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Growth Estimates of Entire Functions Based on Relative $L-(p,q)$th Order

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Abstract

In the paper we study the comparative growth properties of entire functions on the basis of relative $L-(p,q)$th order where p,q are positive integers with $p>q$ and $L=L(r)$ is a slowly changing function.

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1 Introduction, Definitions and Notations.

Let f and g be two entire functions and $F(r)=\max\{|f(z)|:|z|=r\}$, $G(r)=\max\{|g(z)|:|z|=r\}$. If f is non constant then $F(r)$ is strictly increasing and continuous and its inverse $F^{-1}:(|f(0)|,\infty)\to(0,\infty)$ exists and is such that $\lim\limits_{s\to\infty}F^{-1}(s)=\infty$.

Bernal [1] introduced the definition of relative order of f with respect to g, denoted by $\rho_g(f)$ as follows:

$$\begin{aligned}\rho_g(f) &= \inf\{\mu>0:F(r)<G(r^{\mu}) \quad for\ all\ \ r>r_0(\mu)>0\}\\ &= \limsup_{r\to\infty}\frac{\log G^{-1}F(r)}{\log r}.\end{aligned}$$

The definition coincides with the classical one [5] if $g(z) = \exp z$. Similarly one can define the relative lower order of f with respect to g denoted by $\lambda_g(f)$ as follows:

$$\lambda_g(f) = \liminf_{r\to\infty} \frac{\log G^{-1}F(r)}{\log r}.$$

Somasundaram and Thamizharasi [4] introduced the notions of $L-order, L-lower\ order$ and $L-type$ for entire functions where $L = L(r)$ is a positive continuous function increasing slowly *i.e.*, $L(ar) \sim L(r)$ as $r \to \infty$ for every positive constant $'a'$. Their definitions are as follows:

Definition 1. [4] *The $L-order$ ρ_f^L and the $L-lower\ order$ λ_f^L of an entire function f are defined as follows:*

$$\rho_f^L = \limsup_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log\,[rL(r)]} \quad and \quad \lambda_f^L = \liminf_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log\,[rL(r)]},$$

where $\log^{[k]} x = \log(\log^{[k-1]} x)$ *for* $k = 1, 2, 3, ...$ *and* $\log^{[0]} x = x$.

When f is meromorphic, then

$$\rho_f^L = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log\,[rL(r)]} \quad and \quad \lambda_f^L = \liminf_{r\to\infty} \frac{\log T(r,f)}{\log\,[rL(r)]}.$$

Definition 2. [4] *The $L-type$ σ_f^L of an entire function f with $L-order$ ρ_f^L is defined as*

$$\sigma_f^L = \limsup_{r\to\infty} \frac{\log M(r,f)}{[rL(r)]^{\rho_f^L}}, \quad 0 < \rho_f^L < \infty.$$

For meromorphic f, the $L-type$ σ_f^L becomes

$$\sigma_f^L = \limsup_{r\to\infty} \frac{T(r,f)}{[rL(r)]^{\rho_f^L}}, \quad 0 < \rho_f^L < \infty.$$

Juneja, Kapoor and Bajpai [3] defined the $(p,q)th\ order$ and the $(p,q)th\ lower\ order$ of an entire function f respectively as follows:

$$\begin{aligned} \rho_f(p,q) &= \limsup_{r\to\infty} \frac{\log^{[p+1]} M(r,f)}{\log^{[q]} r} \\ and \quad \lambda_f(p,q) &= \liminf_{r\to\infty} \frac{\log^{[p+1]} M(r,f)}{\log^{[q]} r}. \end{aligned}$$

When f is meromorphic, one can easily verify that

$$\begin{aligned} \rho_f(p,q) &= \limsup_{r\to\infty}\frac{\log^{[p]} T(r,f)}{\log^{[q]} r} \\ \text{and} \quad \lambda_f(p,q) &= \liminf_{r\to\infty}\frac{\log^{[p]} T(r,f)}{\log^{[q]} r}, \end{aligned}$$

where p, q are positive integers and $p > q$.

So with the help of the above notion one can easily define the *relative* $L-(p,q)th$ *order* and *relative* $L-(p,q)th$ *lower order* of entire functions.

Definition 3. *The relative $L-(p,q)th$ order and relative $L-(p,q)th$ lower order of an entire function f with respect to another entire function g respectively denoted by ${}^L\rho_g^f(p,q)$ and ${}^L\lambda_g^f(p,q)$ are defined as*

$$\begin{aligned} {}^L\rho_g^f(p,q) &= \limsup_{r\to\infty}\frac{\log^{[p]} G^{-1}F(r)}{\log^{[q]}[rL(r)]} \\ \text{and} \quad {}^L\lambda_g^f(p,q) &= \liminf_{r\to\infty}\frac{\log^{[p]} G^{-1}F(r)}{\log^{[q]}[rL(r)]}, \end{aligned}$$

where p, q are positive integers and $p > q$.

The more generalised concept of $L-order$ and $L-type$ of entire and meromorphic functions are $L^*-order$ and L^*-type respectively. Their definitions are as follows:

Definition 4. *The $L^*-order$, $L^*-lower\ order$ and L^*-type of a meromorphic function f are defined by*

$$\rho_f^{L^*} = \limsup_{r\to\infty}\frac{\log T(r,f)}{\log\left[re^{L(r)}\right]}, \quad \lambda_f^{L^*} = \liminf_{r\to\infty}\frac{\log T(r,f)}{\log\left[re^{L(r)}\right]}$$

$$\text{and} \quad \sigma_f^{L^*} = \limsup_{r\to\infty}\frac{T(r,f)}{\left[re^{L(r)}\right]^{\rho_f^{L^*}}}, \quad 0 < \rho_f^{L^*} < \infty.$$

When f is entire, one can easily verify that

$$\rho_f^{L^*} = \limsup_{r\to\infty}\frac{\log^{[2]} M(r,f)}{\log\left[re^{L(r)}\right]}, \quad \lambda_f^{L^*} = \liminf_{r\to\infty}\frac{\log^{[2]} M(r,f)}{\log\left[re^{L(r)}\right]}$$

$$\text{and} \quad \sigma_f^{L^*} = \limsup_{r\to\infty}\frac{\log M(r,f)}{\left[re^{L(r)}\right]^{\rho_f^{L^*}}}, \quad 0 < \rho_f^{L^*} < \infty.$$

Definition 5. *The relative $L^* - (p,q)th$ order ${}^{L^*}\rho_g^f(p,q)$ and the relative $L^* - (p,q)th$ lower order ${}^{L^*}\lambda_g^f(p,q)$ of an entire function f with respect to another entire function g are defined as*

$$\begin{aligned} {}^{L^*}\rho_g^f(p,q) &= \limsup_{r\to\infty}\frac{\log^{[p]} G^{-1}F(r)}{\log^{[q]}\left[re^{L(r)}\right]} \\ \text{and} \quad {}^{L^*}\lambda_g^f(p,q) &= \liminf_{r\to\infty}\frac{\log^{[p]} G^{-1}F(r)}{\log^{[q]}\left[re^{L(r)}\right]}, \end{aligned}$$

where p, q are positive integers and $p > q$.

In the paper we establish some results on the growth properties of entire functions on the basis of *relative $L - (p,q)th$ order* and *relative $L - (p,q)th$ lower order* where p, q are positive integers with $p > q$. We do not explain the standard definitions and notations in the theory of entire and meromorphic functions as those are available in [6] and [2].

2 Theorems.

In this section we present the main results of the paper.

In the following theorems we see the application of *relative $L - (p,q)th$ order* and *relative $L-(p,q)th$ lower order* in estimating the growth properties of entire functions where p, q are positive integers with $p > q$.

Theorem 1. *Let f, g and h be three entire functions such that $0 < {}^L\lambda_g^f(p,q) \le {}^L\rho_g^f(p,q) < \infty$ and $0 < {}^L\lambda_g^h(m,q) \le {}^L\rho_g^h(m,q) < \infty$. Then*

$$\frac{{}^L\lambda_g^f(p,q)}{{}^L\rho_g^h(m,q)} \le \liminf_{r\to\infty}\frac{\log^{[p]} G^{-1}F(r)}{\log^{[m]} G^{-1}H(r)} \le \frac{{}^L\lambda_g^f(p,q)}{{}^L\lambda_g^h(m,q)}$$

$$\le \limsup_{r\to\infty}\frac{\log^{[p]} G^{-1}F(r)}{\log^{[m]} G^{-1}H(r)} \le \frac{{}^L\rho_g^f(p,q)}{{}^L\lambda_g^h(m,q)},$$

where p, q, m are positive integers with $q < \min\{p, m\}$.

Proof. From the definition of relative $L - (p,q)th$ *order* and *relative $L - (p,q)th$ lower order* we have for arbitrary positive ϵ and for all large values of r,

$$\log^{[p]} G^{-1}F(r) \ge ({}^L\lambda_g^f(p,q) - \epsilon)\log^{[q]}\left[rL(r)\right] \tag{1}$$

$$\text{and} \quad \log^{[m]} G^{-1}H(r) \le ({}^L\rho_g^h(m,q) + \epsilon)\log^{[q]}\left[rL(r)\right]. \tag{2}$$

Now from (1) and (2) it follows for all large values of r,

$$\frac{\log^{[p]} G^{-1}F(r)}{\log^{[m]} G^{-1}H(r)} \geq \frac{{}^L\lambda_g^f(p,q)-\epsilon}{{}^L\rho_g^h(m,q)+\epsilon}.$$

As $\epsilon\ (>0)$ is arbitrary, we obtain that

$$\liminf_{r\to\infty}\frac{\log^{[p]} G^{-1}F(r)}{\log^{[m]} G^{-1}H(r)} \geq \frac{{}^L\lambda_g^f(p,q)}{{}^L\rho_g^h(m,q)}. \tag{3}$$

Again for a sequence of values of r tending to infinity,

$$\log^{[p]} G^{-1}F(r) \leq ({}^L\lambda_g^f(p,q)+\epsilon)\log^{[q]}[rL(r)] \tag{4}$$

and for all large values of r,

$$\log^{[m]} G^{-1}H(r) \geq ({}^L\lambda_g^h(m,q)-\epsilon)\log^{[q]}[rL(r)]\,. \tag{5}$$

So combining (4) and (5) we get for a sequence of values of r tending to infinity,

$$\frac{\log^{[p]} G^{-1}F(r)}{\log^{[m]} G^{-1}H(r)} \leq \frac{{}^L\lambda_g^f(p,q)+\epsilon}{{}^L\lambda_g^h(m,q)-\epsilon}.$$

Since $\epsilon\ (>0)$ is arbitrary it follows that

$$\liminf_{r\to\infty}\frac{\log^{[p]} G^{-1}F(r)}{\log^{[m]} G^{-1}H(r)} \leq \frac{{}^L\lambda_g^f(p,q)}{{}^L\lambda_g^h(m,q)}. \tag{6}$$

Also for a sequence of values of r tending to infinity,

$$\log^{[m]} G^{-1}H(r) \leq ({}^L\lambda_g^h(m,q)+\epsilon)\log^{[q]}[rL(r)]\,. \tag{7}$$

Now from (1) and (7) we obtain for a sequence of values of r tending to infinity,

$$\frac{\log^{[p]} G^{-1}F(r)}{\log^{[m]} G^{-1}H(r)} \geq \frac{{}^L\lambda_g^f(p,q)-\epsilon}{{}^L\lambda_g^h(m,q)+\epsilon}.$$

Choosing $\epsilon\to 0$ we get that

$$\limsup_{r\to\infty}\frac{\log^{[p]} G^{-1}F(r)}{\log^{[q]} G^{-1}H(r)} \geq \frac{{}^L\lambda_g^f(p,q)}{{}^L\lambda_g^h(m,q)}. \tag{8}$$

Also for all large values of r,

$$\log^{[p]} G^{-1}F(r) \leq ({}^L\rho_g^f(p,q)+\epsilon)\log^{[q]}[rL(r)]\,. \tag{9}$$

So from (5) and (9) it follows for all large values of r,

$$\frac{\log^{[p]} G^{-1}F(r)}{\log^{[m]} G^{-1}H(r)} \leq \frac{{}^{L}\rho_g^f(p,q)+\epsilon}{{}^{L}\lambda_g^h(m,q)-\epsilon}.$$

As ϵ (>0) is arbitrary we obtain that

$$\limsup_{r\to\infty}\frac{\log^{[p]} G^{-1}F(r)}{\log^{[m]} G^{-1}H(r)} \leq \frac{{}^{L}\rho_g^f(p,q)}{{}^{L}\lambda_g^h(m,q)}. \tag{10}$$

Thus the theorem follows from (3),(6),(8) and (10).

Theorem 2. *Let f, g and h be three entire functions with* $0 < {}^{L}\lambda_g^f(p,q) \leq {}^{L}\rho_g^f(p,q) < \infty$ *and* $0 < {}^{L}\rho_g^h(m,q) < \infty$, *where p,q,m are positive integers with* $q < \min\{p,m\}$. *Then*

$$\liminf_{r\to\infty}\frac{\log^{[p]} G^{-1}F(r)}{\log^{[m]} G^{-1}H(r)} \leq \frac{{}^{L}\rho_g^f(p,q)}{{}^{L}\rho_g^h(m,q)} \leq \limsup_{r\to\infty}\frac{\log^{[p]} G^{-1}F(r)}{\log^{[m]} G^{-1}H(r)}.$$

Proof. From the definition of relative $L-(p,q)th$ *order* we get for a sequence of values of r tending to infinity,

$$\log^{[m]} G^{-1}H(r) \geq ({}^{L}\rho_g^h(m,q)-\epsilon)\log^{[q]}[rL(r)]. \tag{11}$$

Now from (9) and (11) it follows for a sequence of values of r tending to infinity,

$$\frac{\log^{[p]} G^{-1}F(r)}{\log^{[m]} G^{-1}H(r)} \leq \frac{{}^{L}\rho_g^f(p,q)+\epsilon}{{}^{L}\rho_g^h(m,q)-\epsilon}.$$

As ϵ (>0) is arbitrary we obtain that

$$\liminf_{r\to\infty}\frac{\log^{[p]} G^{-1}F(r)}{\log^{[m]} G^{-1}H(r)} \leq \frac{{}^{L}\rho_g^f(p,q)}{{}^{L}\rho_g^h(m,q)}. \tag{12}$$

Again for a sequence of values of r tending to infinity,

$$\log^{[p]} G^{-1}F(r) \geq ({}^{L}\rho_g^f(p,q)-\epsilon)\log^{[q]}[rL(r)]. \tag{13}$$

So combining (2) and (13) we get for a sequence of values of r tending to infinity,

$$\frac{\log^{[p]} G^{-1}F(r)}{\log^{[m]} G^{-1}H(r)} \geq \frac{{}^{L}\rho_g^f(p,q)-\epsilon}{{}^{L}\rho_g^h(m,q)+\epsilon}.$$

Since $\epsilon\ (>0)$ is arbitrary it follows that

$$\limsup_{r\to\infty}\frac{\log^{[p]} G^{-1}F(r)}{\log^{[m]} G^{-1}H(r)} \geq \frac{{}^{L}\rho_g^f(p,q)}{{}^{L}\rho_g^h(m,q)}. \tag{14}$$

Thus the theorem follows from (12) and (14).

The following theorem is a natural consequence of Theorem 1 and Theorem 2.

Theorem 3. *Let f, g and h be three entire functions with $0 < {}^{L}\lambda_g^f(p,q) \leq {}^{L}\rho_g^f(p,q) < \infty$ and $0 < {}^{L}\lambda_g^h(m,q) \leq {}^{L}\rho_g^h(m,q) < \infty$ where p,q,m are positive integers with $q < \min\{p,m\}$. Then*

$$\begin{aligned}\liminf_{r\to\infty}\frac{\log^{[p]} G^{-1}F(r)}{\log^{[q]} G^{-1}H(r)} &\leq \min\left\{\frac{{}^{L}\lambda_g^f(p,q)}{{}^{L}\lambda_g^h(m,q)}, \frac{{}^{L}\rho_g^f(p,q)}{{}^{L}\rho_g^h(m,q)}\right\}\\ &\leq \max\left\{\frac{{}^{L}\lambda_g^f(p,q)}{{}^{L}\lambda_g^h(m,q)}, \frac{{}^{L}\rho_g^f(p,q)}{{}^{L}\rho_g^h(m,q)}\right\}\\ &\leq \limsup_{r\to\infty}\frac{\log^{[p]} G^{-1}F(r)}{\log^{[q]} G^{-1}H(r)}.\end{aligned}$$

The proof is omitted.

In the following theorems we see some comparative growth properties of entire functions on the basis of relative $L^*-(p,q)$th order and relative $L^*-(p,q)$th lower order where $L=L(r)$ is a slowly changing function and p,q are positive integers with $p>q$.

Theorem 4. *Let f, g and h be three entire functions such that $0 < {}^{L^*}\lambda_g^f(p,q) \leq {}^{L^*}\rho_g^f(p,q) < \infty$ and $0 < {}^{L^*}\lambda_g^h(m,q) \leq {}^{L^*}\rho_g^h(m,q) < \infty$ where p,q,m are positive integers with $q < \min\{p,m\}$. Then*

$$\frac{{}^{L^*}\lambda_g^f(p,q)}{{}^{L^*}\rho_g^h(m,q)} \leq \liminf_{r\to\infty}\frac{\log^{[p]} G^{-1}F(r)}{\log^{[m]} G^{-1}H(r)} \leq \frac{{}^{L^*}\lambda_g^f(p,q)}{{}^{L^*}\lambda_g^h(m,q)}$$

$$\leq \limsup_{r\to\infty}\frac{\log^{[p]} G^{-1}F(r)}{\log^{[m]} G^{-1}H(r)} \leq \frac{{}^{L^*}\rho_g^f(p,q)}{{}^{L^*}\lambda_g^h(m,q)}.$$

The proof is omitted because it can be carried out in the line of Theorem 1.

Theorem 5. *Let f, g and h be three entire functions with $0 < {}^{L^*}\lambda_g^f(p,q) \leq {}^{L^*}\rho_g^f(p,q) < \infty$ and $0 < {}^{L^*}\rho_g^h(m,q) < \infty$ where p, q, m are positive integers with $q < \min\{p, m\}$. Then*

$$\liminf_{r\to\infty} \frac{\log^{[p]} G^{-1}F(r)}{\log^{[m]} G^{-1}H(r)} \leq \frac{{}^{L^*}\rho_g^f(p,q)}{{}^{L^*}\rho_g^h(m,q)} \leq \limsup_{r\to\infty} \frac{\log^{[p]} G^{-1}F(r)}{\log^{[m]} G^{-1}H(r)}.$$

We omit the proof of Theorem 5 because it runs parallel to that of Theorem 2.

The following theorem is a natural consequence of Theorem 4 and Theorem 5.

Theorem 6. *Let f, g and h be three entire functions such that $0 < {}^{L^*}\lambda_g^f(p,q) \leq {}^{L^*}\rho_g^f(p,q) < \infty$ and $0 < {}^{L^*}\lambda_g^h(m,q) \leq {}^{L^*}\rho_g^h(m,q) < \infty$ where p, q, m are positive integers with $q < \min\{p, m\}$. Then*

$$\begin{aligned}
\liminf_{r\to\infty} \frac{\log^{[p]} G^{-1}F(r)}{\log^{[q]} G^{-1}H(r)} &\leq \min\left\{\frac{{}^{L^*}\lambda_g^f(p,q)}{{}^{L^*}\lambda_g^h(m,q)}, \frac{{}^{L^*}\rho_g^f(p,q)}{{}^{L^*}\rho_g^h(m,q)}\right\} \\
&\leq \max\left\{\frac{{}^{L^*}\lambda_g^f(p,q)}{{}^{L^*}\lambda_g^h(m,q)}, \frac{{}^{L^*}\rho_g^f(p,q)}{{}^{L^*}\rho_g^h(m,q)}\right\} \\
&\leq \limsup_{r\to\infty} \frac{\log^{[p]} G^{-1}F(r)}{\log^{[q]} G^{-1}H(r)}.
\end{aligned}$$

The proof is omitted.

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