Int. Journal of Math. Analysis, Vol. 3, 2009, no. 39, 1945 - 1952

Solving Analytically Singular Sixth-Order Boundary Value Problems

Fazhan Geng[1]

Department of Mathematics, Changshu Institute of Technology
Changshu, Jiangsu 215500, China

Yonggang Ye

School of Basic Science, Harbin University of Commerce
Harbin, Heilongjiang 150028, China

Abstract

In this paper, a new method is presented to obtain the analytical and approximate solutions of singular sixth-order boundary value problems. The analytical solution is represented in the form of series in the reproducing kernel space. The approximate solution $u_n(x)$ is obtained by the n-term intercept of the analytical solution. Furthermore, the second derivative $u''(x)$ and forth derivative $u^{(4)}(x)$ of analytical solution $u(x)$ can be also obtained. A test example is studied to demonstrate the accuracy of the present method. Results obtained by the method indicate the method is simple and valid.

Mathematics Subject Classification: 46E22, 47B32

Keywords: Analytical solution; singular sixth-order boundary value problems; reproducing kernel

1 Introduction

In this paper, we consider the following singular sixth-order boundary value problems(BVPs) in the reproducing kernel space

$$\begin{cases} a_0(x)u^{(6)} + a_1(x)u^{(5)} + a_2(x)u^{(4)} + a_3(x)u^{(3)} + a_4(x)u'' + \\ a_5(x)u' + a_6(x)u = f(x), 0 \le x \le 1, \\ u(0) = u(1) = 0,\ u''(0) = u(''1) = 0,\ u^{(4)}(0) = u^{(4)}(1) = 0, \end{cases} \tag{1.1}$$

[1]Corresponding author E-mail address: gengfazhan@sina.com (Fazhan Geng)
The work was supported by the Scientific Research Project of Heilongjiang Education Office (2009-11541098)

where $u \in W_2^3[0,1]$, $f \in W_2^1[0,1]$, $a_i(x)$ are continuous and maybe $a_0(0) = 0$ or $a_0(1) = 0$, i=0,1,2,3,4,5,6. The sixth-order boundary value problems are known to arise in astrophysics; the narrow convecting layers bounded by stable layers which are believed to surround A-type stars may be modelled by sixth-order BVPs[1]. In[2], Glatzmaier also notes that dynamo action in some stars may be modelled by such equations. Chandrasekhar[3] determined that when an infinite horizontal layer of fluid is heated from below and is under the action of ratation, instability sets in.

Agarwal[4] presented the theorys of the existence and uniqueness of solutions of sixth-order BVPs. Scott and Watts[5-6] described the numerical solution of linear BVP using a combination of superposition and orthonormalization, and described several computer codes that were developed using the superposition and orthonormalization technique and invariant imbedding.

There are some other methods to obtain the solutions of sixth-order BVPs. A modified form of the decomposition method was developed by Wazwaz[7-8]. Recently, sixth degree B-Spline technique, quintic spline method, non-polynomial spline technique and variational approach were developed to solve sixth-order BVPs[9-10]. However, most of such methods was used to solve sixth-order BVPs with such form $u^{(6)} = f(x, u(x))$. There are few valid methods to obtain singular sixth-order BVPs with general form.

In this paper, we will give the representation of analytical solution to $Eq.(1.1)$ in the reproducing kernel space under the assumption that the solution to $Eq.(1.1)$ is unique.
Let $v = u''$ and $w = u^{(4)}$. Then $Eq.(1.1)$ can be converted into the following system of three second order BVPs

$$\begin{cases} a_0(x)w'' + a_1(x)w' + a_2(x)w + a_3(x)v' + a_4(x)v + \\ a_5(x)u' + a_6(x)u = f(x), 0 \le x \le 1, \\ u'' - v = 0, 0 \le x \le 1, \\ v'' - w = 0, 0 \le x \le 1, \\ u(0) = u(1) = 0,\ v(0) = v(1) = 0,\ w(0) = w(1) = 0. \end{cases} \tag{1.2}$$

Put $A_{11}u(x) = a_5(x)u' + a_6(x)u$, $A_{12}v(x) = a_3(x)v' + a_4(x)v$, $A_{13}w(x) = a_0(x)w'' + a_1(x)w' + a_2(x)w$,
$A_{21}u(x) = u''$, $A_{22}v(x) = -v$, $A_{23}w(x) = 0$, $A_{31}u(x) = 0$, $A_{32}v(x) = v''$, $A_{33}w(x) = -w$,
$A = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$ and $U = (u, v, w)^\top$, then $Eq.(1.2)$ can be converted into the following form

$$\begin{cases} AU = F(x), 0 \le x \le 1, \\ U(0) = U(1) = 0, \end{cases} \tag{1.3}$$

where $F(x) = (f(x), 0, 0)^\top$, $U \in W_2^3[0,1] \bigoplus W_2^3[0,1] \bigoplus W_2^3[0,1]$, $F \in W_2^1[0,1] \bigoplus W_2^1[0,1] \bigoplus W_2^1[0,1]$. The space $W_2^3[0,1] \bigoplus W_2^3[0,1] \bigoplus W_2^3[0,1]$ is defined as $\{U = (U_1, U_2, U_3)^\top | U_i \in W_2^3[0,1], i = 1,2,3\}$. The inner product and norm are given respectively by $(U,V) = \sum_{i=1}^{3} (U_i, V_i)_{W_2^3}$, $\| U \| = (\sum_{i=1}^{3} \| U_i \|^2)^{\frac{1}{2}}$, $U, V \in W_2^3[0,1] \bigoplus W_2^3[0,1] \bigoplus W_2^3[0,1]$. Clearly, $W_2^3[0,1] \bigoplus W_2^3[0,1] \bigoplus W_2^3[0,1]$ is a Hilbert space. $W_2^1[0,1] \bigoplus W_2^1[0,1] \bigoplus W_2^1[0,1]$ can be defined in like manner and it is also a Hilbert space. $W_2^3[0,1]$ and $W_2^1[0,1]$ are defined in the following section.

2 Several Reproducing Kernel Spaces and Lemmas

1 The reproducing kernel space $W_2^3[0,1]$

The inner product space $W_2^3[0,1]$ is defined as $W_2^3[0,1] = \{u(x) \mid u, u', u''$ are absolutely continuous real valued functions, $u, u', u'', u^{(3)} \in L^2[0,1], u(0) = 0, u(1) = 0\}$. The inner product in $W_2^3[0,1]$ is, given, by

$$(u(y), v(y))_{W_2^3} = \int_0^1 (u''v'' + u^{(3)}v^{(3)})dy, \tag{2.1}$$

and the norm $\| u \|_{W_2^3}$ is denoted by $\| u \|_{W_2^3} = \sqrt{(u,u)_{W_2^3}}$, where $u, v \in W_2^3[0,1]$.

Theorem 2.1. *The space $W_2^3[0,1]$ is a reproducing kernel space. That is, there exists $R_x(y) \in W_2^3[0,1]$, for any $u(y) \in W_2^3[0,1]$ and each fixed $x \in [0,1]$, $y \in [0,1]$, such that $(u(y), R_x(y))_{W_2^3} = u(x)$. The reproducing kernel $R_x(y)$ can be denoted by*

$$R_x(y) = \begin{cases} R_1(x,y), & y \le x \\ R_1(y,x), & y > x \end{cases} \tag{2.2}$$

where

$$R_1(x,y) = \frac{-\left(e^{2-x}+e^x+2\,e^2\,(-1+x)-2\,e\,x\right)}{e^y\,(2-2\,e^2)} - \frac{\left(-1+e^{2-x}+e^x+e^2\,(-1+x)+x-2\,e\,x\right)}{-1+e^2} - \frac{\left(-6\,e^{1-x}+6\,e^x+x\left(4+3\,x-x^2\right)-e\left(-12+20\,x-3\,x^2+x^3\right)\right)y}{6(1+e)} + \frac{(-1+x)\,y^3}{6} - \frac{e^y\left(-2+e^{2-x}+e^x+2\,x-2\,e\,x\right)}{2-2\,e^2}.$$

The method of obtaining reproducing kernel $R_x(y)$ and the proof of Theorem (2.1) refer to Appendix B in [11, 12].

2 The reproducing kernel space $W_2^1[0,1]$

The inner product space $W_2^1[0,1]$ is defined by $W_2^1[0,1] = \{u(x) \mid u$ is absolutely continuous real valued function, $u, u' \in L^2[0,1]\}$. The inner product and

norm in $W_2^1[0,1]$ are given respectively by

$$(u(x),v(x))_{W_2^1}=\int_0^1 (uv+u'v')dx,\ \parallel u \parallel_{W_2^1}=\sqrt{(u,u)_{W_2^1}},$$

where $u(x),v(x)\in W_2^1[0,1]$. In [13], the authors proved that $W_2^1[0,1]$ is a reproducing kernel space and its reproducing kernel is

$$\overline{R}_x(y)=\frac{1}{2\sinh(1)}[\cosh(x+y-1)+\cosh(|x-y|-1)].$$

3 Important Lemma

Lemma 2.1. *If $A_{ij}: W_2^3[0,1]\to W_2^1[0,1], i,j=1,2,3$ are bounded linear operators, then*
$A: W_2^3[0,1]\bigoplus W_2^3[0,1]\bigoplus W_2^3[0,1] \to W_2^1[0,1]\bigoplus W_2^1[0,1]\bigoplus W_2^1[0,1]$ is a bounded linear operator.

Proof. Clearly, A is a linear operator. For $\forall\ u\in W_2^3[0,1]\bigoplus W_2^3[0,1]\bigoplus W_2^3[0,1]$,

$$\begin{aligned}\parallel Au \parallel &= (\sum_{i=1}^{3}\parallel \sum_{j=1}^{3} A_{ij}u_j \parallel^2)^{\frac{1}{2}}\\ &\le [\sum_{i=1}^{3}(\sum_{j=1}^{3}\parallel A_{ij}\parallel\ \parallel u_j\parallel)^3]^{\frac{1}{2}}\\ &\le [\sum_{i=1}^{3}(\sum_{j=1}^{3}\parallel A_{ij}\parallel^2)(\sum_{j=1}^{3}\parallel u_j\parallel^2)]^{\frac{1}{2}}\\ &= (\sum_{i=1}^{3}\sum_{j=1}^{3}\parallel A_{ij}\parallel^2)^{\frac{1}{2}}\parallel u\parallel .\end{aligned}\tag{2.3}$$

The boundedness of A_{ij} implies that A is bounded. The proof is complete. □

It is easy to see that the adjoint operator of A is $A^*=\begin{pmatrix} A_{11}^* & A_{21}^* & A_{31}^*\\ A_{12}^* & A_{22}^* & A_{32}^*\\ A_{13}^* & A_{23}^* & A_{33}^*\end{pmatrix}$, where A_{ij}^* is the adjoint operator of A_{ij}.

3 The analytical and approximate solutions of Eq.(1.3)

In this section, we will give the representation of analytical solution of *Eq.*(1.3) and implementation method in the reproducing kernel space $W_2^3[0,1]\bigoplus W_2^3[0,1]\bigoplus W_2^3[0,1]$.

In view of Lemma(2.1), it is clear that $A: W_2^3[0,1]\bigoplus W_2^3[0,1]\bigoplus W_2^3[0,1]\to W_2^1[0,1]\bigoplus W_2^1[0,1]\bigoplus W_2^1[0,1]$ is a bounded linear operator. Put $\varphi_{ij}(x)=$

$\overline{R}_{x_i}(x)\overrightarrow{e_j} = \begin{cases} (\overline{R}_{x_i}(x),0,0)^\top, j=1 \\ (0,\overline{R}_{x_i}(x),0)^\top, j=2 \\ (0,0,\overline{R}_{x_i}(x))^\top, j=3 \end{cases}$ and $\psi_{ij}(x) = A^*\varphi_{ij}(x)$, $i = 1,2,\cdots, j = 1,2,3$, where $\overline{R}_x(y)$ is the reproducing kernel of $W_2^1[0,1]$ and A^* is the adjoint operator of A .

The orthonormal system $\{\overline{\psi}_{ij}(x)\}_{(1,1)}^{(\infty,3)}$ of $W_2^3[0,1]\bigoplus W_2^3[0,1]\bigoplus W_2^3[0,1]$ can be derived from Gram-Schmidt orthogonalization process of $\{\psi_{ij}(x)\}_{(1,1)}^{(\infty,3)}$,

$$\overline{\psi}_{ij}(x) = \sum_{l=1}^{i}\sum_{k=1}^{j}\beta_{lk}^{ij}\psi_{lk}(x), i = 1,2,\cdots, j = 1,2,3. \tag{3.1}$$

Theorem 3.1. *For Eq.*(1.3)*, if* $\{x_i\}_{i=1}^{\infty}$ *is dense on* $[0,1]$*, then* $\{\psi_{ij}(x)\}_{(1,1)}^{(\infty,3)}$ *is the complete system of* $W_2^3[0,1]\bigoplus W_2^3[0,1]\bigoplus W_2^3[0,1]$.

Proof. For each fixed $U(x) \in W_2^3[0,1]\bigoplus W_2^3[0,1]\bigoplus W_2^3[0,1]$,
let $(U(x),\psi_{ij}(x)) = 0, (i = 1,2,\cdots)$, which means that,

$$(AU(x),\varphi_{ij}(x)) = 0. \tag{3.2}$$

Note that

$$U(x) = \sum_{j=1}^{3} U_j(x)\overrightarrow{e_j} = \sum_{j=1}^{3}(U(\cdot), R_x(\cdot)\overrightarrow{e_j})\overrightarrow{e_j}.$$

Hence, by (3.2), $AU(x_i) = \sum\limits_{j=1}^{3}(AU(y),\varphi_{ij}(y))\overrightarrow{e_j} = 0(i = 1,2,\cdots)$. Since $\{x_i\}_{i=1}^{\infty}$ is dense on $[0,1]$, we must have $(AU)(x) = 0$. It follows that $u \equiv 0$ from the existence of L^{-1}. Therefore, $\{\psi_{ij}(x)\}_{(1,1)}^{(\infty,3)}$ is the complete system of $W_2^3[0,1]\bigoplus W_2^3[0,1]\bigoplus W_2^3[0,1]$. So the proof of the Theorem 3.1 is complete. □

Theorem 3.2. *If* $\{x_i\}_{i=1}^{\infty}$ *is dense on* $[0,1]$ *and the solution of Eq.*(1.3) *is unique, then the solution of Eq.*(1.3) *satisfies the form*

$$U(x) = \sum_{i=1}^{\infty}\sum_{j=1}^{3}\sum_{l=1}^{i}\sum_{k=1}^{j}\beta_{lk}^{ij}F_k(x_l)\overline{\psi}_{ij}(x), \tag{3.3}$$

where $F(x) = (f(x),0,0)^\top = (F_1(x),F_2(x),F_3(x))^\top$.

Proof. Applying Theorem 3.1, it is easy to see that $\{\overline{\psi}_{ij}(x)\}_{(1,1)}^{(\infty,3)}$ is the complete orthonormal basis of $W_2^3[0,1]\bigoplus W_2^3[0,1]\bigoplus W_2^3[0,1]$.

Note that $(V(x), \varphi_{ij}(x)) = V_j(x_i)$ for each $V(x) \in W_2^1[0,1] \bigoplus W_2^1[0,1] \bigoplus W_2^1[0,1]$. Hence, we have

$$\begin{aligned} U(x) &= \sum_{i=1}^{\infty}\sum_{j=1}^{3}(U(x), \overline{\psi}_{ij}(x))\overline{\psi}_{ij}(x) \\ &= \sum_{i=1}^{\infty}\sum_{j=1}^{3}(U(x), \sum_{l=1}^{i}\sum_{k=1}^{j}\beta_{lk}^{ij}\psi_{lk}(x))\overline{\psi}_{ij}(x) \\ &= \sum_{i=1}^{\infty}\sum_{j=1}^{2}\sum_{l=1}^{i}\sum_{k=1}^{j}\beta_{lk}^{ij}(U(x), A^*\varphi_{lk}(x))\overline{\psi}_{ij}(x) \\ &= \sum_{i=1}^{\infty}\sum_{j=1}^{3}\sum_{l=1}^{i}\sum_{k=1}^{j}\beta_{lk}^{ij}(AU(x), \varphi_{lk}(x))\overline{\psi}_{ij}(x) \\ &= \sum_{i=1}^{\infty}\sum_{j=1}^{3}\sum_{l=1}^{i}\sum_{k=1}^{j}\beta_{lk}^{ij}F_k(x_l)\overline{\psi}_{ij}(x) \end{aligned} \tag{3.4}$$

and the proof of the theorem is complete. □

Now, the approximation solution $U_n(x)$ can be obtained by the n-term intercept of the analytical solution $U(x)$and

$$U_n(x) = \sum_{i=1}^{n}\sum_{j=1}^{3}\sum_{l=1}^{i}\sum_{k=1}^{j}\beta_{lk}^{ij}F_k(x_l)\overline{\psi}_{ij}(x). \tag{3.5}$$

Remark: From (3.4),(3.5), the analytical and approximation solutions of *Eq.*(1.1) and their second derivative and forth derivative can be obtained.

4 Test example

Consider the sixth-order singular BVP

$$\begin{cases} 15x(1-x)u^{(6)} + xu^{(5)} + (25+e^{2x})u^{(4)} + (5+e^x)u'' + xu = f(x), 0 \le x \le 1, \\ u(0) = u(1) = 0,\ u''(0) = u(''1) = 0,\ u^{(4)}(0) = u^{(4)}(1) = 0, \end{cases}$$

where $f(x) = \pi^2 x\cos(\pi x) + (-(5+e^x)\pi^2 + (25+e^{2x})\pi^4 + x + 15\pi^6(x-1)x)\sin(\pi x)$. The true solutions $u(x)$ is $\sin(\pi x)$. Using our method, we choose 100 points on $[0,1]$ and obtain approximate solutions $u_{100}(x)$, $u''_{100}(x)$, $u^{(4)}_{100}(x)$ on $[0,1]$. The numerical results are given in the following Tables 1,2,3.

5 Conclusion

A new method of solving singular sixth-order BVPs is developed, and the solution is given analytically. Moreover, the second derivative $u''(x)$ and forth derivative $u^{(4)}(x)$ of analytical solution $u(x)$ can be also obtained using this method. From the above test example, we can see that the method is valid.

Table 1: The numerical results for u_{100}(x)

x	True solution u(x)	Approximate solution u_{100}	Relative error
0.08	0.248690	0.248499	0.00076
0.24	0.684547	0.684024	0.00076
0.40	0.951057	0.950336	0.00075
0.56	0.982287	0.981554	0.00075
0.72	0.770513	0.769952	0.00072
0.88	0.368125	0.367867	0.00070
0.96	0.125333	0.125248	0.00068

Table 2: The numerical results for u''_{100}(x)

x	True solution u''(x)	Approximate solution u''_{100}	Relative error
0.08	-2.45447	-2.45275	0.00070
0.24	-6.75621	-6.75159	0.00068
0.40	-9.38655	-9.38023	0.00067
0.56	-9.69479	-9.68826	0.00067
0.72	-7.60466	-7.59946	0.00068
0.88	-3.63324	-3.63067	0.00070
0.96	-1.23699	-1.23610	0.00072

Table 3: The numerical results for $u^{(4)}_{100}$(x)

x	True solution $u^{(4)}$(x)	Approximate solution $u^{(4)}_{100}$	Relative error
0.08	24.2247	24.2092	0.00063
0.24	66.6811	66.6416	0.00059
0.40	92.6416	92.5891	0.00056
0.56	95.6837	95.6304	0.00055
0.72	75.055	75.0129	0.00055
0.88	35.8587	35.8385	0.00056
0.96	12.2086	12.2022	0.00052

References

[1] J.Toomre, J.R.zahn, J.Latour, E.A.Spiegel, stellar convection theory Π:Single-mode study of the second convection zone in A-type stars, Astophys.J.207(1976)545-563.

[2] G.A.Glatzmaier, Numeircal simulations of stellar convection dynamics at the base of the convection zone, Geophys.Fluid Dynamics 31(1985)137-150.

[3] S.Chandrasekhar, Hydrodynamic and Hydromagnetic stability, Clarendon Press, Oxford, 1961(Reprinted:Dover Books, New York,1981).

[4] R.P.Agarwal, Boundary Value Problems for Higher-Order Differential Equations, World Scientific, Singapore, 1986.

[5] M.R.Scott, H.A.Watts, Computational solution of linear two-point boundary value problem via orthonormalization, SIAM J.Numer.Anal.14(1977)40-70.

[6] M.R.Scott, H.A.Watts, A systematized collection of codes for solving two-point boundary value problems, Numerical Methods of Differential Systems, Academic Press,1976.

[7] A.M.Wazwaz, A reliable modification of Adomian's decomposition method, Appl.Math.Comput. 102(1999)77-86.

[8] A.M.Wazwaz, The numerical solution of sixth-order boundary value problems by the modified decomposition method, Appl.Math.Comput.118(2001)311-325.

[9] G. Akram, S.S. Siddiqi, Solution of sixth order boundary value problems using non-polynomal spline technique. Appl.Math.Comput.118(2006)708-720.

[10] J.H. He, Variational approach to the sixth-order boundary value problems, Appl.Math.Comput.143(2003)537-538.

[11] F.Z. Geng, M.G. Cui, Solving a nonlinear system of second order boundary value problems, J.Math.Anal.Appl.327(2007),1167-1181.

[12] F.Z., M.G. Cui, Solving singular nonlinear second-order periodic boundary value problems in the reproducing kernel space, Appl.Math.Compu. 192 (2007) 389-398.

[13] C.L. Li, M.G. Cui, The exact solution for solving a class nonlinear operator equations in the reproducing kernel space. Appl.Math.Compu.143(2-3)(2003)393-399.

Received: April, 2009