Int. Journal of Math. Analysis, Vol. 3, 2009, no. 37, 1821 - 1834

On the Growth Estimate of Wronskians Generated by Transcendental Meromorphic Functions

Sanjib Kumar Datta

Department of Mathematics, University of North Bengal
Darjeeling, Pin-734013, West Bengal, India
sanjib_kr_datta@yahoo.co.in

Sanjib Mondal

Chaltia Sreeguru Pathsala High School
P.O.-Berhampore, Dist.-Murshidabad, PIN-742101
West Bengal, India
sanjib_mondal_math@yahoo.in

Abstract

In this paper we study the comparative growth of composite entire or meromorphic functions and Wronskians generated by one of the factors.

Mathematics Subject Classification: 30D35, 30D30

Keywords: Transcendental entire and meromorphic function, wronskian, composition, growth

1 Introduction, Definitions and Notations.

Let f and g be two transcendental entire functions defined in the open complex plane $\mathbb{C}$. It is well known that

$$\lim_{r\to\infty}\frac{M(r,f\circ g)}{M(r,f)}=\lim_{r\to\infty}\frac{M(r,f\circ g)}{M(r,g)}=\infty.$$

Clunie [3] discussed on the behaviour of

$$\frac{\log M(r,f\circ g)}{\log M(r,f)}\ \ and\ \ \frac{\log M(r,f\circ g)}{\log M(r,g)}\ \ as\ \ r\to\infty.$$

Song and Yang [9] worked on

$$\frac{\log^{[2]} M(r, f \circ g)}{\log^{[2]} M(r, f)} \ \ and \ \ \frac{\log^{[2]} M(r, f \circ g)}{\log^{[2]} M(r, g)} \ \ as \ \ r \to \infty$$

where $\log^{[k]} x = \log(\log^{[k-1]} x)$ for $k = 1, 2, 3, \dots$ and $\log^{[0]} x = x$.
Replacing maximum modulus functions by Nevanlinna's Characteristic functions Clunie [3] proved for any two transcendental entire functions defined in the open complex plane $\mathbb{C}$,

$$\lim_{r\to\infty} \frac{T(r, f \circ g)}{T(r, f)} = \infty \ \ and \ \ \lim_{r\to\infty} \frac{T(r, f \circ g)}{T(r, g)} = \infty.$$

Singh [10] proved some comparative growth properties of $\log T(r, f \circ g)$ and $T(r, f)$. He [10] also raised the problem of investigating the comparative growth of $\log T(r, f \circ g)$ and $T(r, g)$ and some results on the comparative growth of $\log T(r, f \circ g)$ and $T(r, g)$ are proved in [5]. Since $M(r, f)$ and $M(r, g)$ are increasing functions of r, Singh and Baloria [10] asked whether for any two entire functions f, g and for sufficiently large $R = R(r)$,

$$\limsup_{r\to\infty} \frac{\log^{[2]} M(r, f \circ g)}{\log^{[2]} M(R, f)} \ < \ \infty$$

$$and \ \ \limsup_{r\to\infty} \frac{\log^{[2]} M(r, f \circ g)}{\log^{[2]} M(R, g)} \ < \ \infty.$$

Singh and Baloria [11], Lahiri and Sharma [6], Liao and Yang [7] worked on this question. We do not explain the standard notations and definitions on the theory of entire and meromorphic functions because those are available in [12] and [4].

Since the natural extension of a derivative is a differential polynomial, in this paper we prove our results for a special type of linear differential polynomials viz., the wronskians. We investigate the comparative growths of composite entire and meromorphic functions and wronskians generated by their factors.

The following definitions are well known.

Definition 1. *The order ρ_f and lower order λ_f of a meromorphic function f is defined as*

$$\rho_f = \limsup_{r\to\infty} \frac{\log T(r, f)}{\log r} \ \ and \ \ \lambda_f = \liminf_{r\to\infty} \frac{\log T(r, f)}{\log r}.$$

If f is entire, one can easily verify that

$$\rho_f = \limsup_{r\to\infty} \frac{\log^{[2]} M(r, f)}{\log r} \ \ and \ \ \lambda_f = \liminf_{r\to\infty} \frac{\log^{[2]} M(r, f)}{\log r}.$$

Definition 2. *The type σ_f of a meromorphic function f is defined as follows*

$$\sigma_f = \limsup_{r\to\infty} \frac{T(r,f)}{r^{\rho_f}}, \quad 0 < \rho_f < \infty.$$

When f is entire, then

$$\sigma_f = \limsup_{r\to\infty} \frac{\log M(r,f)}{r^{\rho_f}}, \quad 0 < \rho_f < \infty.$$

Definition 3. *A meromorphic function $a = a(z)$ is called small with respect to f if $T(r,a) = S(r,f)$.*

Definition 4. *Let $a_1, a_2, ..., a_k$ be linearly independent meromorphic functions and small with respect to f. We denote by $L(f) = W(a_1, a_2, ...a_k, f)$ the Wronskian determinant of $a_1, a_2, ...a_k, f$ i.e.,*

$$L(f) = \begin{vmatrix} a_1 & a_2 & .. & .. & a_k & f \\ a_1' & a_2' & .. & .. & a_k' & f' \\ .. & .. & .. & .. & .. & .. \\ .. & .. & .. & .. & .. & .. \\ a_1^{(k)} & a_2^{(k)} & .. & .. & a_k^{(k)} & f^{(k)} \end{vmatrix}.$$

Definition 5. *If $a \,\epsilon\, \mathbb{C} \cup \{\infty\}$, the quantity*

$$\delta(a;f) = 1 - \limsup_{r\to\infty} \frac{N(r,a;f)}{T(r,f)} = \liminf_{r\to\infty} \frac{m(r,a;f)}{T(r,f)}$$

is called Nevanlianna deficiency of the value a.

From the second fundamental theorem it follows that the set of values of $a \,\epsilon\, \mathbb{C} \cup \{\infty\}$ for which $\delta(a;f) > 0$ is countable and

$$\sum_{a\neq\infty} \delta(a;f) + \delta(\infty;f) \leq 2.$$

$(cf.[4, p.43]$. If, in particular, $\sum\limits_{a\neq\infty} \delta(a;f) + \delta(\infty;f) = 2$, we say that f has the maximum deficiency sum.

2 Lemmas.

In this section we present some lemmas which will be needed in the sequel.

Lemma 1. [1] *If f is meromorphic and g is entire then for all sufficiently large values of r,*

$$T(r, f\circ g) \leq \{1 + o(1)\} \frac{T(r,g)}{\log M(r,g)} T(M(r,g), f).$$

Lemma 2. [2] *Let f be meromorphic and g be entire and suppose that $0 < \mu \leq \rho_g \leq \infty$. Then for a sequence of values of r tending to infinity,*

$$T(r, f \circ g) \geq T(\exp(r^{\mu}), f).$$

Lemma 3. [8] *Let f be a transcendental meromorphic function having the maximum deficiency sum. Then*

$$\lim_{r\to\infty} \frac{T(r, L(f))}{T(r, f)} = 1 + k - k\delta\left(\infty; f\right).$$

Lemma 4. *If f be a transcendental meromorphic function with the maximum deficiency sum, then the order and lower order of $L(f)$ are same as those of f.Also the type of $L(f)$ is $\{1 + k - k\delta\left(\infty; f\right)\}$ times that of f when f is of positive finite order.*

Proof. By Lemma 3, $\lim_{r\to\infty} \frac{\log T(r,L(f))}{\log T(r,f)}$ exists and is equal to 1. So

$$\begin{aligned} \rho_{L(f)} &= \limsup_{r\to\infty} \frac{\log T(r, f)}{\log r} . \lim_{r\to\infty} \frac{\log T(r, L(f))}{\log T(r, f)} \\ &= \rho_f .1 = \rho_f . \end{aligned}$$

In a similar manner, $\lambda_{L(f)} = \lambda_f$. Again

$$\begin{aligned} \sigma_{L(f)} &= \limsup_{r\to\infty} \frac{T(r, L(f))}{r^{\rho_{L(f)}}} \\ &= \limsup_{r\to\infty} \frac{T(r, f)}{r^{\rho_f}} . \limsup_{r\to\infty} \frac{T(r, L(f))}{T(r, f)} \\ &= \sigma_f . \left\{1 + k - k\delta\left(\infty; f\right)\right\} . \end{aligned}$$

This proves the lemma.

3 Theorems.

In this section we present the main results of the paper.

Theorem 1. *Let f be transcendental meromorphic and g be entire satisfying the following conditions:*
(i) ρ_f and ρ_g are both finite, (ii) λ_f is positive and (iii) $\sum_{a\neq\infty} \delta\left(a; f\right) + \delta\left(\infty; f\right) = 2$. Then for each $\alpha \in (-\infty, \infty)$,

$$\lim_{r\to\infty} \frac{\{\log T(r, f \circ g)\}^{1+\alpha}}{\log T(\exp(r^p), L(f))} = 0 \quad if \ p > (1 + \alpha)\rho_g.$$

Proof. If $1+\alpha \le 0$, the theorem is trivial. So we take $1+\alpha > 0$. Since $T(r,g) \le \log^+ M(r,g)$, by Lemma 1, we get for all sufficiently large values of r,

$$\begin{aligned} T(r, f\circ g) &\le \{1+o(1)\}T(M(r,g),f) \\ i.e.,\ \log T(r, f\circ g) &\le \log\{1+o(1)\} + \log T(M(r,g),f) \\ i.e.,\ \log T(r, f\circ g) &\le \{1+o(1)\} + (\rho_f+\epsilon)r^{\rho_g+\epsilon} \\ i.e.,\ \log T(r, f\circ g) &\le r^{\rho_g+\epsilon}\{(\rho_f+\epsilon)+o(1)\}. \end{aligned} \tag{1}$$

Again for all sufficiently large values of r, we obtain by Lemma 4,

$$\begin{aligned} \log T(\exp(r^{p'}), L(f)) &\ge (\lambda_{L(f)}-\epsilon)\log(\exp(r^{p'})) \\ i.e.,\quad \log T(\exp(r^{p'}), L(f)) &\ge (\lambda_f-\epsilon)r^{p'}. \end{aligned} \tag{2}$$

Now from (1) and (2) we get for all large values of r,

$$\frac{\{\log T(r, f\circ g)\}^{1+\alpha}}{\log T(\exp(r^{p'}), L(f))} \le \frac{r^{(\rho_g+\epsilon)(1+\alpha)}\{(\rho_f+\epsilon)+o(1)\}^{1+\alpha}}{(\lambda_f-\epsilon)r^{p'}} \tag{3}$$

from which the theorem follows because we can choose ϵ such that

$$0 < \epsilon < \min\{\lambda_f, \frac{p}{1+\alpha} - \rho_g\}. \tag{4}$$

This proves the theorem.

Remark 1. *Theorem 1 improves a result of Lahiri and Sharma* [6].

Remark 2. *If we choose f to be meromorphic and g to be transcendental entire with the maximum deficiency sum satisfying $0 < \lambda_g \le \rho_g < \infty$ and $\rho_f < \infty$, the theorem remains true with $L(f)$ replaced by $L(g)$ in the denominator.*

Remark 3. *If we take $\rho_f > 0$ instead of $\lambda_f > 0$ and the other conditions remain the same, the conclusion of Theorem 1 remains valid with 'limit' replaced by 'limit inferior'.*

Lahiri [5] proved the following theorem on the comparative growth of $\log T(r, f\circ g)$ and $T(r,f)$.

Theorem A Let f and g be two non-constant entire functions such that $\lambda_g < \lambda_f \le \rho_f < \infty$. Then

$$\liminf_{r\to\infty} \frac{\log T(r, f\circ g)}{T(r,f)} = 0.$$

In the following theorem we improve Theorem A by considering f to be transcendental meromorphic. In fact we prove something more.

Theorem 2. *Let f be transcendental meromorphic with the maximum deficiency sum and g be entire with $\lambda_g < \lambda_f \leq \rho_f < \infty$. Then*

$$\liminf_{r\to\infty} \frac{\log\{T(r, f \circ g) \log M(r, g)\}}{T(r, L(f))} = 0.$$

Proof. In view of Lemma 1, for all sufficiently large values of r, we have,

$$\begin{aligned} T(r, f \circ g) \log M(r, g) &\leq \{1 + o(1)\} T(r, g).T(M(r, g), f) \\ i.e., \ \log\{T(r, f \circ g) \log M(r, g)\} &\leq \log\{1 + o(1)\} + \log T(r, g) \\ &\quad + \log T(M(r, g), f). \end{aligned} \tag{5}$$

From above we get for a sequence of values of r tending to infinity,

$$\begin{aligned} &\log\{T(r, f \circ g) \log M(r, g)\} \\ \leq\ & o(1) + (\lambda_g + \epsilon) \log r + (\rho_f + \epsilon) \log M(r, g) \\ \leq\ & o(1) + (\lambda_g + \epsilon) \log r + (\rho_f + \epsilon) r^{\lambda_g + \epsilon}. \end{aligned} \tag{6}$$

Again for all sufficiently large values of r, we obtain by Lemma 4,

$$T(r, L(f)) \geq r^{(\lambda_{L(f)} - \epsilon)} = r^{(\lambda_f - \epsilon)}. \tag{7}$$

In view of (6) and (7) we get for a sequence of values of r tending to infinity,

$$\begin{aligned} &\frac{\log\{T(r, f \circ g) \log M(r, g)\}}{T(r, L(f))} \\ <\ & \frac{o(1) + (\lambda_g + \epsilon) \log r + (\rho_f + \epsilon) r^{\lambda_g + \epsilon}}{r^{(\lambda_f - \epsilon)}}. \end{aligned} \tag{8}$$

Now as $\lambda_g < \lambda_f$, we can choose ϵ (> 0) in such a way that $\lambda_g + \epsilon < \lambda_f - \epsilon$ and the theorem follows from (8).

Remark 4. *If we take $\rho_g < \lambda_f \leq \rho_f < \infty$ instead of $\lambda_g < \lambda_f \leq \rho_f < \infty$ and the other conditions remain the same, the conclusion of Theorem 2 remains valid with 'limit inferior' replaced by 'limit' as we see in the following theorem.*

Theorem 3. *Let f be transcendental meromorphic and g be entire such that $\rho_g < \lambda_f \leq \rho_f < \infty$ and $\sum_{a \neq \infty} \delta(a; f) + \delta(\infty; f) = 2$. Then*

$$\lim_{r\to\infty} \frac{\log\{T(r, f \circ g) \log M(r, g)\}}{T(r, L(f))} = 0.$$

Proof. In view of Lemma 1, we get for all sufficiently large values of r,

$$\begin{aligned}
& \log\{T(r, f\circ g)\log M(r,g)\} \\
\leq\ & \log\{1+o(1)\} + \log T(r,g) + \log T(M(r,g), f) \\
\leq\ & o(1) + (\rho_g+\epsilon)\log r + (\rho_f+\epsilon)\log M(r,g) \\
\leq\ & o(1) + (\rho_g+\epsilon)\log r + (\rho_f+\epsilon)r^{\rho_g+\epsilon}. \qquad (9)
\end{aligned}$$

Now combining (7) and (9) it follows for all sufficiently large values of r,

$$\begin{aligned}
& \frac{\log\{T(r, f\circ g)\log M(r,g)\}}{T(r, L(f))} \\
\leq\ & \frac{o(1) + (\rho_g+\epsilon)\log r + (\rho_f+\epsilon)r^{\rho_g+\epsilon}}{r^{(\lambda_f-\epsilon)}}. \qquad (10)
\end{aligned}$$

As $\rho_g < \lambda_f$ we can choose ϵ (> 0) in such a manner that $\rho_g + \epsilon < \lambda_f - \epsilon$ and thus the theorem follows from (10).

Remark 5. *The condition $\rho_g < \lambda_f$ in Theorem 3 is essential as we see in the following example.*

Example 1. *Let $f = g = \exp z$. Also let $L(f) = f$.*

Then $\lambda_f = \rho_f = 1$ and $\rho_g = 1$. Therefore

$$T(r, f\circ g)\ \Gamma\ \frac{e^r}{(2\pi^3 r)^{\frac{1}{2}}} \quad \text{and} \quad T(r, L(f) = \frac{r}{\pi}.$$

So

$$\begin{aligned}
& \lim_{r\to\infty}\frac{\log\{T(r, f\circ g)\log M(r,g)\}}{T(r, L(f)} \\
=\ & \lim_{r\to\infty}\frac{\log T(r, f\circ g) + \log^{[2]} M(r,g)}{T(r, L(f)} \\
=\ & \lim_{r\to\infty}\frac{r - \frac{1}{2}\log r + O(1) + \log^{[2]} e^r}{\frac{r}{\pi}} \\
=\ & \pi.\lim_{r\to\infty}\left[\frac{r+\frac{1}{2}\log r + O(1)}{r}\right] = \pi \neq 0,
\end{aligned}$$

which is contrary to Theorem 3.

Theorem 4. *Let f be transcendental meromorphic and g be entire such that $0 < \lambda_f \leq \rho_f < \infty$ and $0 < \lambda_g \leq \rho_g < \infty$. If $\sigma_g < \infty$ and*

$$\sum_{a\neq\infty}\delta(a; f) + \delta(\infty; f) = 2 = \sum_{a\neq\infty}\delta(a; g) + \delta(\infty; g),$$

then

$$\limsup_{r\to\infty}\frac{\log T(r,f\circ g)}{\log T(\exp(r^{\rho_g}),L(f))}<\infty$$

$$and\quad \limsup_{r\to\infty}\frac{\log T(r,f\circ g)}{\log T(\exp(r^{\rho_g}),L(g))}<\infty.$$

Proof. Since $T(r,g)\le \log^+ M(r,g)$, by Lemma 1 we get for all sufficiently large values of r,

$$\log T(r,f\circ g)\le \log T(M(r,g),f)+o(1).$$

So in view of Lemma 4 we obtain that

$$\begin{aligned}
&\limsup_{r\to\infty}\frac{\log T(r,f\circ g)}{\log T(\exp(r^{\rho_g}),L(f))}\\
\le\ &\limsup_{r\to\infty}\frac{\log T(M(r,g),f)+o(1)}{\log T(\exp(r^{\rho_g}),L(f))}\\
\le\ &\limsup_{r\to\infty}\frac{\log T(M(r,g),f)}{\log M(r,g)}.\limsup_{r\to\infty}\frac{\log M(r,g)}{r^{\rho_g}}.\limsup_{r\to\infty}\frac{\log(\exp(r^{\rho_g}))}{\log T(\exp(r^{\rho_g}),L(f))}\\
=\ &\rho_f.\sigma_g.\frac{1}{\lambda_{L(f)}}\\
=\ &\rho_f.\sigma_g.\frac{1}{\lambda_f}<\infty.
\end{aligned}$$

Similarly we obtain by Lemma 4 that

$$\begin{aligned}
\limsup_{r\to\infty}\frac{\log T(r,f\circ g)}{\log T(\exp(r^{\rho_g}),L(g))} &\le \rho_f.\sigma_g.\frac{1}{\lambda_{L(g)}}\\
&= \rho_f.\sigma_g.\frac{1}{\lambda_g}<\infty.
\end{aligned}$$

This proves the theorem.

Remark 6. *Theorem* 4 *is an analogue to Theorem* 5 [8] *for composite meromorphic functions.*

Theorem 5. *Let h be meromorphic and k,g be entire such that $\lambda_h>0$ and $\rho_g<\rho_k$. Then for every transcendental meromorphic function f of finite order and satisfying* $\sum\limits_{a\neq\infty}\delta(a;f)+\delta(\infty;f)=2$,

$$\limsup_{r\to\infty}\frac{T(r,h\circ k)}{T(r,f\circ g)T(r,L(f))}=\infty.$$

Proof. In view of Lemma 2, we obtain for a sequence of values of r tending to infinity,

$$T(r, h \circ k) \geq T(\exp(r^{\mu}), h), \quad 0 < \mu < \rho_{k.} \tag{11}$$

Again for all sufficiently large values of r,

$$\begin{aligned} \log T(\exp(r^{\mu}), h) &\geq (\lambda_h - \epsilon) \log\{\exp(r^{\mu})\} \\ i.e., \quad T(\exp(r^{\mu}), h) &\geq \exp\{(\lambda_h - \epsilon) r^{\mu}\}. \end{aligned} \tag{12}$$

Now combining (11) and (12), we get for a sequence of values of r tending to infinity,

$$T(r, h \circ k) \geq \exp\{(\lambda_h - \epsilon) r^{\mu}\}. \tag{13}$$

Since $T(r, g) \leq \log^{+} M(r, g)$, for all sufficiently large values of r we obtain from Lemma 1

$$\begin{aligned} \log T(r, f \circ g) &\leq \log\{1 + o(1)\} + \log T(M(r, g), f) \\ i.e., \quad \log T(r, f \circ g) &\leq \log\{1 + o(1)\} + (\rho_f + \epsilon) \log M(r, g) \\ i.e., \quad \log T(r, f \circ g) &\leq \log\{1 + o(1)\} + (\rho_f + \epsilon) r^{\rho_g + \epsilon} \\ i.e., \quad T(r, f \circ g) &\leq \{1 + o(1)\}. \exp\{(\rho_f + \epsilon) r^{\rho_g + \epsilon}\}. \end{aligned} \tag{14}$$

In view of Lemma 4, we have for all sufficiently large values of r,

$$\begin{aligned} T(r, L(f)) &\leq r^{L(f) + \epsilon} \\ i.e., \quad T(r, L(f)) &\leq r^{\rho_f + \epsilon}. \end{aligned} \tag{15}$$

From (14) and (15) it follows for all large values of r,

$$T(r, f \circ g).T(r, L(f)) \leq \{1 + o(1)\} r^{\rho_f + \epsilon}. \exp\{(\rho_f + \epsilon) r^{\rho_g + \epsilon}\}. \tag{16}$$

Combining (13) and (16) we get for a sequence of values of r tending to infinity,

$$\frac{T(r, h \circ k)}{T(r, f \circ g) T(r, L(f))} \geq \frac{\exp\{(\lambda_h - \epsilon) r^{\mu}\}}{\{1 + o(1)\} r^{\rho_f + \epsilon}. \exp\{(\rho_f + \epsilon) r^{\rho_g + \epsilon}\}}. \tag{17}$$

Since $\rho_g < \rho_k$, we can choose ϵ (> 0) in such a manner that

$$\rho_g + \epsilon < \mu < \rho_k . \tag{18}$$

Thus the theorem follows from (17) and (18).

Theorem 6. *Let f be meromorphic and g be transcendental entire such that $\rho_g < \infty$, $\lambda_{f \circ g} = \infty$ and $\sum_{a \neq \infty} \delta(a; g) + \delta(\infty; g) = 2$. Then for every A (> 0),*

$$\lim_{r \to \infty} \frac{\log T(r, f \circ g)}{\log T(r^A, L(g))} = \infty.$$

Proof. If possible, let there exists a constant β such that for a sequence of values of r tending to infinity,

$$\log T(r, f \circ g) \leq \beta \log T(r^A, L(g)). \tag{19}$$

Again in view of Lemma 4, we obtain for all sufficiently large values of r,

$$\begin{aligned} \log T(r^A, L(g)) &\leq (\rho_{L(g)} + \epsilon) \log r^A \\ i.e., \ \log T(r^A, L(g)) &\leq (\rho_g + \epsilon) A \log r. \end{aligned} \tag{20}$$

Now combining (19) and (20), we have for a sequence of values of r tending to infinity,

$$\begin{aligned} \log T(r, f \circ g) &\leq \beta(\rho_g + \epsilon) A \log r \\ i.e., \ \lambda_{f \circ g} &\leq \beta A(\rho_g + \epsilon), \end{aligned}$$

which contradicts the condition $\lambda_{f \circ g} = \infty$. So for all sufficiently large values of r we get that

$$\log T(r, f \circ g) > \beta \log T(r^A, L(g)),$$

from which the theorem follows.

Corollary 1. *Under the assumptions of Theorem* 6*,*

$$\lim_{r \to \infty} \frac{T(r, f \circ g)}{T(r^A, L(g))} = \infty.$$

Proof. By Theorem 6 we obtain for all sufficiently large values of r and for $K > 1$,

$$\begin{aligned} \log T(r, f \circ g) &> K \log T(r^A, L(g)) \\ i.e., T(r, f \circ g) &> \{T(r^A, L(g))\}^K, \end{aligned}$$

from which the corollary follows.

Remark 7. *If we take* $\rho_f < \infty$ *and* $\sum_{a \neq \infty} \delta(a; f) + \delta(\infty; f) = 2$ *instead of* $\rho_g < \infty$ *and* $\sum_{a \neq \infty} \delta(a; g) + \delta(\infty; g) = 2$ *respectively, then Theorem* 6 *and Corollary* 1 *remain valid with* $L(g)$ *replaced by* $L(f)$ *in the denominator.*

Theorem 7. *If* f *be meromorphic and* g *be transcendental entire satisfying* (i) $\rho_f < \infty$, (ii) $0 < \lambda_g \leq \rho_g < \infty$ *and* (iii) $\sum_{a \neq \infty} \delta(a; g) + \delta(\infty; g) = 2$, *then*

$$\limsup_{r \to \infty} \frac{\log^{[2]} T(r, f \circ g)}{\log T(r, L(g))} \leq \frac{\rho_g}{\lambda_g}.$$

Proof. In view of Lemma 1 and Lemma 4 and since $T(r,g) \leq \log^{+} M(r,g)$, we get for all sufficiently large values of r,

$$\begin{aligned}
\log T(r, f\circ g) &\leq \log T(M(r,g), f) + \log\{1+o(1)\} \\
i.e.,\ \log T(r, f\circ g) &\leq (\rho_f+\epsilon)\log M(r,g) + o(1) \\
i.e.,\ \log T(r, f\circ g) &\leq \{(\rho_f+\epsilon)+o(1)\}\log M(r,g) \\
i.e.,\ \log^{[2]} T(r, f\circ g) &\leq \log^{[2]} M(r,g) + O(1) \\
i.e.,\ \frac{\log^{[2]} T(r, f\circ g)}{\log T(r, L(g))} &\leq \frac{\{\log^{[2]} M(r,g)+O(1)\}}{\log r}.\frac{\log r}{\log T(r, L(g))}
\end{aligned}$$

$$\begin{aligned}
& i.e., \limsup_{r\to\infty} \frac{\log^{[2]} T(r, f\circ g)}{\log T(r, L(g))} \\
\leq\ & \limsup_{r\to\infty} \frac{\log^{[2]} M(r,g)+O(1)}{\log r}.\limsup_{r\to\infty}\frac{\log r}{\log T(r, L(g))} \\
=\ & \rho_g.\frac{1}{\lambda_{L(g)}} = \frac{\rho_g}{\lambda g}.
\end{aligned}$$

This proves the theorem.

Theorem 8. *Let f be transcendental meromorphic and g be transcendental entire satisfying* $\lambda_f > 0,\ 0 < \rho_g < \infty$ *and* $\sum_{a\neq\infty} \delta(a;g) + \delta(\infty;g) = 2$. *Then*

$$\limsup_{r\to\infty} \frac{\log T(r, f\circ g)}{\log T(\exp(r^{\mu}), L(g))} = \infty, \quad where\ \ 0<\mu<\rho_g\ .$$

Proof. Let $0 < \mu < \mu' < \rho_g$ .Then in view of Lemma 2, for a sequence of that values of r tending to infinity, we get that

$$\begin{aligned}
T(r, f\circ g) &\geq T(\exp(r^{\mu'}), f) \\
i.e.,\ \log T(r, f\circ g) &\geq \log T(\exp(r^{\mu'}), f) \\
i.e.,\ \log T(r, f\circ g) &\geq (\lambda_f-\epsilon)\log\{\exp(r^{\mu'})\} \\
i.e.,\ \log T(r, f\circ g) &\geq (\lambda_f-\epsilon) r^{\mu'}. \qquad (21)
\end{aligned}$$

Again in view of Lemma 4, since $\rho_f < \infty$ then for $\epsilon\ (>0)$ and for all sufficiently large values of r,

$$\begin{aligned}
\log T(\exp(r^{\mu}), L(g)) &\leq (\rho_{L(g)}+\epsilon)\log\{\exp(r^{\mu})\} \\
i.e.,\ \log T(\exp(r^{\mu}), L(g)) &\leq (\rho_g+\epsilon) r^{\mu}. \qquad (22)
\end{aligned}$$

So combining (21) and (22) we obtain for a sequence of values of r tending to infinity,

$$\frac{\log T(r, f\circ g)}{\log T(\exp(r^{\mu}), L(g))} \geq \frac{(\lambda_f - \epsilon) r^{\mu'}}{(\rho_g + \epsilon) r^{\mu}}. \tag{23}$$

Since $\mu' > \mu$, the theorem follows from (23).

Corollary 2. *Under the assumptions of Theorem* 8*,*

$$\limsup_{r\to\infty} \frac{T(r, f\circ g)}{T(\exp(r^{\mu}), L(g))} = \infty, \quad 0 < \mu < \rho_g.$$

Proof. In view of Theorem 8, we get for a sequence of values of r tending to infinity,

$$\begin{aligned} \log T(r, f\circ g) &\geq K \log T(\exp(r^{\mu}), L(g)), \quad for\ K > 1. \\ i.e.,\ T(r, f\circ g) &\geq \{T(\exp(r^{\mu}), L(g))\}^{K}, \end{aligned}$$

from which the corollary follows.

Theorem 9. *Let f be a transcendental meromorphic function and g be an entire function such that (i) $0 < \rho_g < \infty$, (ii) $0 < \sigma_g < \infty$, (iii) $\rho_f < \infty$ and (iv) $\sum\limits_{a\neq\infty} \delta(a; g) + \delta(\infty; g) = 2$. Then*

$$\liminf_{r\to\infty} \frac{T(r, f\circ g)}{T(r, L(g))} \leq \frac{\rho_f}{1 + k - k\delta(\infty; f)}.$$

Proof. In view of the inequality $T(r, g) \leq \log^{+} M(r, g)$ and Lemma 1 we get for all sufficiently large values of r,

$$\begin{aligned} T(r, f\circ g) &\leq \{1 + o(1)\} T(M(r, g), f) \\ i.e.,\ \log T(r, f\circ g) &\leq \log\{1 + o(1)\} + \log T(M(r, g), f) \\ i.e.,\ \log T(r, f\circ g) &\leq o(1) + (\rho_f + \epsilon) \log M(r, g) \\ i.e.,\ \liminf_{r\to\infty} \frac{\log T(r, f\circ g)}{T(r, L(g))} &\leq (\rho_f + \epsilon) \liminf_{r\to\infty} \frac{\log M(r, g)}{T(r, L(g))}. \end{aligned} \tag{24}$$

From the definition of type of an entire function for all sufficiently large values of r,

$$\log M(r, g) \leq (\sigma_g + \epsilon) r^{\rho_g}. \tag{25}$$

Again in view of Lemma 4, we obtain for a sequence of values of r tending to infinity,

$$\begin{aligned} T(r, L(g)) &\geq (\sigma_{L(g)} - \epsilon) r^{\rho_{L(g)}} \\ i.e.,\ T(r, L(g)) &\geq \left[\{1 + k - k\delta(\infty; g)\} \sigma_g - \epsilon\right] r^{\rho_g}. \end{aligned} \tag{26}$$

Now it follows from (25) and (26) that for a sequence of values of r tending to infinity,

$$\begin{aligned}\frac{\log M(r,g)}{T(r,L(g))} &\leq \frac{(\sigma_g+\epsilon)}{[\{1+k-k\delta(\infty;g)\}\sigma_g-\epsilon]} \\ i.e.,\ \liminf_{r\to\infty}\frac{\log M(r,g)}{T(r,L(g))} &\leq \frac{(\sigma_g+\epsilon)}{[\{1+k-k\delta(\infty;g)\}\sigma_g-\epsilon]}.\end{aligned}$$

Since ϵ (>0) is arbitrary, it follows from above that

$$\liminf_{r\to\infty}\frac{\log M(r,g)}{T(r,L[g])} \leq \frac{1}{1+k-k\delta(\infty;g)}. \tag{27}$$

Now from (24) and (27) we obtain that

$$\liminf_{r\to\infty}\frac{\log T(r,f\circ g)}{T(r,L(g))} \leq \frac{\rho_f+\epsilon}{1+k-k\delta(\infty;g)}. \tag{28}$$

As ϵ (>0) is arbitrary, the theorem follows from (28).

References

[1] W.Bergweiler: On the Nevanlinna characteristic of a composite function, Complex Variables 10 (1988), pp. 225 – 236.

[2] W.Bergweiler: On the growth rate of composite meromorphic functions, Complex Variables 14 (1990),pp. 187 – 196.

[3] J.Clunie: The Composition of Entire and Meromorphic Functions, Mathematical essays dedicated to A.J. Macintyre, Ohio University Press, 1970, pp. 75 – 92.

[4] W.K.Hayman : Meromorphic Functions, The Clarendon Press, Oxford, 1964.

[5] I.Lahiri : Growth of composite integral functions, Indian J. Pure Appl. Math., 20(9), September (1989), pp. 899 – 907.

[6] I.Lahiri and D.K.Sharma : Growth of composite entire and meromorphic functions, Indian J. Pure Appl. Math., 26(5), May (1995), pp. 451 – 458.

[7] L.Liao and C.C.Yang : On the growth of composite entire functions, Yokohama Math. J. 46 (1999), pp. 97 – 107.

[8] I.Lahiri and A.Banerjee : Value distribution of a Wronskian, Portugaliae Mathematica, Vol. 61 Fasc. 2 (2004), Nova Se′rie pp. 161 – 175.

[9] G.D.Song and C.C.Yang : Further growth properties of composition of entire and meromorphic functions, Indian J. Pure Appl. Math., 15 (1), January (1984), pp. 67 – 82.

[10] A.P Singh.: Growth of composite entire functions, Kodai Math. J., 8 (1985), pp. 99 – 102.

[11] A.P.Singh and M.S. Baloria : On maximum modulus and maximum term of composition of entire functions, Indian J. Pure Appl. Math., 22 (12), December (1991), pp. 1019 – 1026.

[12] G.Valiron : Lectures on the General Theory of Integral Functions, Chelsea Publishing Company, 1949.

Received: April, 2009