nu} \to 0 \quad , \quad \text{as} \quad |y| \to 0. \end{aligned} \tag{4}$$

Now let us introduce the following Lipschitz classes of $\Lambda_{\boldsymbol{\beta}}, \Lambda^{\boldsymbol{\nu}}_{\boldsymbol{\beta}}$ and the local Lipschitz classes $\Lambda_\beta(x_0)$, $\Lambda^\nu_\beta(x_0)$ which we need.

$$\Lambda_\beta = \{f \in L_\infty(\mathbf{R}^n) : \|f(x-y) - f(x)\|_\infty \le c|y|^\beta\} \tag{5}$$

$$\Lambda^{\boldsymbol{\nu}}_{\boldsymbol{\beta}} = \{f \in L_\infty(\mathbf{R}^n_+) : \|T^y f(x) - f(x)\|_\infty \le c|y|^\beta\} \tag{6}$$

$$\Lambda_\beta(x_0) = \{f : |f(x_0 - y) - f(x_0)| \ \le c_f|y|^\beta, \ \forall |y| \le 1, \ x_0, \, y \in \mathbf{R}^n\} \tag{7}$$

$$\Lambda^\nu_\beta(x_0) = \{f : |T^y f(x_0) - f(x_0)| \ \le c_f|y|^\beta, \ \forall |y| \le 1, \ x_0, \, y \in \mathbf{R}^n_+\} \tag{8}$$

where $0 < \beta \le 1$.

The Poisson integrals $\mathcal{P}_t f$ and the generalized Poisson integrals $\mathcal{P}^\nu_t f$ are defined as:

$$(\mathcal{P}_t f)(y) = \int\limits_{\mathbf{R}^n} P(x, \, t) f(y - x) dx, \quad F[P(\cdot, \, t)](\xi) = e^{-t|\xi|}; \tag{9}$$

$$(\mathcal{P}^\nu_t f)(y) = \int\limits_{\mathbf{R}^n_+} P^\nu(x, \, t)(T^x f)(y) x_n^{2\nu} dx, \quad F_\nu[P^\nu(\cdot, \, t)](\xi) = e^{-t|\xi|}. \tag{10}$$

The corresponding Poisson and generalized Poisson kernels in (9) and (10) have the following form :

$$P(x, \, t) \ = \ \frac{\Gamma(\frac{n+1}{2})}{\pi^{(n+1)/2}} \frac{t}{(|x|^2 + t^2)^{(n+1)/2}} \ ; \tag{11}$$

$$P^\nu(x, \, t) \ = \ \frac{2\Gamma(\frac{n+2\nu+1}{2})}{\pi^{n/2}\Gamma(\nu + \frac{1}{2})} \frac{t}{(|x|^2 + t^2)^{(n+2\nu+1)/2}} \tag{12}$$

Furthermore, these kernels have the following properties:

$$\int\limits_{\mathbf{R}^n} P(x, \, t) dy = 1 \quad , \int\limits_{\mathbf{R}^n_+} P^\nu(x, \, t) x_n^{2\nu} dy = 1. \tag{13}$$

More information about these integral operators can be found in [1], [2], [11],[14],[16]. The following lemma gives us some properties of generalized Poisson integral.

Lemma 1.1 *(see [1], [13, p.217]) Let* $f \in L_{p,\nu}$, $1 \le p < \infty$ *and* $\mathcal{P}_t^\nu f$, $t > 0$ *be as in (10). Then*

(a)

$$ess \sup_{x \in \mathbf{R}_+^n} |(\mathcal{P}_t^\nu f)(x)| \le c t^{-\theta} \|f\|_{p,\nu} , \quad \theta = \frac{n+2\nu}{p} ; \tag{14}$$

(b)

$$\sup_{t>0} \|\mathcal{P}_t^\nu f\|_{p,\nu} \le c \|f\|_{p,\nu} ; \tag{15}$$

(c) The maximal operator $(P^* f)(x) = \sup_{t>0} |(\mathcal{P}_t^\nu f)(x)|$ *is weak* (p,p), *i.e.*

$$m\{x : (P^* f)(x) > \lambda\} \le (\frac{c\|f\|_{p,\nu}}{\lambda})^p, \ \forall \ \lambda > 0 ;$$

(d) $\lim_{t\to 0}(\mathcal{P}_t^\nu f)(x) = f(x)$ *in the* $L_{p,\nu}-$*norm and pointwise for almost all* $x \in \mathbf{R}_+^n$. *For* $f \in L_{p,\nu} \cap C_0$, *the convergence is also uniform on* $\mathbf{R}_+^n$.

Remark 1.2 *(see [1], [3, p.694]) Note that Lemma 1.1 is also true for the classical Poisson integral* $\mathcal{P}_t f$, $f \in L_p$, *and in this case the number* θ *in (14) is equal to* $\frac{n}{p}$.

Lemma 1.3 *(see [4, p.27]) Let* (X, μ) *be a measure space and let* $\{T_\varepsilon\}$, $\varepsilon > 0$ *be a family of linear operators on* $L_p(X, \mu)$. *Denote*

$$(T^* f)(x) = \sup_{\varepsilon>0} |(T_\varepsilon f)(x)|.$$

If T^* *is weak* $(p,\ q)$, *i.e.* ,

$$\mu\{y : (T^* f)(y) > \lambda\} \le (\frac{c\|f\|_p}{\lambda})^q, \forall \lambda > 0,$$

then the set

$$\{f :\ f \in L_p(X,\ \mu),\ \lim_{\varepsilon\to 0}(T_\varepsilon f)(x) = f(x),\ a.e\}$$

is closed in $L_p(X, \mu)$.

Using the properties in (13) and the definitions of the Lipschitz classes $\Lambda_{\boldsymbol{\beta}}, \Lambda_{\boldsymbol{\beta}}^{\boldsymbol{\nu}}$ it can be proved the following lemma.

Lemma 1.4 *a) If* $f \in \Lambda_{\boldsymbol{\beta}}$, *then*

$$\|\mathcal{P}_t f - f\|_\infty = O(1) t^\beta \quad \text{as} \ \ \mathrm{t} \to 0. \tag{16}$$

b) If $f \in \Lambda^{\boldsymbol{\nu}}_{\boldsymbol{\beta}}$, *then*

$$\|\mathcal{P}^{\nu}_t f - f\|_{\infty} = O(1)t^{\beta} \quad \text{as t} \to 0. \tag{17}$$

The Flett potentials $\mathcal{F}^{\alpha} f$ and the generalized Flett potentials $\mathcal{F}^{\alpha}_{\nu} f$, initially defined by (1) and (2), have the following integral representations via the Poisson and the generalized Poisson integrals, respectively:

$$(\mathcal{F}^{\alpha} f)(x) = \frac{1}{\Gamma(\alpha)} \int_0^{\infty} t^{\alpha-1} e^{-t} (\mathcal{P}_t f)(x) dt \quad [3, p.694] \tag{18}$$

$$(\mathcal{F}^{\alpha}_{\nu} f)(x) = \frac{1}{\Gamma(\alpha)} \int_0^{\infty} t^{\alpha-1} e^{-t} (\mathcal{P}^{\nu}_t f)(x) dt \quad [15, p.120] \tag{19}$$

2 Main Results

In this section we represent two theorems which give us the approximation properties of the families $\mathcal{F}^{\alpha}_{\nu} f$ and $\mathcal{F}^{\alpha} f$ as the parameter α tends to zero. The first theorem gives the approximation properties of the generalized Flett potentials $\mathcal{F}^{\alpha}_{\nu} f$ at the Lipschitz classes $\Lambda^{\boldsymbol{\nu}}_{\boldsymbol{\beta}}, \Lambda^{\boldsymbol{\nu}}_{\boldsymbol{\beta}}(x_0)$ which are defined as in (6) and (8). Similarly, the second theorem gives the approximation properties of the classical Flett potentials $\mathcal{F}^{\alpha} f$ at the Lipschitz classes $\Lambda_{\beta}, \Lambda_{\beta}(x_0)$ which are defined as in (5) and (7).

Theorem A. *Let* $f \in L_{p,\nu}$, $1 \le p < \infty$, *and* $(\mathcal{F}^{\alpha}_{\nu} f)_{\alpha>0}$ *are defined as in (19). Then*

(A1) $\lim\limits_{\alpha \to 0^+} (\mathcal{F}^{\alpha}_{\nu} f)(x) = f(x)$ *for almost all* $x \in \mathbf{R}^n_+$;

(A2) *If* $f \in L_{p,\nu} \cap C_0$, *then the convergence is uniform on* $\mathbf{R}^n_+$.

(A3) *If* $f \in L_{p,\,\nu} \cap \Lambda^{\nu}_{\beta}$, $0 < \beta \le 1$ *then*

$$\|\mathcal{F}^{\alpha}_{\nu} f - f\|_{\infty} = O(1)\alpha \quad \text{as} \quad \alpha \to 0^+.$$

(A4) *If* $f \in L_{p,\,\nu} \cap \Lambda^{\nu}_{\beta}(x_0)$, $x_0 \in \mathbf{R}^n_+$, $0 < \beta \le 1$ *then*

$$(\mathcal{F}^{\alpha}_{\nu} f)(x_0) - f(x_0) = O(1)\alpha \quad \text{as} \quad \alpha \to 0^+.$$

Proof (A1) According to Lemma 1.1 (d) there exists a point $x \in \mathbf{R}^n_+$ such that

$$\lim_{t \to 0} (\mathcal{P}^{\nu}_t f)(x) = f(x).$$

Therefore given $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|(\mathcal{P}_t^\nu f)(x) - f(x)| < \varepsilon$$

for all $0 < t < \delta$. Using this observation we have

$$\begin{aligned}
|(\mathcal{F}_\nu^\alpha f)(x) - f(x)| &= \left| \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{-t} (\mathcal{P}_t^\nu f)(x) dt - \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{-t} f(x) dt \right| \\
&\leq \frac{1}{\Gamma(\alpha)} \int_0^\delta t^{\alpha-1} e^{-t} |(\mathcal{P}_t^\nu f)(x) - f(x)| dt \\
&+ \frac{1}{\Gamma(\alpha)} \int_\delta^\infty t^{\alpha-1} e^{-t} |(\mathcal{P}_t^\nu f)(x) - f(x)| dt \\
&\equiv i_1(x) + i_2(x),
\end{aligned}$$

and we can estimate $i_1(x)$ as

$$i_1(x) \leq \varepsilon. \tag{20}$$

Now let us estimate $i_2(x)$.

$$\begin{aligned}
i_2(x) &\leq \frac{1}{\Gamma(\alpha)} \int_\delta^\infty t^{\alpha-1} e^{-t} (|\mathcal{P}_t^\nu f(x)| + |f(x)|) dt \\
&\overset{(14)}{\leq} \frac{c}{\Gamma(\alpha)} \int_\delta^\infty t^{\alpha-1} e^{-t} \left(t^{-\frac{n+2\nu}{p}} \|f\|_{p,\nu} + |f(x)| \right) dt \\
&\leq \frac{c\|f\|_{p,\nu}}{\Gamma(\alpha)} \int_\delta^\infty t^{\alpha - \frac{n+2\nu}{p} - 1} e^{-t} dt + \frac{c|f(x)|}{\Gamma(\alpha)} \int_\delta^\infty t^{\alpha-1} e^{-t} dt \\
&\leq \frac{c\|f\|_{p,\nu}}{\Gamma(\alpha)} \frac{\delta^{\alpha - \frac{n+2\nu}{p}}}{\left(\frac{n+2\nu}{p} - \alpha \right)} + \frac{c|f(x)|}{\Gamma(\alpha)} \delta^{\alpha-1} e^{-\delta} = O(1)\alpha \quad \text{as } \alpha \to 0^+.
\end{aligned} \tag{21}$$

Finally

$$|(\mathcal{F}_\nu^\alpha f)(x) - f(x)| \leq \varepsilon + O(1)\alpha \quad \text{as } \alpha \to 0^+$$

and therefore

$$\limsup_{\alpha \to 0^+} |(\mathcal{F}_\nu^\alpha f)(x) - f(x)| \leq \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary we have

$$\lim_{\alpha\to 0^+} |(\mathcal{F}_\nu^\alpha f)(x) - f(x)| = 0.$$

(A2) Let $f \in L_{p,\nu} \cap C_0$. Then using (20) and (21) we can write the following

$$\|\mathcal{F}_\nu^\alpha f - f\|_\infty \le \varepsilon + \frac{c\alpha}{\Gamma(\alpha+1)} \left(\frac{\|f\|_{p,\nu}}{\frac{n+2\nu}{p} - \alpha} \delta^{\alpha - \frac{n+2\nu}{p}} + \|f\|_\infty \delta^{\alpha-1} e^{-\delta} \right).$$

Therefore, $\lim_{\alpha\to 0^+} \|\mathcal{F}_\nu^\alpha f - f\|_\infty = 0$.

(A3) Let $f \in L_{p,\nu} \cap \Lambda_\beta^\nu$. Then

$$\begin{aligned}
(\mathcal{F}_\nu^\alpha f)(x) - f(x) &= \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{-t} [(\mathcal{P}_t^\nu f)(x) - f(x)] dt \\
&= \frac{1}{\Gamma(\alpha)} \int_0^1 t^{\alpha-1} e^{-t} [(\mathcal{P}_t^\nu f)(x) - f(x)] dt \\
&+ \frac{1}{\Gamma(\alpha)} \int_1^\infty t^{\alpha-1} e^{-t} [(\mathcal{P}_t^\nu f)(x) - f(x)] dt \qquad (22)
\end{aligned}$$

Furthermore, using (14) and (17) we get

$$\begin{aligned}
\|\mathcal{F}_\nu^\alpha f - f\|_\infty &\le \frac{1}{\Gamma(\alpha)} \int_0^1 t^{\alpha-1} e^{-t} \|\mathcal{P}_t^\nu f - f\|_\infty dt \\
&+ \frac{1}{\Gamma(\alpha)} \int_1^\infty t^{\alpha-1} e^{-t} (\|\mathcal{P}_t^\nu f\|_\infty + \|f\|_\infty) dt \le \frac{c_1}{\Gamma(\alpha)} \int_0^1 t^{\alpha+\beta-1} dt \\
&+ \frac{1}{\Gamma(\alpha)} \left(c_2 \|f\|_{p,\nu} \int_1^\infty t^{\alpha - \frac{n+2\nu}{p} - 1} dt + \|f\|_\infty \int_1^\infty t^{\alpha-1} e^{-t} dt \right) \\
&= O(1)\alpha \quad \text{as} \quad \alpha \to 0^+.
\end{aligned}$$

Finally, we get the required identity:

$$\|\mathcal{F}_\nu^\alpha f - f\|_\infty = O(1)\alpha \quad \text{as} \quad \alpha \to 0^+.$$

(A4) Let $f \in L_{p,\nu} \cap \Lambda^{\nu}_{\beta}(x_0)$ where $x_0 \in \mathbf{R}^n_+$. Then similar to (22) we have

$$
\begin{aligned}
(\mathcal{F}^{\alpha}_{\nu} f)(x_0) - f(x_0) &= \frac{1}{\Gamma(\alpha)} \int_0^1 t^{\alpha-1} e^{-t} ((\mathcal{P}^{\nu}_t f)(x_0) - f(x_0)) dt \\
&+ \frac{1}{\Gamma(\alpha)} \int_1^{\infty} t^{\alpha-1} e^{-t} \Big((\mathcal{P}^{\nu}_t f)(x_0) - f(x_0) \Big) dt \\
&\leq \frac{1}{\Gamma(\alpha)} \int_0^1 t^{\alpha-1} e^{-t} A_t dt + \frac{1}{\Gamma(\alpha)} \int_1^{\infty} t^{\alpha-1} e^{-t} \left| (\mathcal{P}^{\nu}_t f)(x_0) - f(x_0) \right| dt \\
&\equiv j_1 + j_2 \qquad (23)
\end{aligned}
$$

where $A_t = |(\mathcal{P}^{\nu}_t f)(x_0) - f(x_0)|$. Now we estimate A_t .

$$
\begin{aligned}
A_t &= \left| \int_{\mathbf{R}^n_+} P^{\nu}(y,t) \Big((T^y f)(x_0) - f(x_0) \Big) y_n^{2\nu} dy \right| \\
&\leq \int_{|y|\leq 1} P^{\nu}(y,t) |(T^y f)(x_0) - f(x_0)| y_n^{2\nu} dy \\
&+ \int_{|y|>1} P^{\nu}(y,\, t) |(T^y f)(x_0)| y_n^{2\nu} dy \\
&+ |f(x_0)| \int_{|y|>1} P^{\nu}(y,t) y_n^{2\nu} dy = i_1 + i_2 + i_3.
\end{aligned}
$$

Since x_0 is a Lipschitz point we can write

$$
i_1 \overset{(12)}{\leq} c \int_{|y|\leq 1} \frac{t}{(|y|^2 + t^2)^{(n+2\nu+1)/2}} |y|^{\beta} y_n^{2\nu} dy \leq c_1 t^{\beta}. \qquad (24)
$$

Let us estimate i_2. By making use of Hölder inequality we have

$$
\begin{aligned}
i_2 &\leq \|f\|_{p,\nu} \left(\int_{|y|>1} (P^{\nu}(y,\, t))^{p'} y_n^{2\nu} dy \right)^{1/p'} \\
&\leq c_2 t \left(\int_{|y|>1} |y|^{-p'(n+2\nu+1)} y_n^{2\nu} dy \right)^{1/p'} \leq c_3 t. \qquad (25)
\end{aligned}
$$

Similarly, we have the following estimate about i_3 :

$$i_3 \leq c_4 t. \tag{26}$$

Hence, from (24)-(26) we write

$$A_t = O(1)t^{\beta} \quad \text{as} \quad \alpha \to 0^+. \tag{27}$$

Now using the equality (27) we have the following inequality about the estimation of j_1:

$$j_1 \leq \frac{c_5}{\Gamma(\alpha)} \int_0^1 t^{\alpha+\beta-1} dt = O(1)\alpha \quad \text{as } \alpha \to 0^+. \tag{28}$$

Further, using (14) we get

$$j_2 \leq O(1)\alpha \quad \text{as } \alpha \to 0^+. \tag{29}$$

Finally, from the latest equations (28) and (29) we have

$$(\mathcal{F}_{\nu}^{\alpha} f)(x_0) - f(x_0) = O(1)\alpha \quad \text{as } \alpha \to 0^+.$$

This completes the proof of the Theorem A.

The following theorem is the analogue of Theorem A for the classical Flett potentials $\mathcal{F}^{\alpha} f$ and it can be proved using the same idea as in the proof of the Theorem A .

Theorem B. *Let $f \in L_p$, $1 \leq p < \infty$, and $\mathcal{F}^{\alpha} f$ are defined as in (18). Then*

(B1) $\lim\limits_{\alpha \to 0^+} (\mathcal{F}^{\alpha} f)(x) = f(x)$ *for almost all* $x \in \mathbf{R}^n$;

(B2) *If* $f \in L_p \cap C_0$, *then the convergence is uniform on* $\mathbf{R}^n$.

(B3) *If* $f \in L_p \cap \Lambda_{\beta}$, $0 < \beta \leq 1$ *then*

$$\|\mathcal{F}^{\alpha} f - f\|_{\infty} = \mathrm{O}(1)\alpha \quad \text{as} \quad \alpha \to 0^+.$$

(B4) *If* $f \in L_p \cap \Lambda_{\beta}(x_0)$, $x_0 \in \mathbf{R}^n$, $0 < \beta \leq 1$ *then*

$$(\mathcal{F}^{\alpha} f)(x_0) - f(x_0) = \mathrm{O}(1)\alpha \qquad \text{as} \quad \alpha \to 0^+.$$

Acknowledgment. The research was supported by the Scientific Research Project Administration Unit of the Akdeniz University(Turkey) and TUBITAK.

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Received: March, 2009