Int. Journal of Math. Analysis, Vol. 3, 2009, no. 37, 1845 - 1851

A Note on the $L-(p,q)$th Order of the Derivative of a Meromorphic Function

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Abstract

A single valued function of one complex variable which has no singularities other than poles in the finite complex plane is called a meromorphic function. In this paper we will show that the $L-(p,q)$th order of the derivative of a meromorphic function is the same as that of the original function where p,q are positive integers with $p>q$ and $L=L(r)$ is a slowly changing function.

Mathematics Subject Classification: 30D35

Keywords: Meromorphic function, (p,q)th order, $L-(p,q)$th order, slowly changing function, derivative

1 Introduction, Definitions and Notations.

It is well known $\{[9], p.36\}$ that the order of the derivative of an entire function is equal to the order of the function.The same result is proved for a meromorphic function in $\{[1],[8],[10]\}$. Juneja, Kapoor and Bajpai [4] defined the (p,q)*th order* and (p,q)*th lower order* of an entire function f respectively as follows:

$$\rho_f(p,q)=\limsup_{r\to\infty}\frac{\log^{[p]}M(r,f)}{\log^{[q]}r} \quad \text{and} \quad \lambda_f(p,q)=\liminf_{r\to\infty}\frac{\log^{[p]}M(r,f)}{\log^{[q]}r},$$

where p, q are positive integers and $p > q$.

When f is meromorphic, one can easily verify that

$$\rho_f(p,q) = \limsup_{r\to\infty} \frac{\log^{[p-1]} T(r,f)}{\log^{[q]} r} \quad \textit{and} \quad \lambda_f(p,q) = \liminf_{r\to\infty} \frac{\log^{[p-1]} T(r,f)}{\log^{[q]} r},$$

where $\log^{[k]} x = \log(\log^{[k-1]} x)$, for $k = 1, 2, 3, ...$ and $\log^{[0]} x = x$.

Somasundaram and Thamizharasi [7] introduced the notion of $L-order$ and $L-type$ for entire functions where $L = L(r)$ is a positive continuous function increasing slowly *i.e.*, $L(ar) \sim L(r)$ as $r \to \infty$ for every constant $'a'$. Their definitions are as follows:

Definition 1. [7] *The $L-order$ ρ_f^L and $L-lower\ order$ λ_f^L of an entire function f are defined as follows:*

$$\rho_f^L = \limsup_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log [rL(r)]} \quad \textit{and} \quad \lambda_f^L = \liminf_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log [rL(r)]}.$$

When f is meromorphic, then

$$\rho_f^L = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log [rL(r)]} \quad \textit{and} \quad \lambda_f^L = \liminf_{r\to\infty} \frac{\log T(r,f)}{\log [rL(r)]}.$$

So with the help of the above notion one can easily define the $L-(p,q)th$ order and $L-(p,q)th$ lower order of entire and meromorphic functions.

Definition 2. *The $L-(p,q)th$ order $\rho_f^L(p,q)$ and $L-(p,q)th$ lower order $\lambda_f^L(p,q)$ of an entire function f are respectively defined as :*

$$\rho_f^L(p,q) = \limsup_{r\to\infty} \frac{\log^{[p+1]} M(r,f)}{\log^{[q]} [rL(r)]} \quad \textit{and} \quad \lambda_f^L(p,q) = \liminf_{r\to\infty} \frac{\log^{[p+1]} M(r,f)}{\log^{[q]} [rL(r)]},$$

where p, q are positive integers and $p > q$.

When f is meromorphic, one can easily verify that

$$\rho_f^L(p,q) = \limsup_{r\to\infty} \frac{\log^{[p]} T(r,f)}{\log^{[q]} [rL(r)]} \quad \textit{and} \quad \lambda_f^L(p,q) = \liminf_{r\to\infty} \frac{\log^{[p]} T(r,f)}{\log^{[q]} [rL(r)]}.$$

where p, q are positive integers and $p > q$.

The more generalised concept of $L-(p,q)th$ *order* and $L-(p,q)th$ *lower order* of entire and meromorphic functions are $L^*-(p,q)th$ *order* and $L^*-(p,q)th$ *lower order* respectively. In order to prove our results we require the following definitions:

Definition 3. *The $L^* -$ order, $L^* -$ lower order and $L^* -$ type of a meromorphic function f are defined by*

$$\rho_f^{L^*} = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log\left[re^{L(r)}\right]}, \quad \lambda_f^{L^*} = \liminf_{r\to\infty} \frac{\log T(r,f)}{\log\left[re^{L(r)}\right]}$$

$$\textit{and} \quad \sigma_f^{L^*} = \limsup_{r\to\infty} \frac{T(r,f)}{\left[re^{L(r)}\right]^{\rho_f^{L^*}}}, \quad 0 < \rho_f^{L^*} < \infty.$$

When f is entire, one can easily verify that

$$\rho_f^{L^*} = \limsup_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log\left[re^{L(r)}\right]}, \quad \lambda_f^{L^*} = \liminf_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log\left[re^{L(r)}\right]}$$

$$\textit{and} \quad \sigma_f^{L^*} = \limsup_{r\to\infty} \frac{\log M(r,f)}{\left[re^{L(r)}\right]^{\rho_f^{L^*}}}, \quad 0 < \rho_f^{L^*} < \infty.$$

Definition 4. *The $L^* - (p,q)th$ order $\rho_f^{L^*}(p,q)$ and $L^* - (p,q)th$ lower order $\lambda_f^{L^*}(p,q)$ of an entire function f are defined as*

$$\rho_f^{L^*}(p,q) = \limsup_{r\to\infty} \frac{\log^{[p+1]} M(r,f)}{\log^{[q]}\left[re^{L(r)}\right]} \quad \textit{and} \quad \lambda_f^{L^*}(p,q) = \liminf_{r\to\infty} \frac{\log^{[p+1]} M(r,f)}{\log^{[q]}\left[re^{L(r)}\right]},$$

where p, q are positive integers and $p > q$.

When f is meromorphic, one can easily verify that

$$\rho_f^{L^*}(p,q) = \limsup_{r\to\infty} \frac{\log^{[p]} T(r,f)}{\log^{[q]}\left[re^{L(r)}\right]} \quad \textit{and} \quad \lambda_f^{L^*}(p,q) = \liminf_{r\to\infty} \frac{\log^{[p]} T(r,f)}{\log^{[q]}\left[re^{L(r)}\right]},$$

where p, q are positive integers and $p > q$.

In [5] and [6] Lahiri proved that the generalised order (generalised lower order) of a meromorphic function f is equal to the generalised order of its derivative f'. In the paper we establish a relationship between the $L - (p,q)th$ *order* of the derivative of a meromorphic function and that of the original function. We do not explain the standard notations and definitions in the theory of entire and meromorphic functions because those are available in [9] and [3].

2 Lemmas.

In this section we present some lemmas which will be needed in the sequel.

Lemma 1. [6]*Let f be a transcendental meromorphic function. Then* $T(r, f^{'}) \leq 2T(2r, f) + \circ \{T(2r, f)\}$ *for all large values of* r.

Lemma 2. $\{$*Theorem* $4.1, [11]$; *see also Lemma* $C, [2]\}$. *Let f be a meromorphic function. Then for all large r,* $T(r, f) < C \left\{T\left(2r, f^{'}\right) + \log r\right\}$, *where C is a constant which is only dependent on $f(0)$.*

3 Theorems.

In this section we present the main results of the paper.

Theorem 1. *The $L-(p, q)$th order of a meromorphic function f is equal to the $L-(p, q)$th order of its derivative $f^{'}$ where p, q are positive integers and $p > q$.*

Proof. We suppose that f is a transcendental meromorphic function because otherwise the theorem follows easily. From Lemma 1 we get by taking logarithms $(p-1)$ *times*

$$\log^{[p-1]} T(r, f^{'}) \leq \log^{[p-1]} T(2r, f) + O(1)$$

which gives that

$$\begin{aligned}
\rho^{L}_{f^{'}}(p, q) &\leq \limsup_{r\to\infty} \left\{ \frac{\log^{[p-1]} T(r, f)}{\log^{[q]}[rL(r)]} . \frac{1}{1 - \frac{\log 2}{\log^{[q]}[rL(r)]}} \right\} \\
&= \limsup_{r\to\infty} \frac{\log^{[p-1]} T(r, f)}{\log^{[q]}[rL(r)]} . \lim_{r\to\infty} \frac{1}{1 - \frac{\log 2}{\log^{[q]}[rL(r)]}} \\
&= \rho^{L}_{f}(p, q). \qquad (1)
\end{aligned}$$

Since f is transcendental, we have $\log r = \circ \{T(r, f)\}$. From Lemma 2 we obtain by taking repeated logarithms

$$\log^{[p-1]} T(r, f) + O(1) \leq \log^{[p-1]} T(2r, f^{'})$$

which gives

$$\begin{aligned}
\rho^{L}_{f}(p, q) &\leq \limsup_{r\to\infty} \frac{\log^{[p-1]} T(r, f^{'})}{\log^{[q]}[rL(r)]} . \lim_{r\to\infty} \frac{1}{1 - \frac{\log 2}{\log^{[q]}[rL(r)]}} \\
\text{i.e.,} \quad \rho^{L}_{f}(p, q) &\leq \rho^{L}_{f^{'}}(p, q). \qquad (2)
\end{aligned}$$

Thus the theorem follows from (1) and (2).

Remark 1. *Theorem* 1 *is an analogue of Theorem* 1 [6].

Theorem 2. *The $L-(p,q)$th lower order of a meromorphic function f is equal to the $L-(p,q)$th lower order of its derivative $f^{'}$ where p,q are positive integers and $p>q$.*

We omit the proof of Theorem 2 as it is similar to that of Theorem 1.

Theorem 3. *If f is a transcendental meromorphic function having a finite number of zeros with $f(0)\neq 0,\infty$; $f^{'}(0)\neq 0$ and $\rho_f^L(2,1)<\infty$, then $\rho_{f'}^L(p,q)=\rho_f^L(p,q)$ and $\lambda_{f'}^L(p,q)=\lambda_f^L(p,q)$ where p,q are positive integers and $p>q$.*

Proof. From {Theorem 2.2, [3], p.40} we know that $m(r,\frac{f^{'}}{f})=O(\log r)$. Also by {Theorem 2.3, [3], p.41} we obtain in the present case,

$$\log r=o\left\{T(r,f)\right\} \quad as \quad r\to\infty.$$

So combining the two

$$m(r,\frac{f^{'}}{f})=o\left\{T(r,f)\right\} \quad as \quad r\to\infty.$$

Since f has a finite number of zeros, it is clear that $N(r,\frac{1}{f})=O(\log r)$. Hence,

$$N(r,\frac{1}{f})=o\left\{T(r,f)\right\} \ as\ r\to\infty.$$

Now

$$\begin{aligned} m(r,f^{'}) &\leq m(r,\frac{f^{'}}{f})+m(r,f) \\ i.e., \quad m(r,f^{'}) &\leq m(r,f)+o\left\{T(r,f)\right\} \quad as \quad r\to\infty. \end{aligned}$$

Also if f has a pole of order p at z_o, $f^{'}(z)$ has a pole of order $p+1\leq 2p$, so that $N(r,f^{'})\leq 2N(r,f)$ {p.56, [3]} . Thus by addition we deduce that

$$\begin{aligned} T(r,f^{'}) &\leq m(r,f)+2N(r,f)+o\left\{T(r,f)\right\} \\ i.e., \quad T(r,f^{'}) &\leq 2T(r,f)+o\left\{T(r,f)\right\} \\ i.e., \quad T(r,f^{'}) &\leq \left\{2+o(1)\right\}T(r,f) \quad as \quad r\to\infty. \end{aligned} \tag{3}$$

This gives that

$$\rho_{f'}^L(p,q)\leq\rho_f^L(p,q). \tag{4}$$

Again we have

$$
\begin{aligned}
T(r,f) &= m(r,\frac{1}{f})+N(r,\frac{1}{f})+O(1) \\
i.e., \quad T(r,f) &\leq m(r,\frac{1}{f'})+m(r,\frac{f'}{f})+N(r,\frac{1}{f})+O(1) \\
i.e., \quad T(r,f) &\leq m(r,\frac{1}{f'})+o\left\{T(r,f)\right\} \\
i.e., \quad T(r,f) &\leq T(r,\frac{1}{f'})+o\left\{T(r,f)\right\} \\
i.e., \quad T(r,f) &\leq T(r,f')+o\left\{T(r,f)\right\} \quad as \quad r\to\infty
\end{aligned}
$$

$$
i.e., \quad \left\{1+o(1)\right\}T(r,f)\leq T(r,f') \quad as \quad r\to\infty. \tag{5}
$$

This gives that

$$
\rho_f^L(p,q)\leq\rho_{f'}^L(p,q). \tag{6}
$$

Thus the first part of the theorem follows from (4) and (6).

Similarly, $\lambda_{f'}^L(p,q)=\lambda_f^L(p,q)$.

This proves the theorem.

Remark 2. *Theorem* 3 *is an analogue of Theorem* 2 [5].

Remark 3. *Theorem* 3 *can also be proved with a lesser hypothesis* '$N(r,\frac{1}{f})=O(\log r)$' *than 'having a finite number of zeros'.*

In the paper we also state a relationship without proof between the $L^*-(p,q)$th order of the derivative of a meromorphic function and that of the original function.

Theorem 4. *The $L^*-(p,q)$th order of a meromorphic function f is equal to the $L^*-(p,q)$th order of its derivative f' where p,q are positive integers and $p>q$.*

Theorem 5. *The $L^*-(p,q)$th lower order of a meromorphic function f is equal to the $L^*-(p,q)$th lower order of its derivative f' where p,q are positive integers and $p>q$.*

The proof is omitted.

Theorem 6. *If f is a transcendental meromorphic function having a finite number of zeros with $f(0)\neq 0,\infty$; $f'(0)\neq 0$ and $\rho_f^{L^*}(2,1)<\infty$, then $\rho_{f'}^{L^*}(p,q)=\rho_f^{L^*}(p,q)$ and $\lambda_{f'}^{L^*}(p,q)=\lambda_f^{L^*}(p,q)$ where p,q are positive integers and $p>q$.*

The proof of Theorem 6 is omitted because it can be carried out in the line of Theorem 3.

Remark 4. *Theorem* 6 *can also be proved with a lesser hypothesis* '$N(r,\frac{1}{f})=O(\log r)$' *than 'having a finite number of zeros'.*

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Received: April, 2009