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The Symmetric Positive Solutions of Four-Point Problems for Nonlinear Boundary Value Second-Order Differential Equations

Haidong Qu

Department of Mathematics, Hanshan Normal University
Chaozhou, Guangdong, 521041, P. R. China
qhaidong@163.com

Abstract

In this paper, we are concerned with the existence of symmetric positive solutions for second-order differential equations. Under the suitable conditions, the existence and symmetric positive solutions are established by using Krasnoselskii's fixed-point theorems.

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1 Introduction

Recently, there are many results about the existence and multiplicity of positive solutions for nonlinear second-order differential equations(see[7],[5],[3]). Henderson and Thompson(see[4]), Li and Zhang (see[2]) studied the multiple symmetric positive and nonnegative solutions of second-order ordinary differential equations. Yao (see[6]) considered the existence and iteration of n symmetric positive solutions for a singular two-point boundary value problem(BVP). Sun(see[8]) considered the existence and multiplicity of symmetric positive solutions for three-point boundary value problem. Inspired by the works mentioned above, in this paper, we study the existence of symmetric positive solutions of second-order four-point differential equations as follows,

$$\begin{cases} -u''(t) = f(t,v), \\ -v''(t) = g(t,u), 0 \le t \le 1, \end{cases} \tag{1}$$

subject to the boundary conditions

$$\begin{cases} u(t) = u(1-t), u'(0) - u'(1) = u(\xi_1) + u(\xi_2), \\ v(t) = v(1-t), v'(0) - v'(1) = v(\xi_1) + v(\xi_2), 0 < \xi_1 < \xi_2 < 1, \end{cases} \tag{2}$$

where $f, g : [0,1] \times R^+ \to R^+$ are continuous, both $f(\cdot, u)$ and $g(\cdot, u)$ are symmetric on $[0,1]$,$f(x,0) \equiv g(x,0) \equiv 0$. To the best of author's knowledge, there is no such result involving this problem. In this paper, we intend to fill in such gaps in the literature. The arguments for establishing the symmetric positive solutions of (1) and (2) involve the properties of the functions in Lemma 2.1 that play a key role in defining some cones. A fixed point theorem due to Krasnoselskii is applied to yield the existence of symmetric positive solutions of (1) and (2).

2 Preliminary Notes

In this section, we present some necessary definitions and preliminary lemmas that will be used in the proof of the results.

Definition 2.1. Let E be a real Bananch space. A nonempty closed set $P \subset E$ is called a cone of E if it satisfies the following conditions:
(I) $x \in P, \lambda > 0$ implies $\lambda x \in P$;
(II) $x \in P, -x \in P$ implies $x = 0$.

Definition 2.2. The function u is called to be concave on $[0,1]$ if $u(rt_1 + (1-r)t_2) \geq ru(t_1) + (1-r)u(t_2), r, t_1, t_2 \in [0,1]$.

Definition 2.3. The function u is symmetric on $[0,1]$ if $u(t) = u(1-t), t \in [0,1]$.

Definition 2.4. The function (u, v) is called a symmetric positive solution of the equation (1) if u and v are symmetric and positive on $[0,1]$, and satisfy the equation (2).

We shall consider the real Banach space $C[0,1]$, equipped with norm $\| u \| = \max_{0 \leq t \leq 1} | u(t) |$. Denote $C^+[0,1] = \{u \in C[0,1] : u(t) \geq 0, t \in [0,1]\}$.

Lemma 2.1. Let $y \in C[0,1]$ be symmetric on $[0,1]$, then the four-point BVP

$$\begin{cases} u''(t) + y(t) = 0, 0 < t < 1, \\ u(t) = u(1-t), u'(0) - u'(1) = u(\xi_1) + u(\xi_2), \end{cases} \tag{3}$$

has a unique symmetric solution $u(t) = \int_0^1 G(t,s)y(s)ds$, where $G(t,s) = G_1(t,s) + G_2(s)$, here

$$G_1(t,s) = \begin{cases} t(1-s), 0 \leq t \leq s \leq 1, \\ s(1-t), 0 \leq s \leq t \leq 1, \end{cases}$$

$$G_2(s) = \begin{cases} \frac{1}{2}[(\xi_1 - s) + (\xi_2 - s) - \xi_1(1-s) - \xi_2(1-s) + 1], 0 \leq s \leq \xi_1, \\ \frac{1}{2}[(\xi_2 - s) - \xi_1(1-s) - \xi_2(1-s) + 1], \xi_1 \leq s \leq \xi_2, \\ \frac{1}{2}[-\xi_1(1-s) - \xi_2(1-s) + 1], \xi_2 \leq s \leq 1. \end{cases}$$

Proof.From (3), we have $u''(t) = -y(t)$. For $t \in [0,1]$, integrating from 0 to t we get

$$u'(t) = -\int_0^t y(s)ds + A_1, \tag{4}$$

since $u'(t) = -u'(1-t)$, we obtain that $-\int_0^t y(s)ds + A_1 = \int_0^{1-t} y(s)ds - A_1$, which leads to

$$\begin{aligned} A_1 &= \frac{1}{2}\int_0^t y(s)ds + \frac{1}{2}\int_0^{1-t} y(s)ds \\ &= \frac{1}{2}\int_0^t y(s)ds - \frac{1}{2}\int_0^{1-t} y(1-s)d(1-s) \\ &= \frac{1}{2}\int_0^t y(s)ds + \frac{1}{2}\int_t^1 y(s)ds \\ &= \frac{1}{2}\int_0^1 y(s)ds \\ &= \int_0^1 (1-s)y(s)ds. \end{aligned}$$

Integrating again we obtain

$$u(t) = -\int_0^t (t-s)y(s)ds + t\int_0^1 (1-s)y(s)ds + A_2.$$

From (3) and (4) we have

$$\begin{aligned} \int_0^1 y(s)ds = &-\int_0^{\xi_1} (\xi_1 - s)y(s)ds + \xi_1\int_0^1 (1-s)y(s)ds + A_2 \\ &-\int_0^{\xi_2} (\xi_2 - s)y(s)ds + \xi_2\int_0^1 (1-s)y(s)ds + A_2. \end{aligned}$$

Thus

$$\begin{aligned} A_2 = \frac{1}{2}\int_0^{\xi_1} &[(\xi_1 - s) + (\xi_2 - s) - \xi_1(1-s) - \xi_2(1-s) + 1]y(s)ds \\ &+\frac{1}{2}\int_{\xi_1}^{\xi_2} [(\xi_2 - s) - \xi_1(1-s) - \xi_2(1-s) + 1]y(s)ds \\ &+\frac{1}{2}\int_{\xi_2}^1 [-\xi_1(1-s) - \xi_2(1-s) + 1]y(s)ds. \end{aligned}$$

From above we can obtain the BVP(3) has a unique symmetric solution

$$\begin{aligned} u(t) = &-\int_0^t (t-s)y(s)ds + t\int_0^1 (1-s)y(s)ds \\ &+\frac{1}{2}\int_0^{\xi_1} [(\xi_1 - s) + (\xi_2 - s) - \xi_1(1-s) - \xi_2(1-s) + 1]y(s)ds \end{aligned}$$

$$+\frac{1}{2}\int_{\xi_1}^{\xi_2}[(\xi_2-s)-\xi_1(1-s)-\xi_2(1-s)+1]y(s)ds$$
$$+\frac{1}{2}\int_{\xi_2}^{1}[-\xi_1(1-s)-\xi_2(1-s)+1]y(s)ds$$
$$=\int_0^1 G_1(t,s)y(s)ds+\int_0^1 G_2(s)y(s)ds=\int_0^1 G(t,s)y(s)ds.$$

This complete the proof.

Lemma 2.2. Let $m_{G_2}=min\{G_2(\xi_1),G_2(\xi_2)\}$,$L=\frac{4m_{G_2}}{4m_{G_2}+1}$,then the function $G(t,s)$ satisfies $LG(s,s)\le G(t,s)\le G(s,s)$ for $t,s\in[0,1]$.

Proof. For any $t\in[0,1]$ and $s\in[0,1]$, we have

$$G(t,s)=G_1(t,s)+G_2(s)\ge G_2(s)=\frac{1}{4m_{G_2}+1}G_2(s)+\frac{4m_{G_2}}{4m_{G_2}+1}G_2(s)$$
$$\ge\frac{1}{4}\cdot\frac{4m_{G_2}}{4m_{G_2}+1}+\frac{4m_{G_2}}{4m_{G_2}+1}G_2(s)\ge s(1-s)\frac{4m_{G_2}}{4m_{G_2}+1}+\frac{4m_{G_2}}{4m_{G_2}+1}G_2(s)$$
$$\ge LG_1(s,s)+LG_2(s)=LG(s,s).$$

It is obvious that $G(s,s)\ge G(t,s)$ for $t,s\in[0,1]$. The proof is complete.

Lemma 2.3. Let $y\in C^+[0,1]$, then the unique symmetric solution $u(t)$ of the BVP (3) is nonnegative on $[0,1]$.

Proof. Let $y\in C^+[0,1]$. From the fact that $u''(t)=-y(t)\le 0, t\in[0,1]$, we know that the graph of $u(t)$ is concave on $[0,1]$. From (3). We have that

$$u(0)=u(1)=\frac{1}{2}\int_0^{\xi_1}[(\xi_1-s)+(\xi_2-s)-\xi_1(1-s)-\xi_2(1-s)+1]y(s)ds$$
$$+\frac{1}{2}\int_{\xi_1}^{\xi_2}[(\xi_2-s)-\xi_1(1-s)-\xi_2(1-s)+1]y(s)ds$$
$$+\frac{1}{2}\int_{\xi_2}^{1}[-\xi_1(1-s)-\xi_2(1-s)+1]y(s)ds\ge 0.$$

Note that $u(t)$ is concave, thus $u(t)\ge 0$ for $t\in[0,1]$. This complete the proof.

Lemma 2.4. Let $y\in C^+[0,1]$, then the unique symmetric solution $u(t)$ of BVP (3) satisfies

$$\min_{t\in[0,1]}u(t)\ge L\parallel u\parallel. \tag{5}$$

Proof. For any $t\in[0,1]$, on one hand, from Lemma 2.2 we have that $u(t)=\int_0^1 G(t,s)y(s)ds\le\int_0^1 G(s,s)y(s)ds$. Therefore,

$$\parallel u\parallel\le\int_0^1 G(s,s)y(s)ds. \tag{6}$$

On the other hand, for any $t \in [0,1]$, from Lemma 2.2 we can obtain that

$$u(t) = \int_0^1 G(t,s)y(s)ds \geq L \int_0^1 G(s,s)y(s)ds \geq L \parallel u \parallel . \tag{7}$$

From (6) and (7) we know that (5) holds. Obviously, $(u,v) \in C^2[0,1] \times C^2[0,1]$ is the solution of (1) and (2) if and only if $(u,v) \in C[0,1] \times C[0,1]$ is the solution of integral equations

$$\begin{cases} u(t) = \int_0^1 G(t,s)f(s,v(s))ds, \\ v(t) = \int_0^1 G(t,s)g(s,u(s))ds. \end{cases} \tag{8}$$

Integral equations (8) can be transferred to the nonlinear integral equation

$$u(t) = \int_0^1 G(t,s)f(s, \int_0^1 G(s,\xi)g(\xi,u(\xi))d\xi)ds \tag{9}$$

Let $P = \{u \in C^+[0,1] : u(t)$ is symmetric, concave on $[0,1]$ and $min_{0 \leq t \leq 1} u(t) \geq L \parallel u \parallel\}$. It is obvious that P is a positive cone in $C[0,1]$. Define an integral operator $A : P \to C$ by

$$Au(t) = \int_0^1 G(t,s)f(s, \int_0^1 G(s,\xi)g(\xi,u(\xi))d\xi)ds. \tag{10}$$

It is easy to see that the BVP (1) and (2) has a solution $u = u(t)$ if and only if u is a fixed point of the operator A defined by (10).

Lemma 2.5. If the operator A is defined as (10), then $A : P \to P$ is completely continuous.

Proof. It is obvious that Au is symmetric on $[0,1]$. Note that $(Au)''(t) - f(t,v(t)) \leq 0$, we have that Au is concave, and from Lemma 2.3, it is easily known that $Au \in C^+[0,1]$. Thus from lemma 2.2 and non-negativity of f and g,

$$\begin{aligned} Au(t) &= \int_0^1 G(t,s)f(s, \int_0^1 G(s,\xi)g(\xi,u(\xi))d\xi)ds \\ &\leq \int_0^1 G(s,s)f(s, \int_0^1 G(s,\xi)g(\xi,u(\xi))d\xi)ds, \end{aligned}$$

then

$$\parallel Au \parallel \leq \int_0^1 G(s,s)f(s, \int_0^1 G(s,\xi)g(\xi,u(\xi))d\xi)ds.$$

For another hand,

$$Au \geq L \int_0^1 G(s,s)f(s, \int_0^1 G(s,\xi)g(\xi,u(\xi))d\xi)ds \geq L \parallel Au \parallel .$$

Thus, $A(P) \subset P$. Since $G(t,s)$, $f(t,u)$ and $g(t,u)$ are continuous, it is easy to know that $A : P \to P$ is completely continuous. The proof is complete.

Lemma 2.6.(see[1]) Let E be a Banach space and $P \subset E$ is a cone in E. Assume that Ω_1 and Ω_2 are open subsets of E with $0 \in \Omega_1$ and $\overline{\Omega_1} \subset \Omega_2$. Let $A : P \bigcap (\overline{\Omega_2} \backslash \Omega_1) \to P$ be a completely continuous operator. In addition suppose either

(I) $\| Au \| \leq \| u \|, \forall u \in P \bigcap \partial\Omega_1$ and $\| Au \| \geq \| u \|, \forall u \in P \bigcap \partial\Omega_2$ or

(II) $\| Au \| \leq \| u \|, \forall u \in P \bigcap \partial\Omega_2$ and $\| Au \| \geq \| u \|, \forall u \in P \bigcap \partial\Omega_1$

holds. Then A has a fixed point in $P \bigcap (\overline{\Omega_2} \backslash \Omega_1)$.

Lemma 2.7.(see[1]) Let E be a Banach space and $P \subset E$ is a cone in E. Assume that Ω_1, Ω_2 and Ω_3 are open subsets of E with $0 \in \Omega_1$, $\overline{\Omega_1} \subset \Omega_2$,$\overline{\Omega_2} \subset \Omega_3$ and let $A : P \bigcap (\overline{\Omega_3} \backslash \Omega_1) \to P$ be a completely continuous operator. In addition suppose either

(I) $\| Au \| \geq \| u \|, \forall u \in P \bigcap \partial\Omega_1$;

(II) $\| Au \| \leq \| u \|, Au \neq u$, $\forall u \in P \bigcap \partial\Omega_2$;

(III) $\| Au \| \geq \| u \|, \forall u \in P \bigcap \partial\Omega_3$

holds.Then A has at least two fixed-points x_1,x_2 in $P \bigcap (\overline{\Omega_3} \backslash \Omega_1)$, and furthermore $x_1 \in P \bigcap (\Omega_2 \backslash \Omega_1)$,$x_2 \in P \bigcap (\overline{\Omega_3} \backslash \overline{\Omega_2})$.

3 Main Results

In this section, we study the existence of positive solutions for BVP (1) and (2). First we give the following assumptions:

(H_1) $\lim_{u\to 0^+} \sup_{0\leq t\leq 1} \dfrac{f(t,u)}{u} = 0$,$\lim_{u\to 0^+} \sup_{0\leq t\leq 1} \dfrac{g(t,u)}{u} = 0$;

(H_2) $\lim_{u\to \infty} \inf_{0\leq t\leq 1} \dfrac{f(t,u)}{u} = \infty$,$\lim_{u\to \infty} \inf_{0\leq t\leq 1} \dfrac{g(t,u)}{u} = \infty$;

(H_3) $\lim_{u\to 0^+} \inf_{0\leq t\leq 1} \dfrac{f(t,u)}{u} = \infty$,$\lim_{u\to 0^+} \inf_{0\leq t\leq 1} \dfrac{g(t,u)}{u} = \infty$;

(H_4) $\lim_{u\to \infty} \sup_{0\leq t\leq 1} \dfrac{f(t,u)}{u} = 0$,$\lim_{u\to \infty} \sup_{0\leq t\leq 1} \dfrac{g(t,u)}{u} = 0$;

(H_5)There exists a constant $R_1 > 0$, such that $f(s,u) \leq \dfrac{R_1}{\int_0^1 G(s,s)ds}$ for every $(s,u) \in [0,1] \times [LR_1, R_1]$.

Theorem 3.1. If (H_1) and (H_2) are satisfied, then BVP (1) and (2) have at least one symmetric positive solution $(u,v) \in C^2([0,1],R^+) \times C^2([0,1],R^+)$ satisfying $u(t) > 0, v(t) > 0$.

Proof. From (H_1) there is a number $N_1 \in (0,1)$ such that for each $(s,u) \in [0,1] \times (0,N_1)$, one has $f(s,u) \leq \eta_1 u$, $g(s,u) \leq \eta_1 u$, where $\eta_1 > 0$ satisfies $\eta_1 \int_0^1 G(s,s)ds \leq 1$, for every $u \in P$ and $\| u \| = \frac{N_1}{2}$, note that $\int_0^1 G(s,\xi)g(\xi,u(\xi))d\xi \leq \int_0^1 G(\xi,\xi)g(\xi,u(\xi))d\xi \leq \int_0^1 \eta_1 G(\xi,\xi)u(\xi)d\xi \leq \| u \| =$

$\frac{N_1}{2} < N_1$,then

$$\begin{aligned} Au(x) &\leq \int_0^1 G(t,s)f(s,\int_0^1 G(s,\xi)g(\xi,u(\xi))d\xi)ds \\ &\leq \eta_1 \int_0^1 G(s,s)\int_0^1 G(s,\xi)g(\xi,u(\xi))d\xi ds \\ &\leq \eta_1^2 \int_0^1 G(s,s)\int_0^1 G(\xi,\xi)u(\xi)d\xi ds \leq \| u \| . \end{aligned}$$

Let

$$\Omega_1 = \{u \in C^+[0,1], \| u \| < \frac{N_1}{2}\},$$

then

$$\| Au \| \leq \| u \|, u \in P \bigcap \partial\Omega_1. \tag{11}$$

From (H_2) there is a number $N_2 > \sqrt{L}N_1$ for each $(s,u) \in [0,1] \times (N_2, +\infty)$, one has $f(s,u) \geq \eta_2 u$, $g(s,u) \geq \eta_2 u$ where $\eta_2 > 0$ satisfies $\eta_2 L^{\frac{3}{2}} \int_0^1 G(s,s)ds \geq 1$, then, for every $u \in P$ and $\| u \| = \frac{2N_2}{\sqrt{L}}$, from Lemma 2.2 and Lemma 2.4, we have

$$\begin{aligned} \int_0^1 G(s,\xi)g(\xi,u(\xi))d\xi &\geq L^2 \int_0^1 \eta_2 G(\xi,\xi) \| u \| d\xi \\ &\geq 2\sqrt{L} \| u \| = 2N_2 > N_2, \end{aligned}$$

then

$$\begin{aligned} \| Au \| &= \int_0^1 G(t,s)f(s,\int_0^1 G(s,\xi)g(\xi,u(\xi))d\xi)ds \\ &\geq L\eta_2 \int_0^1 G(s,s)\int_0^1 G(s,\xi)g(\xi,u(\xi))d\xi ds \\ &\geq L^2\eta_2^2 \int_0^1 G(s,s)\int_0^1 G(\xi,\xi)u(\xi)d\xi ds \\ &\geq L^3\eta_2^2 \int_0^1 G(s,s)\int_0^1 G(\xi,\xi) \| u \| d\xi ds \geq \| u \| . \end{aligned}$$

Let

$$\Omega_2 = \{u \in C^+[0,1], \| u \| < \frac{2N_2}{\sqrt{L}}\},$$

then

$$\| Au \| \geq \| u \|, u \in P \bigcap \partial\Omega_2. \tag{12}$$

Thus from(11),(12) and Lemma 2.6,we know that the operator A has a fixed point in $P\bigcap(\overline{\Omega_2}\backslash\Omega_1)$. The proof is complete.

Theorem 3.2. If (H_3) and (H_4) are satisfied, then BVP (1) and (2) have at least one symmetric positive solution $(u,v)\in C^2([0,1],R^+)\times C^2([0,1],R^+)$ satisfying $u(t)>0, v(t)>0$.

Proof. From (H_3) there is a number $\overline{N_3}\in(0,1)$ such that for each $(x,u)\in[0,1]\times(0,\overline{N_3})$, one has $f(s,u)\geq\eta_3 u$, $g(s,u)\geq\eta_3 u$ where $\eta_3>0$ satisfies $L^{\frac{3}{2}}\eta_3\int_0^1 G(s,s)ds\geq 1$. From $g(x,0)\equiv 0$ and the continuity of $g(s,u)$, we know that there exists number $N_3\in(0,\overline{N_3})$ such that $g(s,u)\leq\frac{\overline{N_3}}{\int_0^1 G(s,s)ds}$ for each $(s,u)\in[0,1]\times(0,N_3]$. Then for every $u\in P$ and $\parallel u\parallel=N_3$, note that

$$\int_0^1 G(s,\xi)g(\xi,u(\xi))d\xi\leq\int_0^1 G(\xi,\xi)\frac{\overline{N_3}}{\int_0^1 G(s,s)ds}d\xi=\overline{N_3}.$$

Thus

$$\begin{aligned}Au(x)&=\int_0^1 G(t,s)f(s,\int_0^1 G(s,\xi)g(\xi,u(\xi))d\xi)ds\\&\geq L\eta_3\int_0^1 G(s,s)\int_0^1 G(s,\xi)g(\xi,u(\xi))d\xi ds\\&\geq L^2\eta_3^2\int_0^1 G(s,s)\int_0^1 G(\xi,\xi)u(\xi)d\xi ds\\\geq L^3\eta_3^2\int_0^1 G(s,s)\int_0^1 G(\xi,\xi)\parallel u\parallel d\xi ds&\geq\parallel u\parallel .\end{aligned}$$

Let

$$\Omega_3=\{u\in C^+[0,1],\parallel u\parallel<N_3\},$$

then

$$\parallel Au\parallel\geq\parallel u\parallel, u\in P\bigcap\partial\Omega_3. \tag{13}$$

From (H_4), there exist $C_1>0$ and $C_2>0$ such that $f(s,u)\leq\eta_4 u+C_1$, $g(s,u)\leq\eta_4 u+C_2$ for $\forall(s,u)\in[0,1]\times(0,\infty)$, where $\eta_4>0$, and $\eta_4\int_0^1 G(\xi,\xi)d\xi\leq 1$. Then, for $u\in C^+[0,1]$ we have

$$\begin{aligned}Au&=\int_0^1 G(s,t)f(s,\int_0^1 G(s,\xi)g(\xi,u(\xi))d\xi)ds\\&\leq\int_0^1 G(s,s)(\eta_4\int_0^1 G(s,\xi)g(\xi,u(\xi))d\xi+C_1)ds\end{aligned}$$

$$\begin{aligned}
&\leq \eta_4 \int_0^1 G(s,s) \int_0^1 G(\xi,\xi) g(\xi,u(\xi)) d\xi ds + C_3 \\
&\leq \eta_4 \int_0^1 G(s,s) \int_0^1 G(\xi,\xi)(\eta_4 u + C_2) d\xi ds + C_3 \\
&\leq (\eta_4)^2 \int_0^1 G(s,s) \int_0^1 G(\xi,\xi) \parallel u \parallel d\xi ds + C_4 \leq \parallel u \parallel + C_4
\end{aligned}$$

Thus $\parallel Au \parallel \leq \parallel u \parallel$ with $\parallel u \parallel \to \infty$.
Let $\Omega_4 = \{u \in E, \parallel u \parallel < N_4\}$. For each $u \in P$ and $\parallel u \parallel = N_4 > N_3$ large enough, we have

$$\parallel Au \parallel \leq \parallel u \parallel, u \in P \bigcap \partial\Omega_4. \tag{14}$$

Thus from(13),(14) and Lemma 2.6,we know that the operator A has a fixed point in $P \bigcap (\overline{\Omega_4} \backslash \Omega_3)$. The proof is complete.
Theorem 3.3. If (H_2), $(H_3)$and (H_5) are satisfied, then BVP (1) and (2) have at least two symmetric positive solutions $(u_1, v_1), (u_2, v_2) \in C^2([0,1], R^+) \times C^2([0,1], R^+)$ satisfying $u_1(t) > 0, v_1(t) > 0, u_2(t) > 0, v_2(t) > 0$.
Proof.Let

$$\Omega_5 = \{u \in C^+[0,1], \parallel u \parallel < R_1\},$$

then $\forall u \in P \bigcap \partial\Omega_5$, we have $u(s) \in [LR_1, R_1]$. From lemma 2.2, Lemma 2.4 and (6) we can obtain

$$\int_0^1 G(s,\xi) g(\xi,u(\xi)) d\xi \geq L \int_0^1 G(\xi,\xi) g(\xi,u(\xi)) d\xi \geq L \parallel u \parallel$$

and

$$\begin{aligned}
\int_0^1 G(s,\xi) g(\xi,u(\xi)) d\xi &\leq \int_0^1 G(\xi,\xi) g(\xi,u(\xi)) d\xi \\
&\leq \int_0^1 G(\xi,\xi) d\xi \frac{R_1}{G(s,s) ds} = R_1.
\end{aligned}$$

Thus $Au = \int_0^1 G(s,t) f(s, \int_0^1 G(s,\xi) g(\xi,u(\xi)) d\xi) ds \leq \int_0^1 G(s,s) \frac{R_1}{\int_0^1 G(\xi,\xi) d\xi} ds = R_1 = \parallel u \parallel$. Then

$$\parallel Au \parallel \leq \parallel u \parallel, u \in P \bigcap \partial\Omega_5. \tag{15}$$

For another hand, from (H_2) and (H_3), we can choose two right numbers $\widetilde{N_2} \in (R_1, \infty)$, $\widetilde{N_3} \in (0, R_1)$ satisfy

$$\parallel Au \parallel \geq \parallel u \parallel, u \in P \bigcap \partial\widetilde{\Omega_2}, \tag{16}$$

$$\parallel Au \parallel \geq \parallel u \parallel, u \in P \bigcap \partial\widetilde{\Omega_3}, \tag{17}$$

where $\widetilde{\Omega_2} = \{u \in C^+[0,1], \parallel u \parallel < \widetilde{N_2}\}$, $\widetilde{\Omega_3} = \{u \in C^+[0,1], \parallel u \parallel < \widetilde{N_3}\}$.
Then from Lemma 2.7, (15), (16) and (17), A has at least two fixed points in $P \bigcap (\overline{\widetilde{\Omega_2}} \backslash \overline{\Omega_5})$ and $P \bigcap (\Omega_5 \backslash \widetilde{\Omega_3})$, respectively. The proof is complete.

4 Examples

In this section, we give three examples to illustrate our results.

Examples 4.1. Let $f(t,v) = v^2 + \frac{[1+t(1-t)]v^2}{1+v^2}$,$g(t,u) = 2u^2 + \frac{2[1+t(1-t)]u^2}{1+u^2}$, $\xi_1 = \frac{1}{4}, \xi_2 = \frac{1}{2}$, we can choose $L = \frac{5}{16}$, then conditions of Theorem 3.1 are satisfied. From Theorem 3.1, BVP (1) and (2) have at least one symmetric positive solution.

Examples 4.2. Let $f(t,v) = v^{\frac{1}{2}} + \frac{[1+t(1-t)]v^2}{1+v^2}$,$g(t,u) = 2u^{\frac{1}{2}} + \frac{2[1+t(1-t)]u^2}{1+u^2}$, $\xi_1 = \frac{1}{4}, \xi_2 = \frac{1}{2}$, we can choose $L = \frac{5}{16}$,, then conditions of Theorem 3.2 are satisfied. From Theorem 3.2, BVP (1) and (2) have at least one symmetric positive solution.

Examples 4.3. Let $f(t,v) = \frac{45[t(1-t)+1]}{32}(v^{\frac{1}{2}} + v^2)$,$g(t,u) = \frac{43[t(1-t)+1]}{32}(u^{\frac{1}{2}} + u^2)$, $\xi_1 = \frac{1}{4}, \xi_2 = \frac{1}{2}$, we can choose $L = \frac{5}{16}$ and $R_1 = 1$, then conditions of Theorem 3.3 are satisfied. From Theorem 3.3, BVP (1) and (2) have at least two symmetric positive solutions.

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