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A Novel Explicit Formula for Computing $\pi(x)$ the Number of Primes $\leq x$

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Abstract. The author gives an explicit formula for the function $\prod(x)$ at the both cases $x \neq P^n$ and $x = P^n$ where "P" denotes a prime and "n" a natural number. This function is an essential function for the computation of $\pi(x)$ (the number of the primes less than or equal to "x"). A novel explicit formulation for $\pi(x)$ is derived from the function $\prod(x)$. To obtain the function $\prod(x)$, the author uses the Riemann Zeta Function and investigates its poles then applies the complex functions and analytic number theory for the determination of the explicit formula.

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1. Introduction

In this paper, the author gives an explicit formula for an important function in mathematics where is called $\Pi(n)$, then using this function he gives an explicit formula for the function $\pi(x)$, the number of the primes less than or equal to "x".

Investigation about the distribution of primes in a certain interval was first begun by Euler, Gauss, and Dirichlet. The Prime Number Theorem was given by Bernhard Riemann [1] representing the Riemann Zeta Function in 1859. This theorem says that $\pi(n)$ can be approximated by $\frac{n}{\log n}$ i.e. $\pi(n) \sim \frac{n}{\log n}$ or $\Pi(n) \sim \frac{n}{\log n}$ for sufficiently large integers. It was first proved by Hadamard [2] and de la Valleé Poussin [3] in 1896 independently. Chebyshev [4] proved that if the limit in the Prime Number Theorem exists, its value must be one. Erdös [5] and Selberg [6] gave "elementary" proofs for it that bypassed the Zeta Function. From those times to date, much more methods have been given for approaching to

a proof of the Prime Number Theorem and its conclusions. V. Ramaswami represented in a paper [7], the method on the number of positive integers less than "x" and free of prime divisors greater than x^c where c is every real fixed positive, in 1948. Lawrence Washington [8] discussed on Class numbers of Elliptic Function Fields and their relations to the distribution of prime numbers between n^2 and $(n+1)^2$ for all positive integers "n".

Ramanujan [9] claimed that $\pi(x)=\int_0^{\infty}\frac{(logx)^u du}{u\zeta(u+1)\Gamma(u+1)}+O(1)$ but Hardy [9] showed that it has a high accuracy over range $1\le x\le 10^9$ and it is not true for the all real numbers.

2. A Novel Formula for the Function $\prod(x)$ by the Riemann Zeta Function

Lemma 2.1 *Prove the following Explicit Formula for* $\prod(x)$

If
$$Li(x)=\frac{x}{\log(x)}+\int_2^x\frac{du}{(\log u)^2}+O(1) \tag{2-1}$$

Then:
$$\prod(x)=Li(x)-\sum_{\gamma>0}\left\{Li(x^{\rho})+Li(x^{1-\rho})\right\}+\int_x^{\infty}\frac{dt}{t(t^2-1)\log t}-\log 2 \tag{2-2}$$

***Proof*:**

The proof is comprised of the two sections, the first for $x\neq P^n$ and the second for $x=P^n$ where "P" denotes a prime, "n" a natural number, $\rho=\beta+i\gamma$ a pole of the Riemann Zeta function, and $x\ge 2$.

The first section of the proof for the Case $x\neq P^n$

Based on the definition of $\prod(x)$ could be written:

$$\prod(x)=\sum_{n\le x}\frac{\Lambda(n)}{\log n} \tag{2-3}$$

The following theorems are used for the proof of the Lemma 2.1.

Theorem 2.2 *The Abel Summation Formula*.(For the proof, see [10]).

Theorem 2.3 *Prove* $\prod(x)=\frac{\psi(x)}{\log(x)}+\int_2^x(\frac{\psi(u)}{u(\log u)^2})du$ *that* $\psi(x)=\sum_{n\le x}\Lambda(n)$
(For the proof, see [10]).

Theorem 2.4 *1) If* $R(s)>0$ *and* $E(x)=\begin{cases}0 & if \quad x\le 1\\ x-1 & if \quad x>1\end{cases}$ *then we have* $\frac{1}{s(s+1)}=\int_0^{\infty}\frac{E(x)}{x^{s+2}}$ *2) If* $c>0$ *be any fixed real number, then for all* $x>0$ *we will have* $E(x)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{x^{s+1}}{s(s+1)}ds$ (For the proof, see [10]). $R(s)$ is the real part of "s"

Using the above theorems, the contour C_n is defined according to the Fig.1 for $n\ge 3$. Suppose $N(T)$ be the number of zeros of the type $\rho=\beta+i\gamma$ from the function $\xi(s)$ and $0<\gamma<T$, $0<R(s)<1$, thus the zeros of the function $\xi(s)$ is equal to the zeros of $\zeta(s)$ the Riemann Zeta Function together with the simple zeros at the points $s=-2r$, $r=1,2,...$, then assume $\{T_n\}$ be an infinite series of the real numbers where $n<T_n<n+1$ so $s=\sigma+iT_n$ and $-n\le\sigma\le 2$ then

$$I_n=\frac{1}{2\pi i}\int_{C_n}\frac{Z(n)}{s(s+1)}x^{s+1}ds \tag{2-4}$$

$$I_n=J_n+k_n \tag{2-5}$$

J_n denotes integral along the line connected between $2-iT_n$ to $2+iT_n$

k_n denotes integral along the other three sides of the rectangle. The poles of the function $\frac{Z(s)}{s(s+1)}$ should be obtained.

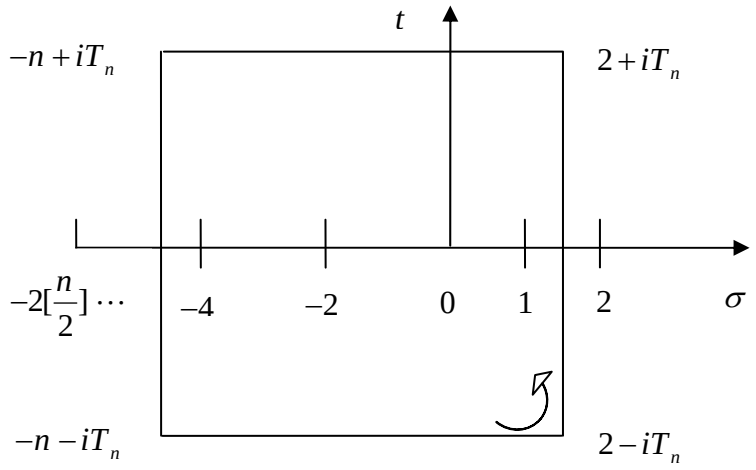

Fig.1 Representation of the poles of the function $\frac{Z(s)}{s(s+1)}$

To obtain the function $Z(s)$ the functions $\zeta(s)$ and $\xi(s)$ should be used [10].

$$Z(s)=-\frac{\zeta'(s)}{\zeta(s)} \tag{2-6}$$

$$\zeta(s)=\frac{\xi(s)\pi^{s/2}}{(s-1)\Gamma(1+\frac{s}{2})} \tag{2-7}$$

where $\Gamma(s)$ is the gamma function.
If $s=\sigma+it$ and $\sigma>0$ then [10]

$$\Gamma(s)=\int_0^{\infty}e^{-u}u^{s-1}du \tag{2-8}$$

And for all values s could be written [10]:

$$\frac{1}{s\Gamma(s)}=e^{\gamma s}\prod_{n=1}^{\infty}(1+\frac{s}{n})e^{-s/n} \tag{2-9}$$

Where γ is the Euler's constant.

$$\Gamma(s+1)=s\Gamma(s) \tag{2-10}$$

$$\Gamma(\frac{s}{2}+1)=\frac{s}{2}\Gamma(\frac{s}{2}) \tag{2-11}$$

$$\Gamma(\frac{s}{2}+1)=\frac{\frac{s}{2}}{(\frac{s}{2})e^{\gamma s/2}\prod_{n=1}^{\infty}(1+\frac{s}{2n})e^{-s/2n}}=\frac{1}{e^{\gamma s/2}\prod_{n=1}^{\infty}(1+\frac{s}{2n})e^{-s/2n}} \tag{2-12}$$

$$\zeta(s)=\frac{\pi^{\frac{s}{2}}\xi(s)e^{\frac{\gamma s}{2}}\prod_{n=1}^{\infty}(1+\frac{s}{2n})e^{\frac{-s}{2n}}}{(s-1)} \tag{2-13}$$

(2-14)

$$\zeta'(s)=\frac{\left\{\begin{array}{l}\frac{1}{2}(\log\pi)\pi^{\frac{s}{2}}\xi(s)e^{\frac{\gamma s}{2}}\prod_{n=1}^{\infty}(1+\frac{s}{2n})e^{\frac{-s}{2n}}+\xi'(s)\pi^{\frac{s}{2}}e^{\frac{\gamma s}{2}}\prod_{n=1}^{\infty}(1+\frac{s}{2n})e^{\frac{-s}{2n}}+\frac{\gamma}{2}e^{\frac{\gamma s}{2}}\pi^{\frac{s}{2}}\xi(s)\prod_{n=1}^{\infty}(1+\frac{s}{2n})e^{\frac{-s}{2n}}+\\ +e^{\frac{\gamma s}{2}}\pi^{\frac{s}{2}}\xi(s)\left(\begin{array}{l}\frac{1}{2}e^{\frac{-s}{2}}\prod_{n=2}^{\infty}(1+\frac{s}{2n})e^{\frac{-s}{2n}}+\frac{1}{4}e^{\frac{-s}{4}}\prod_{n=1,3,4,5,\ldots}^{\infty}(1+\frac{s}{2n})e^{\frac{-s}{2n}}+\ldots+\\ +\frac{1}{2n}e^{\frac{-s}{2n}}\prod_{j=1,\ldots,n-1}^{\infty}(1+\frac{s}{2j})e^{\frac{-s}{2j}}-\sum_{j=1}^{\infty}\frac{1}{2j}\prod_{n=1}^{\infty}(1+\frac{s}{2n})e^{\frac{-s}{2n}}\end{array}\right)\end{array}\right\}(s-1)-}{(s-1)^2}$$

$$-e^{\frac{\gamma s}{2}}\pi^{\frac{s}{2}}\xi(s)\prod_{n=1}^{\infty}(1+\frac{s}{2n})e^{\frac{-s}{2n}}$$

$$-\frac{\zeta'(s)}{\zeta(s)}=-\lim_{n\to\infty}\left[\begin{array}{l}\frac{\log\pi}{2}+\frac{\xi'(s)}{\xi(s)}+\frac{\gamma}{2}+\frac{1}{2(1+s/2)}+...+\\ \frac{1}{2n(1+s/2n)}-(\frac{1}{2}+\frac{1}{4}+...+\frac{1}{2n})-\frac{1}{(s-1)}\end{array}\right]= \tag{2-15}$$

$$=-\left[\begin{array}{l}\frac{\log\pi}{2}+\frac{\xi'(s)}{\xi(s)}+\frac{\gamma}{2}+\sum_{j=1}^{\infty}\frac{1}{(s+2j)}\\ -\sum_{j=1}^{\infty}\frac{1}{2j}-\frac{1}{(s-1)}\end{array}\right]$$

If assume

$$\xi(s)=(s-\rho_1)(s-\rho_2)...(s-\rho_\gamma) \tag{2-16}$$

Then

$$Z(s)=-\frac{\zeta'(s)}{\zeta(s)}=-[\frac{\log\pi}{2}+\frac{\gamma}{2}+\sum_{j=1}^{\gamma}\frac{1}{(s-\rho_j)}+\sum_{j=1}^{\infty}\frac{1}{(s+2j)} -\sum_{j=1}^{\infty}\frac{1}{2j}-\frac{1}{(s-1)}] \tag{2-17}$$

And the poles of $Z(s)$ are:

$$s=1,\ \rho_1,...,\ \rho_\gamma,-2r\ (r=1,2,...,\infty) \tag{2-18}$$

The poles of $\frac{Z(s)x^{s+1}}{s(s+1)}$ are:

$$s=0,1,-1,\ \rho_1,...,\ \rho_\gamma,\ -2r\ (r=1,2,...,\infty) \tag{2-19}$$

Because $\rho=\beta+\gamma i$ and the poles $\rho_1,...,\ \rho_\gamma$ are defined so that $|\gamma|<T_n$ and $2r<n$. Cauchy's residue theorem now allows us to conclude

$$I_n=Z(0)x+\frac{x^2}{2}-z(-1)-\sum_{2r<n}\frac{x^{1-2r}}{(2r)(1-2r)}-\sum_{|\gamma|<T_n}\frac{x^{\rho+1}}{\rho(\rho+1)}+k_n \tag{2-20}$$

At the relation (2-20), one should show that for $x>1\ \wedge\ n\to\infty$ the value $k_n\to 0$. In the last sum, if any of the zeros ρ are multiple, then the corresponding term occurs with the multiplicity of ρ. In this paper, we forbear from these types of terms.

Theorem2.5*Suppose* $R(\delta)=\{s\mid\ R(s)\le-\gamma,\ |s+2r|\ge\gamma\ ,r=1,2,...\}$ *and* $s\in R(\delta)$ *P rove* $|Z(s)|<A_2(\delta)(\log|s|+1)$ *that* $A_2(\delta)$ *is an explicitly computable function of* δ (For the proof, referring to the [10]) Using the above theorem considering $n<T_n<n+1$, $s=\sigma+iT_n$, $-1\le\sigma\le 2$ will be concluded $Z(s_n)=O\{\log^2 n\}$ then $T\to\infty$

with $\sigma=\frac{1}{2}$ $|Z(s)|<A_1\log^2 n,|s|\ge n$, and $|s+1|\ge n\ \Rightarrow\ |x^{s+1}|\le x^3$

Therefore could be written:

$$|k_n| \le \frac{1}{2\pi}\int \frac{|Z(s)|}{|s(s+1)|}.|x^{s+1}|.|ds| = O\left\{\frac{\log^2 n}{n^2}(2n+4+2T_n)x^3\right\} \qquad (2\text{-}21)$$

Thus, for $x>1 \ \wedge \ n\to\infty$, k_n will be close to zero and the proof is completed.

Theorem 2.6 *As T tends to infinity show that $N(T+1)-N(T)=O(\log T)$, $\sum_{0<\gamma\le T}\frac{1}{\gamma}=O(\log^2 T)$,and $\sum_{\gamma>T}\frac{1}{\gamma^2}=O(\frac{\log T}{T})$ particular, the series $\sum\gamma^{-2}$ is convergent.* (For the proof, referring to the [10]).The absolute values of the general terms of two series $\sum_{r=1}^{\infty}\frac{x^{1-2r}}{2r(2r-1)}$ and $\sum_{\rho}\frac{x^{1+\rho}}{\rho(1+\rho)}$ on any fixed interval $[1,X]$ are less than r^{-2} and $X^2|\gamma|^{-2}$ respectively.

Hence both series should be absolutely, uniformly convergent by appeal to theorem 2.6. The uniform convergence of the series $\sum_{\rho}\frac{x^{1+\rho}}{\rho(1+\rho)}$ and the continuity of g'(g is any function with a continuous derivative on the interval $[a,b]$) on the interval $[a,b]$ is obvious, therefore we know because

$$\psi_1 = Lim_{n\to\infty}\left\{Z(0)x+\frac{x^2}{2}-z(-1)-\sum_{|\gamma|<T_n}\frac{x^{\rho+1}}{\rho(\rho+1)}-\sum_{2r<n}\frac{x^{1-2r}}{(2r)(1-2r)}\right\} \qquad (2\text{-}22)$$

$$\sum_{\rho}\int_a^b g'(x)\frac{x^{\rho+1}}{\rho(\rho+1)}dx \qquad (2\text{-}23)$$

The relation (2-23) will be convergent.
Therefore

$$-\sum_{\rho}\int_a^b g'(x)\frac{x^{\rho+1}}{\rho(\rho+1)}dx = \int_a^b g'(x)\left\{\psi_1(x)-\frac{x^2}{2}-xZ(0)+Z(-1)+\sum_{r=1}^{\infty}\frac{x^{1-2r}}{2r(2r-1)}dx\right\} \qquad (2\text{-}24)$$

$$=\int_a^b g'(x)\,f_1(x)dx$$

Then

$$f_1(x)=\psi_1(x)-\frac{x^2}{2}-xZ(0)+Z(-1)+\sum_{r=1}^{\infty}\frac{x^{1-2r}}{2r(2r-1)}dx \qquad (2\text{-}25)$$

and for $x\neq P^n$

$$f(x)=f_1'(x) \qquad (2\text{-}26)$$

Because $\psi(x)$ is discontinuous at $x=P^n$
Then

$$\psi_0(x)=\frac{1}{2}\{\psi(x+0)+\psi(x-0)\} \tag{2-27}$$

If $x\neq P^n$ $$\psi_0(x)=\psi(x) \tag{2-28}$$

If $x=P^n$ $$\psi_0(x)=\psi(x)-\frac{1}{2}\Lambda(x) \tag{2-29}$$

If $x>1$

$$\psi_0(x)=x+Z(0)-\frac{1}{2}\log(1-x^{-2})-\lim_{\substack{\sum_{|\gamma|<T}\\ T\to\infty}}\frac{x^{\rho}}{\rho} \tag{2-30}$$

$$\prod(x)=\frac{\psi(x)}{\log(x)}+\int_2^x(\frac{\psi(u)}{u(\log u)^2})du \tag{2-31}$$

Substituting the relation (2-30) into the relation (2-31) for $x\neq P^n$ could be written:

$$\prod(x)=\frac{x+Z(0)-\frac{1}{2}\log(1-x^{-2})-\lim_{\substack{\sum_{|\gamma|<T}\\ T\to\infty}}\frac{x^{\rho}}{\rho}}{\log x}+\int_2^x\frac{(u+Z(0)-\frac{1}{2}\log(1-u^{-2})-\lim_{\substack{\sum_{|\gamma|<T}\\ T\to\infty}}\frac{u^{\rho}}{\rho})du}{u(\log u)^2}=$$

$$=\frac{x}{\log x}+\frac{Z(0)}{\log x}-\frac{\log(1-x^{-2})}{2\log x}-\lim_{\substack{\sum_{|\gamma|<T}\\ T\to\infty}}\frac{\frac{x^{\rho}}{\rho}}{\log x}+\int_2^x\frac{du}{(\log u)^2}+Z(0)\int_2^x\frac{du}{u(\log u)^2}-$$

$$\frac{1}{2}\int_2^x\frac{\log(1-u^{-2})du}{u(\log u)^2}-\lim_{\substack{\sum_{|\gamma|<T}\\ T\to\infty}}\int_2^x\frac{u^{\rho}du}{\rho u(\log u)^2}$$

because $$\int_2^x\frac{du}{u(\log u)^2}=\frac{1}{\log 2}-\frac{1}{\log x} \tag{2-32}$$

$$Li(x)=\frac{x}{\log x}+\int_2^x\frac{du}{(\log u)^2}+O(1) \tag{2-33}$$

$$\frac{\log(1-x^{-2})}{2\log x}+\frac{1}{2}\int_2^x\frac{\log(1-u^{-2})du}{u(\log u)^2}=\frac{\log(1-x^{-2})}{2\log x}+\frac{1}{2}\int_2^x\log(1-u^{-2})d(\frac{-1}{\log u}) \tag{2-34}$$

By integration by parts, from the relation (2-34), the relation (2-35) will be concluded:

(2-35)

$$=\frac{\log(1-x^{-2})}{2\log x}+\frac{1}{2}[\frac{-\log(1-u^{-2})}{\log u}]_2^x-\frac{1}{2}\int_2^x\frac{-2du}{u(\log u)(u^2-1)}=\frac{\log(1-2^{-2})}{2\log 2}+\int_2^x\frac{du}{u(u^2-1)(\log u)}$$

Therefore, the relation (2-31) could be written as below:

$$\left\{\frac{x}{\log x}+\int_2^x\frac{du}{(\log u)^2}\right\}+\left\{\frac{Z(0)}{\log x}+Z(0)\int_2^x\frac{du}{u(\log u)^2}\right\}-$$
$$\left\{\frac{\log(1-x^{-2})}{2\log x}+\frac{1}{2}\int_2^x\frac{\log(1-u^{-2})du}{u(\log u)^2}\right\}-\lim_{\substack{|\gamma|<T\\T\to\infty}}\sum\left\{\frac{x^{\rho}}{\rho\log x}+\int_2^x\frac{u^{\rho}du}{\rho u(\log u)^2}\right\} \quad (2\text{-}36)$$

Also from the relation (2-36) will be concluded:

$$\lim_{\substack{|\gamma|<T\\T\to\infty}}\sum\left\{\frac{x^{\rho}}{\rho\log x}+\int_2^x\frac{u^{\rho}du}{\rho u(\log u)^2}\right\}=\lim_{\substack{0<\gamma<T\\T\to\infty}}\sum\left\{\frac{x^{\rho}}{\rho\log x}+\int_2^x\frac{u^{\rho}du}{\rho u(\log u)^2}\right\}+$$
$$\lim_{\substack{-T<\gamma<0\\T\to\infty}}\sum\left\{\frac{x^{\rho}}{\rho\log x}+\int_2^x\frac{u^{\rho}du}{\rho u(\log u)^2}\right\} \quad (2\text{-}37)$$

Because the function $\xi(s)$ is symmetric about the two lines $R(s)=\frac{1}{2}$ and $\mathrm{Im}(s)=0$ hence:

$$\rho=\beta+i\gamma\ ,\xi(\beta-i\gamma)=\xi(1-\beta+i\gamma)\ ,\xi(\beta+i\gamma)=\xi(1-\beta-i\gamma) \quad (2\text{-}38)$$

Therefore: The first term of the right hand side of the relation (2-37) is:
Making the substitution $u^{\rho}=w$ in the integral

(2-39)

$$\lim_{\substack{0<\gamma<T\\T\to\infty}}\sum\left\{\frac{x^{\rho}}{\rho\log x}+\int_2^x\frac{u^{\rho}du}{\rho u(\log u)^2}\right\}=\lim_{\substack{0<\gamma<T\\T\to\infty}}\sum\left\{\frac{x^{\rho}}{\rho\log x}+\frac{1}{\rho}\int_{2^{\rho}}^{x^{\rho}}\frac{u^{\rho-1}dw}{\rho u^{\rho-1}(\frac{1}{\rho})^2(\log w)^2}\right\}$$
$$=\lim_{\substack{0<\gamma<T\\T\to\infty}}\sum\left\{\frac{x^{\rho}}{\rho\log x}+\int_{2^{\rho}}^{2}\frac{dw}{(\log w)^2}+\int_2^{x^{\rho}}\frac{dw}{(\log w)^2}\right\}=\lim_{\substack{0<\gamma<T\\T\to\infty}}\sum\left\{Li(x^{\rho})+\int_{2^{\rho}}^{2}\frac{dw}{(\log w)^2}-O(1)\right\}$$

The second term of the right hand side of the relation (2-37) is similar to the first term supposing $\beta=\frac{1}{2}$:

$$\lim_{\substack{-T<\gamma<0\\T\to\infty}}\sum\left\{\frac{x^{\rho}}{\rho\log x}+\int_2^x\frac{u^{\rho}du}{\rho u(\log u)^2}\right\}=\lim_{\substack{0<\gamma<T\\T\to\infty}}\sum\left\{Li(x^{1-\rho})+\int_{2^{1-\rho}}^{2}\frac{dw}{(\log w)^2}-O(1)\right\} \quad (2\text{-}40)$$

$$Z(0)=-(\frac{\log\pi}{2}+\frac{\xi'(0)}{\xi(0)}+\frac{\gamma}{2}+1)=-\frac{1}{2} \quad (2\text{-}41)$$

Hence, using the relation (2-36), the above expressions, and the relation (2-41), the equation (2-42) could be obtained: (γ is Euler's Constant))

$$\prod(x)=Li(x)-O(1)+\frac{Z(0)}{\log 2}-\frac{\log(\frac{3}{4})}{2\log 2}-\int_2^x\frac{du}{u(u^2-1)\log u}-$$
$$\sum_{\gamma>0}\left\{Li(x^{\rho})+Li(x^{1-\rho})\right\}-\lim_{\substack{0<\gamma<T\\T\to\infty}}\sum\left\{\int_{2^{\rho}}^{2}\frac{dw}{(\log w)^2}-O(1)\right\}- \qquad (2\text{-}42)$$
$$\lim_{\substack{0<\gamma<T\\T\to\infty}}\sum\left\{\int_{2^{1-\rho}}^{2}\frac{dw}{(\log w)^2}-O(1)\right\}$$

The proof is finally completed:

$$\prod(x)=Li(x)-\sum_{\gamma>0}\left\{Li(x^{\rho})+Li(x^{1-\rho})\right\}+\int_x^{\infty}\frac{du}{u(u^2-1)\log u}-\log 2 \qquad (2\text{-}43)$$

The second section of the proof for the Case $x=P^n$

From the relations (2-27) and (2-29) can be written:

$$\begin{aligned}\psi(x+0)&=\psi_0(x)+\frac{1}{2}\Lambda(x)\\ \psi(x-0)&=\psi_0(x)-\frac{1}{2}\Lambda(x)\end{aligned} \qquad (2\text{-}44)$$

$$\begin{aligned}\prod(x+0)&=\frac{\psi(x+0)}{\log(x)}+\int_2^{x+0}(\frac{\psi(u)}{u(\log u)^2})du=\\ &=\frac{\psi_0(x)+\frac{1}{2}\Lambda(x)}{\log x}+\int_2^x(\frac{\psi(u+0)}{u(\log u)^2})du= \qquad (2\text{-}45)\\ &=\frac{\psi_0(x)+\frac{1}{2}\Lambda(x)}{\log x}+\int_2^x(\frac{\psi_0(u)+\frac{1}{2}\Lambda(u)}{u(\log u)^2})du\end{aligned}$$

Also:

$$\begin{aligned}\prod(x-0)&=\frac{\psi(x-0)}{\log(x)}+\int_2^{x-0}(\frac{\psi(u)}{u(\log u)^2})du=\\ &=\frac{\psi_0(x)-\frac{1}{2}\Lambda(x)}{\log x}+\int_2^x(\frac{\psi_0(u)-\frac{1}{2}\Lambda(u)}{u(\log u)^2})du\end{aligned} \qquad (2\text{-}46)$$

$$\prod_0(x)=\prod(x)=\frac{1}{2}[\ \prod(x+0)+\prod(x-0)\ ]= \qquad (2\text{-}47)$$

$$= \frac{1}{2}\left\{\frac{2\psi_0(x)}{\log x} + 2\int_2^x \frac{\psi_0(u)du}{u(\log u)^2}\right\} =$$

$$= \frac{\psi_0(x)}{\log x} + \int_2^x \frac{\psi_0(u)du}{u(\log u)^2}$$

3. A Novel Formula for the Function $\pi(x)$ by the Function $\prod(x)$

Lemma 3.1 *Prove*

$$\pi(x) = \sum_{r=1}^{\infty} \frac{\mu(r)}{r} \prod(x^{\frac{1}{r}}) \tag{3-1}$$

Proof.

$$\frac{\mu(1)}{1}\prod(x) = \prod(x) = \pi(x) + \frac{1}{2}\pi(x^{\frac{1}{2}}) + \frac{1}{3}\pi(x^{\frac{1}{3}}) + \frac{1}{4}\pi(x^{\frac{1}{4}}) + \ldots$$

$$\frac{\mu(2)}{2}\prod(x^{\frac{1}{2}}) = -\frac{1}{2}\prod(x^{\frac{1}{2}}) = -\frac{1}{2}\pi(x^{\frac{1}{2}}) - \frac{1}{4}\pi(x^{\frac{1}{4}}) - \frac{1}{6}\pi(x^{\frac{1}{6}}) + \ldots$$

$$\frac{\mu(3)}{3}\prod(x^{\frac{1}{3}}) = -\frac{1}{3}\prod(x^{\frac{1}{3}}) = -\frac{1}{3}\pi(x^{\frac{1}{3}}) - \frac{1}{6}\pi(x^{\frac{1}{6}}) - \frac{1}{9}\pi(x^{\frac{1}{9}}) + \ldots$$

$$\frac{\mu(4)}{4}\prod(x^{\frac{1}{4}}) = 0 \times \frac{1}{4}\prod(x^{\frac{1}{4}}) = 0$$

$$\frac{\mu(5)}{5}\prod(x^{\frac{1}{5}}) = -\frac{1}{5}\prod(x^{\frac{1}{5}}) = -\frac{1}{5}\pi(x^{\frac{1}{5}}) - \frac{1}{10}\pi(x^{\frac{1}{10}}) - \frac{1}{15}\pi(x^{\frac{1}{15}}) + \ldots$$

$$\frac{\mu(6)}{6}\prod(x^{\frac{1}{6}}) = +\frac{1}{6}\prod(x^{\frac{1}{6}}) = +\frac{1}{6}\pi(x^{\frac{1}{6}}) + \frac{1}{12}\pi(x^{\frac{1}{12}}) + \frac{1}{18}\pi(x^{\frac{1}{18}}) + \ldots$$

$$\ldots \quad = \quad \ldots \quad = \quad \ldots \qquad \ldots \qquad \ldots \qquad + \ldots$$

By summing separately right and left hand sides of the equations, the proof is completed. Therefore our novel explicit formula is:

(3-2)

$$\pi(x) = \sum_{r=1}^{\infty} \frac{\mu(r)}{r}\left\{Li(x^{\frac{1}{r}}) - \sum_{\gamma>0}\left\{Li(x^{\frac{\rho}{r}}) + Li(x^{\frac{1-\rho}{r}})\right\} + \int_{x^{\frac{1}{r}}}^{\infty} \frac{du}{u(u^2-1)\log u} - \log 2\right\}$$

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