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Generalized Contractions on Spaces with Two Metrics

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Abstract

We establish a convergence theorem for iterative processes of the type: $x_{n+1} = Tx_n,\ n = 0, 1, 2, \cdots$, where $T : D \subset X \to X$ is an operator on a complete metric space (X, d') satisfying:

$$d(Tx, T^2x) \leq \varphi(d(x, Tx)) \text{ for all } x \in D, Tx \in D \text{ with } d(x, Tx) \in J,$$

φ is a Bianchini-Grandolfi gauge function on an interval J and d is another metric on X.

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1 Introduction and Preliminaries

Throughout this paper J denotes an interval on $\mathbb{R}_+$ containing 0, that is an interval of the form $[0, A], [0, A)$ or $[0, \infty)$. We use the abbreviation φ^n for the nth iterate of a function $\varphi : J \to J$. Let $r \geq 1$, a function $\varphi : J \to J$ is said to be a gauge function [7, Definition 1.1] of order r on J if it satisfies the following conditions: (i) $\varphi(\lambda t) \leq \lambda^r \varphi(t)$ for all $\lambda \in (0, 1)$ and $t \in J$; (ii) $\varphi(t) < t$ for all $t \in J - \{0\}$. It is easy to see that the first condition of this definition is equivalent to the following: $\varphi(0) = 0$ and $\varphi(t)/t^r$ is nondecreasing on $J - \{0\}$. A nondecreasing function $\varphi : J \to J$ is said to be a Bianchini-Grandolfi gauge function [2, 7, Definition 2.3] on J if

$$\sigma(t) = \sum_{n=0}^{\infty} \varphi^n(t) < \infty, \text{ for all } t \in J. \tag{1}$$

Let (X, d') be a complete metric space, d another metric on X. For $x_0 \in X$, $r > 0$ and $S(x_0, r) = \{x \in X : d(x, x_0) < r\}$ we denote by $\overline{S(x_0, r)^{d'}}$ the d'-closure of $S(x_0, r)$ and by $\overline{S(x_0, r)^{d}}$ the d-closure of $S(x_0, r)$. In this paper we establish a convergence theorem for iterative processes of the type:

$$x_{n+1} = Tx_n, \ n = 0, 1, 2, \cdots \tag{2}$$

where $T : D \subset X \to X$ is an operator satisfying:

$$d(Tx, T^2x) \leq \varphi(d(x, Tx)) \ \text{ for all } \ x \in D, Tx \in D \ \text{ with } \ d(x, Tx) \in J, \tag{3}$$

and φ is a Bianchini-Grandolfi gauge function on an interval J. Our results generalize, extend and improve some recent results by Agarwal and O'Regan [1] and thus generalize some results of Ciric [3], Hardy and Rogers[4], Kannan [5], Maia [6], Precup [8] and Reich [9]. We use the machinery of Proinov [7] to prove our results. For convenience we define the function $E : D \to \mathbb{R}_+$ by $E(x) = d(x, Tx)$. Therefore condition (3) can be rewritten in the form

$$E(Tx) \leq \varphi(E(x)) \text{ for all } x \in D \text{ with } Tx \in D \text{ and } E(x) \in J. \tag{4}$$

Suppose $x_0 \in D$ and $E(x_0) \in J$. Then for every iterate x_n $(n \geq 0)$ which belongs to D we define the closed ball $\overline{S(x_n, \rho_n)^{d}}$ with center x_n and radius $\rho_n = \sigma(E(x_n))$, where $\sigma : J \to \mathbb{R}_+$ is defined by (1). A point $x_0 \in D$ will be called an initial point [7, Definition 3.2] of T if $E(x_0) \in J$ and all of the iterates $x_0, x_1, x_2, \cdots$ are well-defined and belong to D. We state following results for convenience.

Lemma 1.1 *[7, Lemma 2.4] Every gauge function of order $r \geq 1$ on J is a Bianchini-Grandolfi gauge function on J.*

Theorem 1.2 *[1, Theorem 2.1] Let (X, d') be a complete metric space, d another metric on X, $x_0 \in X$, $r > 0$ and $F : \overline{B(x_0, r)^{d'}} \to X$. Suppose there exists $q \in (0, 1)$ such that for $x, y \in \overline{B(x_0, r)^{d'}}$ we have*

$$d(Fx, Fy) \leq q \max\{d(x, y), d(x, Fx), d(y, Fy), \frac{1}{2}[d(x, Fy) + d(y, Fx)]\}.$$

In addition assume the following three properties hold:

$$d(x_0, Fx_0) < (1 - q)r \tag{5}$$

$$\text{If } d \not\geq d' \text{ assume } F \text{ is uniformly continuous from } (B(x_0, r), d) \text{ into} (X, d'), \tag{6}$$

and

$$\text{If } d \neq d' \text{ assume } F \text{ is continuous from } (\overline{B(x_0, r)^{d'}}, d') \text{ into} (X, d'), \tag{7}$$

Then F has a fixed point. That is, there exists $x \in \overline{B(x_0, r)^{d'}}$ with $x = Fx$.

Theorem 1.3 *[1, Theorem 2.3] Let (X, d') be a complete metric space, d another metric on X and $F : X \to X$. Suppose there exists $q \in (0, 1)$ such that for $x, y \in X$ we have*

$$d(Fx, Fy) \leq q \max\{d(x, y), d(x, Fx), d(y, Fy), \frac{1}{2}[d(x, Fy) + d(y, Fx)]\}.$$

In addition assume the following two properties hold:

$$\text{If } d \not\geq d' \text{ assume } F \text{ is uniformly continuous from } (X, d) \text{ into} (X, d'), \tag{8}$$

and

$$\text{If } d \neq d' \text{ assume } F \text{ is continuous from } (X, d') \text{ into} (X, d'), \tag{9}$$

Then F has a fixed point.

2 Main Results

Theorem 2.1 *Let (X, d') be a complete metric space, d another metric on X and $T : D \subset X \to X$ is an operator satisfying condition (3), with a Bianchini-Grandolfi gauge function φ on an interval J. Then starting from an initial point x_0 of T the iterative sequence (2) remains in $\overline{S(x_0, \rho_0)^d}$ and converges to a point $\xi \in \overline{S(x_0, \rho_0)^{d'}}$ which is a fixed point of T provided that the following three conditions hold:*

$$\text{If } d \not\geq d' \text{ then } T \text{ is uniformly continuous from } (S(x_0, \rho_0), d) \text{ into } (X, d') \tag{10}$$

$$\text{If } d \neq d' \text{ then } T \text{ is continuous from } (\overline{S(x_0, \rho_0)^{d'}}, d') \text{ into } (X, d'), \tag{11}$$

$$\text{If } d = d' \text{ then } T \text{ is continuous at } \xi. \tag{12}$$

Proof: Since x_0 is an initial point of T, it follows from [7, Lemma 3.2] that

$$\overline{S(x_{n+1}, \rho_{n+1})^d} \subset \overline{S(x_n, \rho_n)^d} \text{ for all } n \geq 0, \tag{13}$$

which implies that $x_n \in \overline{S(x_0, \rho_0)^d}$ for all $n \geq 0$. Using the definition of ρ_n, from [7, Lemma 3.6], we get

$$\begin{aligned} \rho_n = \sigma(E(x_n)) &\leq \sigma(\varphi^n(E(x_0))) \\ &= \sum_{j=0}^{\infty} \varphi^j(\varphi^n(E(x_0))) \\ &= \sum_{j=n}^{\infty} \varphi^j(E(x_0)) \text{ for all } n \geq 0, \end{aligned} \tag{14}$$

since σ is non-decreasing. As φ is Bianchini-Grandolfi gauge function on J and $E(x_0) \in J$, from (14), we get

$$\rho_n \to 0 \text{ as } n \to \infty. \tag{15}$$

From (13) and (15) it follow that $\{\overline{S(x_n, \rho_n)^d}\}$ is nested sequence of closed spheres such that $\rho_n \to 0$ as $n \to \infty$. Therefore, it follows from Cantor's Theorem that the sequence (2) is a Cauchy sequence with respect to d i.e., there exists $N \in \{1, 2, \cdots\}$ with

$$d(x_n, x_m) < \delta \quad \text{whenever} \quad n, m \geq N. \tag{16}$$

We now claim that, (2) is a Cauchy sequence with respect to d'. If $d \geq d'$ this is trivial. Next suppose $d \ngeq d'$. Let $\epsilon > 0$ be given. Then (10) guarantees that there exists $\delta > 0$ such that

$$d'(Tx, Ty) < \epsilon \quad \text{whenever} \quad x, y \in S(x_0, \rho_0) \quad \text{and} \quad d(x, y) < \delta \tag{17}$$

Now (16) and (17) imply

$$d'(x_{n+1}, x_{m+1}) = d'(Tx_n, Tx_m) < \epsilon \quad \text{whenever} \quad n, m \geq N,$$

and as a result our claim is proved. Now since (X, d') is complete there exists $\xi \in \overline{S(x_0, \rho_0)^{d'}}$ with $d'(x_n, \xi) \to 0$ as $n \to \infty$. We claim that $\xi = T\xi$. First consider the case when $d \neq d'$.

$$d'(\xi, T\xi) \leq d'(\xi, x_n) + d'(x_n, T\xi) = d'(\xi, x_n) + d'(Tx_{n-1}, T\xi) \tag{18}$$

Let $n \to \infty$ then (11) insures that $d'(\xi, x_n) \to 0$ implies $d'(Tx_{n-1}, T\xi) \to 0$ and so $\xi = T\xi$. Next suppose $d = d'$.

$$d(\xi, Tx_n) \leq d(\xi, x_n) + d(x_n, Tx_n) = d(\xi, x_n) + d(x_n, x_{n+1}).$$

Taking limit as $n \to \infty$ we get,

$$lim_{n\to\infty} d(\xi, Tx_n) \leq 0.$$

From (12) since T is continuous at ξ so we have $d(\xi, T\xi) = 0$ which simply means that $\xi = T\xi$.

Remark 2.2 *Theorem 2.1 remains true if φ is a gauge function of order $r \geq 1$.*

Corollary 2.3 *Let (X, d') be a complete metric space, d another metric on X and $T : D \subset X \to X$ is an operator satisfying*

$$d(Tx, Ty) \leq \varphi(d(x, y)) \text{ for all } x, y, Tx, Ty \in D \text{ with } d(x, y) \in J \tag{19}$$

where φ is a gauge function of order $r \geq 1$ on an interval J. Assume that x_0 is an initial point of T such that $d(x_0, Tx_0) \in J$. Then the following statements hold true:

(a) The iterative sequence (2) converges to a fixed point ξ of T.

(b) The operator T has a unique fixed point in $S = \{x \in X : d(x, \xi) \in J\}$,

provided that (10) and (11) hold.

Proof: From (19) $d(Tx, T^2x) \leq \varphi(d(x, Tx))$ holds for all $x \in D, y = Tx \in D$ and $d(x, Tx) \in J$. Further, by definition of φ we have $d(Tx, Ty) \leq \varphi(d(x, y)) < d(x, y)$ for $x \neq y$. Hence, T is d continuous. Thus *(a)* follows immediately from Theorem 2.1. To show the uniqueness, i.e., *(b)*, assume that $\eta \in S$ is another fixed point of T. Then $d(\xi, \eta) \in J$. It follows from (19) that

$$d(\xi, \eta) = d(T\xi, T\eta) \leq \varphi(d(\xi, \eta)) < d(\xi, \eta)$$

Hence $\xi = \eta$.

Corollary 2.4 *Let (X, d') be a complete metric space, d another metric on X and $T : D \subset X \to X$ is an operator satisfying*

$$d(Tx, Ty) \leq \varphi(m(x, y)) \text{ for all } x, y, Tx, Ty \in D \tag{20}$$

where φ is a gauge function of order $r \geq 1$ on an interval $J = [0, \infty)$ and

$$m(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2}\}. \tag{21}$$

Then for initial point x_0 of T, the iterative sequence (2) converges to a unique fixed point ξ of T provided that (10),(11) and (12) hold. Moreover, if φ is continuous, then continuity of T in (12) can be omitted.

Proof: First we shall prove that, $m(x, Tx) = d(x, Tx)$ for all $x \in D$. By (21) and triangle inequality, we get

$$\begin{aligned} m(x, Tx) &= \max\{d(x, Tx), d(x, Tx), d(Tx, T^2x), d(x, T^2x)\} \\ &= \max\{d(x, Tx), d(Tx, T^2x)\} = \max\{E(x), E(Tx)\} \end{aligned}$$

where $E(x) = d(x, Tx)$. If $x = Tx$ then $m(x, Tx) = E(x) = d(x, Tx)$. If $x \neq Tx$ then since $\varphi(t) < t$ for $t > 0$ so we have from (20)

$$E(Tx) = d(Tx, T^2x) \leq \varphi(m(x, Tx)) < m(x, Tx) = \max\{E(x), E(Tx)\}$$

which means $m(x, Tx) = E(x)$. Thus condition (20) implies (3). Applying Theorem 2.1, we deduce that the iterative sequence (2) converges to the fixed point ξ of T.

If η is another fixed point then from (21) we have $m(\xi, \eta) = d(\xi, \eta)$ and hence uniqueness follows immediately from Corollary 2.3.

Finally, suppose that $d = d'$ and φ is continuous then it follows from (20) that

$$d(x_{n+1}, T\xi) = d(Tx_n, T\xi) \leq \varphi(m(x_n, \xi)) \text{ for all } n \geq 0$$

Since $m(x_n, \xi) = d(x_n, \xi)$ so passing limit as $n \to \infty$ we obtain $d(\xi, T\xi) \leq \varphi(0) = 0$ which implies $\xi = T\xi$.

Remark 2.5 *If $\varphi(t) = qt \ \ (0 < q < 1)$, then Corollary 2.4 generalizes Theorem 1.2 (resp. Theorem 1.3) in the following manner.*

(i) Taking $D = B(x_0, r)^{d'}$ (resp. $D = X$).

(ii) Uniform Continuity of the operator T in (6) (resp. in (8)) is required from $(B(x_0, r), d)$ into (X, d') (resp. from (X, d) into (X, d')) while it is required from $(S(x_0, \rho_0), d)$ into (X, d') in (10).

(iii) Continuity f the operator T in (7) (resp. in (9)) is required from $\overline{B(x_0, r)^{d'}}$ into (X, d') (resp. from (X, d') into (X, d')) while it is required from $(\overline{S(x_0, \rho_0)^{d'}}, d')$ into (X, d') in (10).

(iv) Corollary 2.4 concludes with a unique fixed point.

(v) (12) can be omitted because φ is continuous in this case.

Remark 2.6 *Note that in the hypothesis of Corollary 2.4 we assumed that x_0 is an initial point of T. This assumption was not the part of the hypothesis of Theorem 1.2. But a closer look at condition (5) in Theorem 1.2 in-fact implies that x_0 is an initial point of T.*

Remark 2.7 *If $d = d'$, then Corollary 2.4 coincides with the first part of [7, Corollary 4.5]. In addition, if $\varphi(t) = qt \ \ (0 < q < 1)$ then it reduces to [1, Corollary 2.2].*

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