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# An Efficient Three-Step Iterative Method for Non-Linear Equations

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**Abstract**

In this paper, we present three-step iterative method for solving non-linear equations. The convergence analysis of the method is discussed. It is established that the new method has convergence order eight. Numerical examples are given to test the efficiency and performance of the method. Numerical tests show that the new method is comparable with the well known existing methods and in many cases gives better results. Our results can be considered as an improvement and refinement of the previously known results in the literature.

**Mathematics Subject Classification:** 34A34

**Keywords:** Iterative methods; three-step methods; Predictor-corrector methods;Convergence order; Nonlinear equations

## 1 Introduction

Let us consider a non-linear equation

$$f(x) = 0. \tag{1.1}$$

Finding zeros of a single variable nonlinear equation (1.1) efficiently, is an interesting and very old problem in numerical analysis and has many applications in physical sciences.

In recent years, researchers have developed many iterative methods for solving equation (1.1). These methods can be classified as one-step, two-step and three-step methods, see[$1-12$]. These methods have been proposed using Taylor series, decomposition techniques, error analysis and quadrature rules, etc. .Abbasbandy[1], Chun[3] and Grau[7] have proposed many two-step and three-step methods.

In this paper, we present three step iterative method for solving non-linear single variable equations. We prove that the new method has order of convergence eight.The method and its algorithm are described in section 2.The convergence analysis is discussed in section 3. Finally, in section 4, the method is tested on well known numerical examples given in the literature. Numerical examples are given to test the efficiency and performance of the method. Numerical tests show that the new method is comparable with the well known existing methods and in many cases gives better results. Our results can be considered as an improvement and refinement of the previously known results in the literature.

## 2 The Iterative Method

We suggest the following modification to Mamta et al. method for solving non-linear equations:

Consider the Mamta et al. method[10] :

$$x_{n+1} = x_n - h(x_n),$$

where

$$h(x_n) = \frac{2f(x_n)}{f'(x_n) + \sqrt{(f'(x_n))^2 + 4(f(x_n))^2}}, \tag{2.1}$$

We observe that

$$\frac{f(x_n)}{f'(x_n)} = \frac{h(x_n)}{1-(h(x_n))^2}.$$

Following the analogy of Newton's method, the modified Mamta et al. method for solving non-linear equations can be written as

$$x_{n+1} = x_n - \frac{h(x_n)}{1-(h(x_n))^2}, \tag{2.2}$$

where $h(x_n)$ is provided by (2.1).This formulation allows to suggest the following three-step iterative method:

**Algorithm 2.1** For a given initial guess $x_o$,find the approximate solution by the iterative scheme:

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{2.3}$$

$$z_n = y_n - \frac{h(y_n)}{1-(h(y_n))^2}, \tag{2.4}$$

$$x_{n+1} = z_n - (y_n - z_n)\frac{f(z_n)}{f(y_n) - 2f(z_n)}, \tag{2.5}$$

$$\text{where } h\left(y_n\right) = \frac{2f\left(y_n\right)}{f'\left(y_n\right) + \sqrt{(f'(y_n))^2 + 4(f(y_n))^2}}. \tag{2.6}$$

We now prove that the algorithm 2.1 has eighth order convergence.

# 3 Convergence Analysis

**Theorem 3.1** *Let $w \in I$ be a simple zero of sufficiently differentiable function $f : I \sqsubseteq \mathbb{R} \to \mathbb{R}$ for an open interval $I$. If $x_0$ is sufficiently close to $w$, then the algorithm 2.1 has eighth order convergence.*

**Proof.**

Let $w$ be a simple zero of $f$ and $x_n = w + e_n$. By Taylor's expansion, we have:

$$f(x_n) = f(w)(e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + c_5e_n^5 + c_6e_n^6) + O(e_n^7) \tag{3.1}$$

$$f^{'}(x_n) = f^{'}(w)(1 + 2c_2e_n + 3c_3e_n^2 + 4c_4e_n^3 + 5c_5e_n^4 + 6c_{65}e_n^5) + O(e_n^6), \tag{3.2}$$

where

$$c_k = (\frac{1}{k!})\frac{f^{(k)}(w)}{f'(w)}, k = 2, 3, ...\text{and } e_n = x_n - w \tag{3.3}$$

Using (3.1) and (3.2), we have

$$\frac{f(x_n)}{f'(x_n)} = e_n - c_2e_n^2 + 2(c_2^2 - c_3)e_n^3 + (7c_2c_3 - 3c_4 - 4c_2^3)e_n^4 + O(e_n^5) \qquad (3.4)$$

Using (3.4) in (2.3), we thus have

$$y_n = w + c_2e_n^2 + (-2c_2^2 + 2c_3)e_n^3 - (7c_2c_3 - 4c_2^3 - 3c_4)e_n^4 + O(e_n^5) \qquad (3.5)$$

Expanding $f(y_n)$ by Taylor's series about $w$,we have:

$$f(y_n) = f(w) + c_2e_n^2 + 2(c_3 - c_2^2)e_n^3 + (-7c_2c_3 + 3c_4 + 5c_2^3)e_n^4 + O(e_n^5). \qquad (3.6)$$

By Taylor's series, we have

$$f'(y_n) = f'(w)(1 + 2c_2^2e_n^2 + (-4c_2^3 + 4c_2c_3)e_n^3 + (-11c_3c_2^2 + 8c_2^4$$

$$+6c_2c_4)e_n^4) + O(e_n^5), \qquad (3.7)$$

From 2.6), we have

$$h(y_n) = \frac{2f\,(y_n)}{f'\,(y_n) + \sqrt{(f'(y_n))^2 + 4(f(y_n))^2}}$$

$$= c_2e_n^2 + (-2c_2^2 + 2c_3)e_n^3 + (-7c_2c_3 + 3c_4 + 3c_2^3)e_n^4$$

$$+(-10c_2c_4 + 4c_5 - 4c_2^4 - 6c_3^2 + 16c_3c_2^2)e_n^5 + O(e_n^6) \qquad (3.8)$$

Using (3.8) , we have:

$$\frac{h(y_n)}{1 - (h(y_n))^2} = c_2e_n^2 + (-2c_2^2 + 2c_3)e_n^3 + (-7c_2c_3 + 3c_4 + 3c_2^3)e_n^4$$

$$+(-10c_2c_4 + 4c_5 - 4c_2^4 - 6c_3^2 + 16c_3c_2^2)e_n^5$$

$$+(29c_2c_3^2 - 13c_2c_5 + 5c_6 + 22c_4c_2^2 + 6c_2^5$$

$$-17c_4c_3 - 32c_3c_2^3)e_n^6 + O\left(e_n^7\right). \qquad (3.9)$$

Using (3.9) in (2.4) , we have

$$z_n = w + c_2^3 e_n^4 + (-4c_2^4 + 4c_3 c_2^2) e_n^5 + (4c_2 c_3^2 + 6c_4 c_2^2$$

$$+10c_2^5 - 20c_3 c_2^3) e_n^6 + O\left(e_n^7\right), \tag{3.10}$$

Expanding $f(z_n)$ by Taylor's series about $w$, we have

$$f(z_n) = f'(w)\,(c_2^3 e_n^4 - 4c_2^2(c_2^2 - c_3) e_n^5 + 2c_2(5c_2^4 - 10c_3 c_2^2 + 3c_4 c_2 + 2c_3^2) e_n^6$$

$$+(8c_2 c_6 + 4c_4^2 - 9c_3^3 - c_7 + 24c_2^3 c_4 - 16c_2^2 c_5 + 62c_2^2 c_3^2$$

$$-52c_2^4 c_3 + 12c_2^6 + 8c_3 c_5 - 40c_2 c_4 c_3) e_n^7) + O\left(e_n^8\right) \tag{3.11}$$

Using (3.11), (3.10), (3.6) and (3.5), we have

$$\frac{(y_n - z_n) f(z_n)}{f(y_n) - 2f(z_n)} = c_2^3 e_n^4 - 4c_2^2(c_2^2 - c_3) e_n^5 + 2c_2(5c_2^4 - 10c_3 c_2^2 + 3c_4 c_2$$

$$+2c_3^2) e_n^6 + (8c_2 c_6 + 4c_4^2 - 9c_3^3 - c_7 + 24c_2^3 c_4 - 16c_2^2 c_5$$

$$+62c_2^2 c_3^2 - 52c_2^4 c_3 + 12c_2^6 + 8c_3 c_5 - 40c_2 c_4 c_3) e_n^7$$

# 4 Numerical Examples

We consider here some numerical examples to demonstrate the performance of the new developed three-step iterative method, namely algorithm 2.1.We compare the Ostrowski's method $(OM)$ [13], the method of Grau$(GM)$ [7] and the new developed three-step method$(MN)$, in this paper. All the values are computed with 350 significant digits. We take $\in = 10^{-15}$ as tolerance. The following criteria is used for estimating the zero:

(i) $\delta = |x_{n+1} - x_n| < \in$

(ii) $|f(x_n)| < \in$

The following examples as in Chun[3] are used for numerical testing:

| **Example** | **Exact Zero** |
|---|---|
| $f_1 = x^3 + 4x^2 - 15,$ | $w = 1.6319808055660636,$ |
| $f_2 = xe^{x^2} - \sin^2(x) + 3\cos(x) + 5,$ | $w = -1.207647827130919,$ |

$f_3 = \sin(x) - \frac{1}{2}x,$ $w = 1.8954942670339809,$

$f_4 = 10xe^{-x^2} - 1,$ $w = 1.67963061042845,$

$f_5 = \cos(x) - x,$ $w = 0.73908513321516067,$

$f_6 = \sin^2(x) - x^2 + 1,$ $w = 1.4044916482153411,$

$f_7 = e^{-x} + \cos(x),$ $w = 1.7461395304080124176\mathbf{5}.$

For convergence criteria, it was required that $\delta$, the distance between two consecutive iterates was less than $10^{-15}, n$ represents the number of iterations and $f(x_n)$ is the absolute value of the function at $x_n$. The numerical comparison is given in Table 4.1.

| | $n$ | $f(x_n)$ |
|---|---|---|
| $f_1, x_0 = 2$ | | |
| OM | 3 | 1.03e-228 |
| GM | 3 | 4.46e-179 |
| MN | 3 | 1.0e-348 |
| $f_2, x_0 = -1$ | | |
| OM | 3 | 8.82e-223 |
| GM | 3 | 2.54e-155 |
| MN | 3 | 1.25e-317 |
| $f_3, x_0 = 2$ | | |
| OM | 3 | 5.12e-313 |
| GM | 3 | 8.44e-252 |
| MN | 3 | 3.0e-350 |
| $f_4, x_0 = 1.8$ | | |
| OM | 3 | 1.16e-236 |
| GM | 3 | 9.37e-187 |
| MN | 3 | 0 |
| $f_5, x_0 = 1$ | | |
| OM | 3 | 7.05e-296 |
| GM | 3 | 4.12e-237 |
| MN | 3 | 0 |
| $f_6, x_0 = 1.6$ | | |
| OM | 3 | 3.26e-226 |
| GM | 3 | 7.54e-178 |
| MN | 3 | 1.0e-349 |
| $f_7, x_0 = 2$ | | |
| OM | 3 | 1.05e-279 |
| GM | 3 | 1.58e-223 |
| MT | 3 | 3.0e-350 |

**Table 4.1.**

# 5 CONCLUSION

From Table 4.1, we observe that our three-step iterative method is comparable with all the methods cited in the table and in almost all the cases gives better results in terms of the function evaluation $f(x_n)$. The algorithm (2.1) established in this paper, has computational efficiency $8^{\frac{1}{5}} = 1.515717$.

# References

[1] S. Abbasbandy, Improving Newton-Raphson method for nonlinear equations by modified Adomian decomposition method, Appl. Math. Comput. 145(2003)887-893.

[2] R. L. Burden, J. D. Faires, Numerical Analysis, PWS publishing company, Boston USA, 2001.

[3] C. Chun, Iterative methods improving Newton's method by the decomposition method, Comput & Math with Appl. 50(2005)1559-1568.

[4] J.E.Dennis, R.B. Schnable, Numerical methods of unconstrained optimization and non-linear equations, Prentice Hall, 1983.

[5] M. Frontini, E. Sormani, Some variants of Newton's method with third order convergence and multiple roots, J. Comput. Appl. Math. 156 (2003), 345-354.

[6] M. Frontini, E. Sormani, Third order methods for quadrature formulae for solving system of nonlinear equations, Appl. Math. Comput. 149 (2004), 771-782.

[7] M. Grau, J.L. Diaz-Barrero, An improvement to Ostrowski root-finding method, Appl. Math. Comput. 173(2006) 450-456.

[8] Jisheng Kou, Yitian Li and Xiuhua Wang, Third order modification of Newton's method, Appl.Math. and Comput.,(2006, in press).

[9] M. V. Kanwar, V. K. Kukreja, S. Singh, On a class of quadratically convergent iteration formulae, Appl. Math. Comput. 166 (3) (2005), 633-637.

[10] Mamta, V. Kanwar, V. K. Kukreja, S. Singh, On some third order iterative methods for solving non-linear equations, Appl. Math. Comput. 171 (2005), 272-280.

[11] Gyurhan Nedzhibov, On a few iterative methods for solving nonlinear equations, Application of Mathematics in Engineering and Economics, in: Proceedings of the XXVIII Summer School Sozopol 2002, Heron Press, Sofia, 2002.

[12] S. Weera koon, T. G. I. Fernando, A variant of Newton's method with accelerated third order convergence, Appl. Math. Lett. 13(2000), 87-93.

[13] A.M.Ostrowski, Solutions of Equations and Sytems of Equations, Academis Press, New York, 1960.

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