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Portfolio Optimization Based on Spectral Risk Measures

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Abstract

Since VaR method is absent of general subadditivity, and of description to tail information, the coherent risk measure is proposed recently. Under this framework, many risk measure technologies are established, one of which is the spectral risk measure. The spectral risk measure combines the distribution of profit and loss with the subjective risk aversion from the investors. This paper considers the portfolio problems based on spectral risk measures and gets some useful results. This model can be regarded as an natural generalization of the classical Markowitz's portfolio optimization problem.

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1 Introduction

Financial market risk is the main risk which investors face in the process of investment and is also the management emphasis of the world's financial regulatory authorities. The financial market of China is an emerging market. As a result of the non-standardization of the information disclosure, macroeconomic policy and regulatory system, the financial markets fluctuate frequently and generate the great market risk. This not only shakes the confidence of the small and medium-sized investors in financial markets but also has the significant negative impacts on the stability of financial markets and the healthy development of the national economy. Thus, the study of the risk of financial markets is of great practical significance.

Before Markowitz [11, 12], the financial risk has been seen as the modified coefficient of expected returns. These original measures are beneficial for the investors to determine their investment priorities. Since 1952, when Markowitz put forward the optimal portfolio choice theory based on the risk of variance, the variance (average variance) has become a highly influential and classic measure of financial risks. The calculation of variance is simple, easy-to-use, and the theory has been quite mature. However, the risk in this model includes the variance of some income over the average, which is inconsistent with the facts; And it only considers the

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average deviation, not giving full consideration to the issue of the left tail of the earnings which is widespread concern. So it is not suitable to describe the loss resulted by the small probability events.

In 1991, Harlow [9] put forward the risk measurement index LPMq(Lower Partial Moments Quantity) and the resource optimal regrouping model. He made empirical comparison between the risk index on the LPM and the risk index of variance. The researches, over the past several decades, have shown that the efficient boundary of portfolio calculated on mean - LPMs is in the upper left corner of the one obtained by mean-variance curve in the coordinate system. This indicates that in the circumstances of the same expected revenue, the optimization scheme whose object function is LPMs is better than that whose objective is variance. Harlow's study also pointed out that, when the rate of return obeys normal distribution, the two methods have the same result; when the return series are not normally distributed, the optimization scheme based on the LPMs risk measurement is superior to the one based on the variance risk measurement. Therefore, from the view of empirical studies, the risk measurement index LPMq is better than the risk measurement index of variance-type.

In 1993, J. P. Morgon, the G30 put forward a new risk measure methodVaR (Value at Risk), which is also a downside risk measure approach. The main goal of this method is to answer the question: if the probability is determinate, how does the investor expect the loss money in a certain day in the future and how many of his assets is in risk? Artzner[3, 4], Pflug [14], Basak and Shapiro [5] studied the portfolio selection problems in the setting of VaR being regarded as the risk metric.

The market risk is summarized as a simple figure in VaR and the economic significance is easy to understand. But this does not mean that it is a reasonable and effective measure method. In the recent years, the theoretical researches and practical results have proved that it has defects as follows: VaR does not satisfy subadditivity. It means thatusing VaR to measure the risk, the risk of portfolio is not always less than the risk of the securities portfolio. which runs counter to the market phenomenon of risk diversification, and is unreasonable in an economic sense. Therefore, to solve these problems, it is important to design a more reasonable and more comprehensive risk measure, and provide new ideas and methods in order to control the market risk effectively.

The non-subadditivity of the measure VaR impel one to search for the better risk measure index, which is requested to be the coherent risk measure. The theory of coherent risk measures was proposed by Artzner etc [3] in the form of axioms. This theory demands that a risk measure should satisfy the following four axioms: monotonicity, translation invariance, subadditive and positive homogeneous. This article gives us the concept of tail conditions expectations (TEC), which is also known as the tail VaR (tail VaR). TEC considers the risk resulted from the part which exceeds VaR, but in general it does not belong to the coherent risk measure. Furthermore, Artzner etc [3] proposed the worst conditional expectation (WCE). WCE is the coherent risk measure, but it is more difficult to implement because of the WCE's depending on the structure of probability space in concept. The coherent risk measure theory of Artzner[3] evoked strong response and the related researches has been carried out. In 2002, Delbaen [6] popularized the finite probability space requested by the coherent risk measure theory to the arbitrary probability space. They have also combined the coherent risk measure, the game theory and the distortion probability measure, which all have been illustrated by giving examples. Because the requirement of the axioms of subadditive and positive homogeneous in the coherent risk measure theory is too strict, Follmer and Schied [7] put forward the axiom of convexity with weak requirement, and called the risk measurement which satisfies monotonicity, translation invariance and the axiom of convexity as convex risk measure. Convex risk measure is also known as the weakly coherent risk measure. Acerbi [1, 2] and other scholars put forward the spectral risk measure theory. The theory requires that the risk measure is the coherent risk measure, and have spectral density of risk with good nature to describe the characteristics of investor. This article holds that the spectral risk measure theory will have good prospect. This theory makes clear requirement on risk measure from the technical aspect, describes the risk aversion characteristics of investor in the form of spectral density, also provide an important theoretical basis for the improvement of existing indicators. In the framework of the theory of spectral measures of risk, CvaR and VaR are not good indicators of the risk measure.

Recently, many authors pay their attention to portfolio optimization problems in the setting of different risk constraints such as Entropic risk measures [18], generalized CVaR measures [8], and of cost constraints, for instance, jump-cost constraint [16], and of different approaches such as [13, 10, 17]. The authors also studied the relationship between portfolio optimization problem and no-arbitrage [15].

This manuscript will focus its attention on studying the applications of spectral risk measures to portfolios. We give first the model based on mean-spectral risk measures and obtain the efficient frontier of this model in the sense of the normal distribution. We also arrive at the economic significance for this model. Then, we get some interesting results after the contrast with the classical mean variance efficient frontier. In the view of practical decision of the investor, we establish the optimal model of portfolios based on the risk spectrum restriction. At last risk spectrum restriction control risk is proven to be effective after a simple example.

The arrangement of this paper is as follows. The first two sections are preparation knowledge, which give the backgrounds and some necessary notations and terminologies. The third section considers portfolio optimization problem based on the mean-spectral risk measures, and some interesting results are obtained in this section.

2 Coherent Risk Measures

Just considering the uncertainty of one monment between 0 and T, for the time period $[0, T]$. Assume that the set of all possible state at the end of the period T is a finite set, denoted by Ω. The net-value in the future of the initial position is denoted by the random variable X of the Ω. The set of all the real functions on Ω as the risk set, denoted by χ. Let $L_+ = \{X|X(\omega) \geq 0, \forall\omega \in \Omega\}$, $L_- = \{X|X(\omega) \leq 0, \forall\omega \in \Omega\}$, and define $L_{--} = \{X|X(\omega) < 0, \forall\omega \in \Omega\}$.

Artzner [1] and others firstly give out the following axiom which the acceptable set A should be subjected to: (1) $L_+ \subset A$; (2) $A \cap L_{--} = \emptyset$; (3) $A \cap L_- = \{0\}$; (4) A is of a convex cone.

Definition 2.1 *The risk measure corresponding to the acceptable set A is defined as $\rho_A(X) = \inf\{c|X + c \cdot r \in A\}$, where r is of the risk-free interest rate.*

Definition 2.2 *Let ρ be a risk measure and $A_\rho = \{X \in \chi|\rho(X) \leq 0\}$, then we call A_ρ the acceptable set associated with risk measure ρ.*

It is well known that for the risk measure there holds the following four axioms

(1) Translate invariance: for all $X \in \chi$, and all the real α, there is $\rho(X+\alpha \cdot r) = \rho(X) - \alpha$;
(2) Sub-additivity: for all $X_1, X_2 \in \chi$, there is $\rho(X_1 + X_2) \leq \rho(X_1) + \rho(X_2)$;
(3) Positive homogeneous: for all $\lambda \geq 0$, and all $X \in \chi$, there is $\rho(\lambda X) = \lambda \rho(X)$;
(4) Monotonicity: for all $X, Y \in \chi$ with $X \leq Y$, there is $\rho(X) \leq \rho(Y)$.
In this setting, we call $\rho(X)$ the coherent risk measure.

First of all, let us recall the definition of the Expected Shortfall (ES). Assume that X is the real valued random variables of the probability of space $(\Omega, \mathcal{F}, P)$, and it denotes a variable of profit - loss of a given portfolio. $F_Z(x) = P[X \leq x]$ is the distribution function of X, and the confidence level is $1-\alpha \in (0,1)$.

Definition 2.3 *Define the below and up α-quantiles of X, respectively, as*

$$x_{(\alpha)} = q_\alpha(X) = \inf\{x \in \mathbb{R} : P[X \leq x] \geq \alpha\};$$

$$x^{(\alpha)} = q^\alpha(X) = \inf\{x \in \mathbb{R} : P[X \leq x] > \alpha\} = \sup\{x \in \mathbb{R} : P[X \leq x] \leq \alpha\}.$$

It is easy to see that $x_{(\alpha)} \leq x^{(\alpha)}$, and there is $x_{(\alpha)} = x^{(\alpha)}$ if and only if there exists at most one x such that $P[X \leq x] = \alpha$.

Definition 2.4 *Assumed that $E[X^-] < \infty$, then the ES of X under the confidence level $1-\alpha$ is $ES_\alpha(X) = -\frac{1}{\alpha}(E[X 1_{\{X \leq x_{(\alpha)}\}}] + x_{(\alpha)}(\alpha - P[X \leq x_{(\alpha)}]))$.*

It is obvious to get that, for any $x \in \mathbb{R}$, there holds the following

$$1^{(\alpha)}_{\{X \leq x\}} = \begin{cases} 1_{\{X \leq x\}}, \text{ if } P[X = x] = 0; \\ 1_{\{X \leq x\}} + \frac{\alpha - P[X < x]}{P[X = x]} 1_{\{X = x\}}, \text{ if } P[X = x] > 0. \end{cases}$$

By virtue of this formula, we arrive at the following $1^{(\alpha)}_{\{X \leq x_\alpha\}} \in [0,1]$, $E[1^{(\alpha)}_{\{X \leq x_\alpha\}}] = \alpha$ and $ES_{(\alpha)}(X) = -\frac{1}{\alpha} E[X 1^{(\alpha)}_{\{X \leq x_\alpha\}}]$. It is easy to see that ES is a coherent risk measure.

Theorem 2.1[1] *Let X be the real valued random variable of the probability space (Ω, F, P) with $E(X^-) < \infty$. Then, for the fixed $\alpha \in (0,1)$, there is $ES_\alpha(X) = -\frac{1}{\alpha}\int_0^\alpha \chi_{(u)} du$.*

When $\alpha = 0$, it is denoted that the worst happens, at this time $ES_{(0)}(X) = -\text{ess}\inf\{X\}$.

Theorem 2.2[2] *If $\rho_i (i = 1, 2, \cdots, n)$ are the coherent risk measure, then the combination of convex $\rho = \sum\limits_{i=1}^{n} \lambda_i \rho_i$ (where $\lambda_i \geq 0, \sum\limits_i \lambda_i = 1$) is slao a coherent risk measure; Similarly, if $\rho_\alpha(\alpha \in [a,b])$ is the set of one-parameter coherent risk measures, and for any measure $d\mu_{(\alpha)}$ on $[a,b]$, this measure satisfy $\int_a^b d\mu_{(\alpha)} = 1$, then $\rho = \int_a^b d\mu_{(\alpha)} \rho_\alpha$ is of a coherent risk measure.*

Introducing the measure $d\mu_{(\alpha)}$ on $[0,1]$, which satisfies the integral condition of the normalization $\int_0^1 \alpha d\mu_{(\alpha)} = 1$, then we get $M_\mu(X) = \int_0^1 \alpha ES_{(\alpha)}(X) d\mu(\alpha) = -\int_0^1 d\mu(\alpha) \int_0^\alpha F_X^{\leftarrow}(p) dp$ is of a coherent risk measure from Theorem 2.2, where $F_X^{\leftarrow}(p) = \inf\{\eta | F_X(\eta) \geq p\}$ is the letf-continuous inverse of the cumulative distribution function F_X. By using the integral transformation of Fubini-Tonelli, one can obtain $M_\mu(X) = -\int_0^1 F_X^{\leftarrow}(p) \int_p^1 d\mu(\alpha) dp = -\int_0^1 F_X^{\leftarrow}(p) \phi(p) dp =$

$M_\phi(X)$, $\phi(p) = \int_p^1 d\mu(\alpha)$ is called the spectrum risk, and it is easy to see that $\int_0^1 \phi(p)dp = \int_0^1 dp \int_p^1 d\mu(\alpha) = \int_0^1 d\mu(\alpha) \int_0^\alpha dp = \int_0^1 \alpha d\mu(\alpha) = 1$. Then, we know that $\phi(p) \in L^1[0,1]$ and the norm $\|\phi\| = \int_0^1 |\phi(p)|dp$.

Definition 2.5 *$\phi \in L^1[a,b]$ is called to be positive definite, if there is $\int_I \phi(p)dp \geq 0$ for any $I \subset [a,b]$; $\phi \in L^1[a,b]$ is called to be descending, if there is $\int_{q-\epsilon}^q \phi(p)dp \geq \int_q^{q+\epsilon} \phi(p)dp$ for any $q \in [a,b]$ and any $\epsilon > 0$ such that $[q-\epsilon, q+\epsilon] \subset [a,b]$.*

Definition 2.6 *$\phi \in L^1[a,b]$ is called the admissible spectrum risk , if it meets the following three conditions: (1) ϕ is the positive definite; (2) ϕ is descending; (3) $\|\phi\| = 1$.*

Definition 2.7 *The admissible spectrum risk $\phi \in L^1[0,1]$ is called to be the risk aversion function of the risk measure $M_\phi(X) = -\int_0^1 F_X^{\leftarrow}(p)\phi(p)dp$; Conversely, the risk measure M_ϕ is called to be the spectral risk measure generated by ϕ.*

Theorem 2.3[2] *Suppose $\phi \in L^1[0,1]$, $M_\phi(X) = \int_0^1 F_X^{\leftarrow}(p)\phi(p)dp$. Then, $M_\phi(X)$ is coherent risk measure if and only if ϕ is the admissible risk spectrum.*

Theorem 2.4[17] *Suppose $f(\alpha,p)$ is a separable function, i.e., $f(\alpha,p) = g(\alpha)h(p)$ with $g(\alpha) > 0$, $h(p) \geq 0$. Then, $\phi(p)$ is the admissible risk spectrum if and only if $h(p)$ is a non-creasing function, and $g(\alpha) = 1/\int_0^\alpha h(p)dp$.*

The coherent risk measure which uses the objective probability and obtains the objective risk can't reflect the subject preference. Taking the most basic coherent risk measure ES for the basic element, Acerbi (2002) who used the method of spanning space which includes the subjective risk preference get the risk measure which is coherent risk measure simultaneously. From the definition of the spectrum measure of risk, the function $\phi(p)$ associates different weights for different $F_X^{\leftarrow}(p)$, namely the tail numerical. Specifically, $\phi(p)$ puts up the higher weights to the higher risk level. So the characteristics of the subjective risk aversion of rational investors could be depicted by the function $\phi(p)$, and there is bigger penalty given as there is higher risk level. Like this the theory of the spectrum measure of risk and the characteristics of the risk preference of investors are combined perfectly by the theory of spectral measures of risk.

3 The Investment Portfolio Based on Spectral Risk Measures and its Empirical Analysis

In this section, we will discuss the efficient frontier of the portfolio of risk assets with normal distributions in the sense of mean- spectral risk measures. For a portfolio, the return $R(x,y) = x^T y$, where $x = (x_1, x_2 \cdots, x_n)^T$, x_i is the weight of the ith financial asset, and $\sum_{i=1}^n x_i = x^T I = 1, I = (1,1,\cdots,1)^T, y = (y_1, y_2, \cdots, y_n)^T$, where y_i represents the return rate of ith financial asset. The feasible set is $X = \{x | x \in \mathbb{R}, x^T I = 1\}$. Assume that the return rate $y \sim N(\mu, V)$, the return of the portfolio $R(x,y) = x^T y \sim N(x^T \mu, x^T V x)$. For convenience, let $R(x,y) = r_x$.

Example 3.1 Calculate $ES_\alpha(X)$ corresponding to $T(\alpha)$ under assumption of standard normal distribution. In fact, one arrives at by virtue of Definition of spectral risk measures the following

$$T(\alpha) = -\frac{1}{\alpha}\int_{-\infty}^{\Phi^{-1}(\alpha)} xf(x)dx = \frac{1}{\alpha}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}|_{-\infty}^{\Phi^{-1}\alpha} = \frac{1}{\alpha}f(\Phi^{-1}(\alpha)).$$

Mean-spectral Risk Measure Boundary and its Efficient Frontier

Definition 3.1 *The portfolio $x^* \in X$ belonging to the mean- spectral risk measure boundary with confidence level $1-\alpha$ means that for a $r \in R$, x^* is the solution of the following problem*

$$\begin{cases} \min\limits_{x\in X} M(r_x) = T(\alpha)\sigma(r_x) - E(r_x) \\ \qquad\qquad\quad = T(\alpha)\sqrt{x^TVx} - x^T\mu \\ E(r_x) = E(x^Ty) = x^T\mu = r, \; x^TI = 1 \end{cases} \tag{3.3.1}$$

Definition 3.2 *The portfolio $x^* \in X$ belonging to the mean- variance efficient boundary means that for a $r \in R$, x^* is the solution of the following problem*

$$\begin{cases} \min\limits_{x\in X} \sigma^2(r_x) = x^TVx \\ E(r_x) = E(x^Ty) = x^T\mu = r, \; x^TI = 1 \end{cases} \tag{3.3.2}$$

It is well known, by solving (3.3.2), that one can get the equation of mean-variance boundary is

$$\frac{\sigma^2(r_x)}{1/C} - \frac{[E(r_x) - A/C]^2}{D/C^2} = 1,$$

where $A = I^TV^{-1}\mu$, $B = \mu^TV^{-1}\mu$, $C = I^TV^{-1}I$, $D = BC - A^2$, and $x = (Cr - A)V^{-1}\mu/D + (B - Ar)V^{-1}I/D$. From these statements, we have the following

Theorem 3.1 *The portfolio x is being in the mean-spectral risk measure efficient boundary if and only if this portfolio belongs to the efficient boundary of mean-variances, and the boundary of mean-spectral risk measure optimal portfolio problems satisfies*

$$\frac{[M(r_x) + E(r_x)/T(\alpha)]^2}{1/C} - \frac{[E(r_x) - A/C]^2}{D/C^2} = 1.$$

As a result, we study the relations of $T(\alpha)$ and α first.

Theorem 3.2 *If $f(\alpha, p)$ is the risk spectrum given by Theorem 2.4, then $T(\alpha)$ is a decreasing function of α, and $T(\alpha) \to +\infty$ as $\alpha \to 0^+$.*

Proof In fact, it is not hard to see that there holds the following

$$T(\alpha) = -\int_0^\alpha f(\alpha, p)\Phi^{-1}(p)dp = -g(\alpha)\int_0^\alpha h(p)\Phi^{-1}(p)dp, \; g(\alpha) = 1/\int_0^\alpha h(p)dp.$$

Thus, we get $g'(\alpha) = -g^2(\alpha)h(\alpha)$, and also arrive at

$$\begin{aligned}\frac{dT}{d\alpha} &= -g'(\alpha)\int_0^\alpha h(p)\Phi^{-1}(p)dp - g(\alpha)h(\alpha)\Phi^{-1}(\alpha)\\ &= -g(\alpha)h(\alpha)[-g(\alpha)\int_0^\alpha h(p)\Phi^{-1}(p)dp + \Phi^{-1}(\alpha)]\\ &= -g(\alpha)h(\alpha)(M(r_x) - VaR).\end{aligned}$$

In the case of normal distributions, there holds $M(r_x) > VaR$, and $\frac{dT}{d\alpha} < 0$, then $T(\alpha)$ is a deceasing function *w.r.t.* α. Subsequently, by a direct computation, one can derive that the limit is tenable. Since we have

$$\begin{aligned}\lim_{\alpha\to 0^+} T(\alpha) &= -\lim_{\alpha\to 0^+} g(\alpha)\int_0^\alpha h(p)\Phi^{-1}(p)dp = \lim_{\alpha\to 0^+} T(\alpha)\frac{\int_0^\alpha h(p)\Phi^{-1}(p)dp}{\int_0^\alpha h(p)dp}\\ &= \lim_{\alpha\to 0^+}\frac{h(p)\Phi^{-1}(p)dp}{h(\alpha)} = -\lim_{\alpha\to 0^+}\Phi^{-1}(\alpha) = +\infty.\end{aligned}$$

This ends the Proof of Theorem 3.2. □

Portfolio of Minimum Spectral Risk Measures

Theorem 3.3 *If the portfolio of minimum spectral risk measure exists under the given confidence level* $1-\alpha(0 < \alpha < 0)$*, it must be located on the efficient frontier of mean-variance protfolios.*

Proof Reduction to absurdity is used here. Assume that the portfolio $x \in X$ is that of the minimum of spectral risk measure, which is not the mean-variance efficient frontier. Then, there exists a portfolio $x^* \in X$ such that $E(r_{x^*}) \geq E(r_x)$ and $\sigma(r_{x^*}) \leq \sigma(r_x)$, and at least one inequality is strictly tenable. Then, one has the following

$$M(r_x) = T(\alpha)\sigma(r_x) - E(r_x) \geq T(\alpha)\sigma(r_{x^*}) - E(r_{x^*}) = M(r_{x^*}).$$

Thus, for a given α, the risk $M(r_x)$ of portfolio $x \in X$ can't be the minimum, so $x \in X$ can't be the portfolio of minimum spectral risk measures. This shows the assumption is not tenable, Theorem 3.3 is proved. □

Theorem 3.4 *The portfolio of minimum spectral risk measures exists under the given confidence level* $1-\alpha(0 < \alpha < 1) \Leftrightarrow T(\alpha) > \sqrt{D/C}$*, and when* $T(\alpha) > \sqrt{D/C}$*, the portfolio* x^* *of minimum spectral risk measures is given by* $x^* = m + nE(r_{x^*})$*, where* $m = -\frac{1}{D}[B(V^{-1}I) - A(V^{-1}\mu)]$*,* $n = \frac{1}{D}[C(V^{-1}I) - A(V^{-1}I)]$*. The expected return of the portfolio with minimum spectral risk measures is* $E(r_{x^*}) = \frac{A}{C} + \sqrt{\frac{D}{C}[\frac{T^2(\alpha)}{CT^2(\alpha)-D} - \frac{1}{C}]}$*, and the corresponding minimum spectral risk measure is* $\min M = T(\alpha)\sqrt{\frac{T^2(\alpha)}{CT^2(\alpha)-D}} - E(r_{x^*})$*.*

Proof By Theorem 3.3, we know that if $\min M$ exists, it must be on the mean-variance efficient frontier

$$E(r_x) = A/C + \sqrt{\frac{D}{C}[\sigma^2(r_x) - \frac{1}{C}]} \tag{3.3.3}$$

On the other hand, the following optimization problem has a solution

$$\min_{x\in X} M(r_x) = T(\alpha)\sigma(r_x) - E(r_x) \tag{3.3.4}$$

From [17], we know that the solution x^*_σ to $\min_{x\in X}\sigma(r_x)$ satisfies $E(r_{x^*_\sigma}) = A/C, \sigma(r_{x^*_\sigma}) = 1/\sqrt{C}$. By using Theorem 3.3, we know that solving (3.3.4) is equal to solving the following

$$\begin{aligned}\min_{x\in X} M(r_x) &= \min_{\sigma\in[1/\sqrt{C},+\infty]} g(\sigma) = \min_{\sigma\in[1/\sqrt{C},+\infty]} [T(\alpha)\sigma - (A/C + \sqrt{\frac{D}{C}[\sigma^2(r_x) - \frac{1}{C}]})] \\ g'(\sigma) &= T(\alpha) - \sqrt{D/C}\frac{\sigma}{\sqrt{\sigma^2 - 1/C}}\end{aligned}$$

Since $\sigma^2 - 1/C \to 0^+$ as $\sigma \to (1/\sqrt{C})^+$, therefore

$$\min_{\sigma\to(1/\sqrt{C})^+} g'(\sigma) = \min_{\sigma\to(1/\sqrt{C})^+} [T(\alpha) - \sqrt{D/C}\frac{\sigma}{\sqrt{\sigma^2 - 1/C}}] \to -\infty$$

Thus, $g(\sigma) = M(r_x)$ can't be minimum at $\sigma = 1/\sqrt{C}$. By virtue of $\lim_{\sigma\to+\infty} g(\sigma) = +\infty$, so $g(\sigma)$ can't be minimum at infinity. In other words, if $g(\sigma)$ arrives at the minimization in $(1/\sqrt{C}, +\infty)$, it will achieve the minimal value at the critical point in this region. Let $g'(\sigma) = 0$, one has $g'(\sigma) = T(\alpha) - \sqrt{D/C}\frac{\sigma}{\sqrt{\sigma^2-1/C}} = 0$. By solving this equation, we get

$$\sigma(r_{x^*}) = \sqrt{\frac{T^2(\alpha)}{CT^2(\alpha) - D}} > \frac{1}{\sqrt{C}} \tag{3.3.5}$$

It is easy to see that there must be $CT^2(\alpha) - D > 0$ if the critical point exists, and vice versa. By a direct computation, we also have

$$\begin{aligned}g''(\sigma) &= [T(\alpha) - \sqrt{D/C}\frac{\sigma}{\sqrt{\sigma^2 - 1/C}}]' = \frac{\sqrt{D/C}[\sqrt{\sigma^2 - 1/C} - \frac{\sigma^2}{\sqrt{\sigma^2-1/C}}]}{-(\sigma^2 - 1/C)^2} \\ &= \frac{\sqrt{D/C}}{C(\sigma^2 - 1/C)^{\frac{5}{2}}}, \quad \forall\sigma \in [\frac{1}{\sqrt{C}}, +\infty).\end{aligned}$$

Therefore, if the critical point of $g(\sigma)$ does exist, it must be the local minimum point, which is also the global minimum point. So the fact that the critical point exists $\Leftrightarrow$ that the minimum value of $M(r_x)$ exists $\Leftrightarrow T(\alpha) > \sqrt{D/C}$.

In the following subsection, we solve the portfolio x^* of $\min M(r_x)$ in the setting of $T(\alpha) > \sqrt{D/C}$. According to Theorem 3.3, x^* to $\min M(r_x)$ must be the mean-variance efficient frontier portfolio when $T(\alpha) > \sqrt{D/C}$, as well as it is the mean-variance boundary portfolio. By the investment portfolio theory, the mean value $E(r_{x^*})$ and the mean-variance boundary portfolio x^* corresponding to the critical point $\sigma(r_{x^*})$ are of unique. From (3.3.3) and (3.3.5),

$$\begin{aligned}E(r_{x^*}) &= A/C + \sqrt{\frac{D}{C}[\frac{T^2(\alpha)}{CT^2(\alpha) - D} - \frac{1}{C}]} \\ x^* &= m + nE(r_{x^*})\end{aligned} \tag{3.3.6}$$

where $m = \frac{1}{D}[B(V^{-1}I) - A(V^{-1}\mu)], n = \frac{1}{D}[C(V^{-1}\mu) - A(V^{-1}I)]$. The minimum value related to $M(r_x)$ is $\min M(r_x) = T(\alpha)\sigma(r_{x^*}) - E(r_{x^*})$, $\sigma(r_{x^*})$ and $E(r_{x^*})$ are given respectively by (3.3.5) and (3.3.6). This completes the proof of Theorem 3.4. □

The Efficient Frontier of Mean-Spectral Risk Measure Portfolios

By using the similar arguments for mean-variance portfolio choice problems, it is not hard to show that there holds the following statements. A portfolio $x^* \in X$ belongs to the efficient frontier of mean-spectral risk measure under the given confidence level $1-\alpha(0 < \alpha < 1) \Leftrightarrow$ there doesn't exist a portfolio $x \in X$ so that $E(r_x) \geq e(r_{x^*})$ and $M(r_x) \leq M(r_{x^*})$ are tenable at the same time, and there exists one at least being a strict inequality, while the point $(E(r_{x^*}), M(r_{x^*}))$ corresponding to x^* is on the efficient frontier.

Theorem 3.5 *(1) If $T(\alpha) > \sqrt{D/C}$, the portfolio belongs to the efficient frontier of mean-spectral risk measures under the given confidence level $1 - \alpha \Leftrightarrow$ the portfolio is on the mean-spectral risk measure boundary and the expected return is larger than or equal to that of the minimum risk measure portfolio under the given confidence level $1 - \alpha$;*

(2) If $T(\alpha) \leq \sqrt{D/C}$, there doesn't exist an efficient frontier of mean-mean-spectral risk measures under the corresponding confidence level $1 - \alpha$.

Proof (1) Necessity. Since the portfolio belonging to the efficient frontier of mean-spectral risk measure portfolios is certainly belonging to the mean-spectral risk measure boundary. When $T(\alpha) > \sqrt{D/C}$, by using Theorem 3.4, the minimum spectral risk measure portfolio does exist and is denoted by x^*, if a portfolio x belongs to the efficient frontier of mean-spectral risk measure under the given confidence level $1 - \alpha$, there must be $M(r_x) \geq M(r_{x^*})$. This implies that $E(r_x) \geq E(r_{x^*})$ in terms of the definition of efficient frontiers.

Sufficiency. Assume that $T(\alpha) > \sqrt{D/C}$, a portfolio $x \in X$ belongs to the mean-spectral risk measure boundary, and $E(r_x) \geq E(r_{x^*})$, $x^* \in X$ is the minimum spectral risk measure portfolio under the confidence level $1 - \alpha$. In order to prove that x belongs to the efficient frontier of mean-spectral risk measure portfolio under the confidence level $1 - \alpha$, we only need to prove the following two inequalities are tenable simultaneously

$$\frac{dM(r_x)}{dE(r_x)} > 0; \ \frac{d^2M(r_x)}{dE(r_x)^2} > 0 \tag{3.3.7}$$

Let $g(\sigma) = M(r_x)$, for the portfolio x^* minimizing $M(r_x)$, by using Theorem 3.3, one gets that its mean and variance satisfy $\sigma^* > 1/\sqrt{C}, g'(\sigma^*) = 0$, and there also holds the following

$$g''(\sigma^*) > 0, \ \ \forall\sigma > 1/\sqrt{C} \tag{3.3.8}$$

Thus, we have

$$g'(\sigma) > g'(\sigma^*), \text{ if } \sigma > \sigma^* \tag{3.3.9}$$

Since the portfolios x^* and x belong to the mean-spectral risk measure boundary, so they are on the mean-variance boundary. According to Theorem 3.3 and $E(r_x) \geq E(r_{x^*})$, one can obtain that x^* is on the mean-variance efficient frontier, and x is also on the frontier naturally. Thus we have $\frac{d\sigma(r_x)}{dE(r_x)} > 0$. By using this inequality and (3.3.9), we know that

$$\frac{dM(r_x)}{dE(r_x)} = \frac{dM(r_x)}{d\sigma(r_x)}\frac{d\sigma(r_x)}{dE(r_x)} = g'(\sigma)\frac{d\sigma(r_x)}{dE(r_x)} > 0$$

Since x is a portfolio on mean-spectral risk measure boundary, we can get

$$\sigma(r_x) = \sqrt{\frac{1}{C} + \frac{[E(r_x) - A/C]^2}{D/C}}$$

According to the above equation, we have

$$\begin{aligned}\frac{dM(r_x)}{dE(r_x)} &= \frac{d[T(\alpha)\sigma(r_x) - E(r_x)]}{dE(r_x)} = T(\alpha)\Big[\sqrt{\frac{1}{C} + \frac{[E(r_x) - A/C]^2}{D/C}}\Big]' - 1 \\ &= T(\alpha)\frac{E(r_x) - A/C}{D/C}\Big/\sqrt{\frac{1}{C} + \frac{[E(r_x) - A/C]^2}{D/C}} - 1\end{aligned}$$

$$\frac{d^2M(r_x)}{dE(r_x)^2} = T(\alpha)\frac{\sqrt{\frac{1}{C} + \frac{[E(r_x)-A/C]^2}{D/C}} - \frac{\frac{[E(r_x)-A/C]^2}{D/C}}{\sqrt{\frac{1}{C} + \frac{[E(r_x)-A/C]^2}{D/C}}}}{\frac{1}{C} + \frac{[E(r_x)-A/C]^2}{D/C}} = \frac{T(\alpha)/D}{\sqrt[3]{[\frac{1}{C} + \frac{[E(r_x)-A/C]^2}{D/C}]}} > 0$$

This shows that (3.3.7) is tenable.

(2) When $T(\alpha) < \sqrt{D/C}$, it is easy to see that there holds $\frac{dM(r_x)}{dE(r_x)} < 0$. Since the efficient frontier of mean-spectral risk measures must belong to the mean-spectral risk measure boundary, and for any portfolio $x \in X$ on the mean-spectral risk measure boundary, it means that the slope of mean-spectral risk measure boundary is negative everywhere. As a result, for every portfolio $x \in X$, there exists another portfolio $y \in X$, whose expected return is higher but the spectral risk measure is lower, which shows that x is not the efficient frontier portfolio of the mean-spectral risk measure. This ends the proof of Theorem 3.5. □

According to Theorem 3.5, for any $0 < \alpha < 1$, there exists the expected return vector $\mu \in R^n$ and the covariance matrix V such that $T(\alpha) \leq \sqrt{D/C}$, thereby it produces an empty efficient frontier of mean-spectral risk measure under the confidence level $1 - \alpha$.

Theorem 3.6 *Let the efficient sets of mean-spectral risk measure and mean-variance portfolios be denoted by X_M, X_σ, respectively, under a given $\alpha(0 < \alpha < 1)$, then we can get $X_M \subseteq X_\sigma \subseteq X$.*

Proof When $T(\alpha) \leq \sqrt{D/C}$, the minimum spectral risk measure portfolio doesn't exist. By using Theorem 3.5, we know the mean-spectral risk measure efficient portfolio is an empty set, so it's a trivial subset of the mean-spectral risk measure efficient portfolio set.

When $T(\alpha) > \sqrt{D/C}$, Theorem 3.5 shows that the efficient portfolio set of mean-spectral risk measures under the confidence level $1 - \alpha$ consists of the portfolios whose expected returns are larger than or equal to those consisting of the minimum mean-variance efficient frontier with expected return $E(r_{x^*})$. Since $E(r_{x^*}) > E(r_{x^*_\sigma}) = A/C$, the mean-spectral risk measure efficient frontier is the subset of the mean-variance efficient frontier, this implies that there holds the following $X_M \subseteq X_\sigma \subseteq X$. This completes the proof of Theorem 3.6. □

Emperical Analysis of Portfolios Based on Spectral Risk Measures

Choose four good growth stocks and study the potential risk of their portfolios in China. The four chosen stocks are: AAAA Shares, BBBB Shares, CCCC Shares and DDDD Shares.

The time interval is from January, 2002 to December, 2007, 72 months in total. The above data is from http://www.business.sohu.com.

The steps of solving the efficient frontier of security portfolios are as follows:

(1) Solve the monthly yield serials of each security. Get the yearly yield by multiplying the monthly rate by 12.

(2) Estimate the expected yearly yields of different securities. The sample expected yields of AAAA Shares, BBBB Shares, CCCC Shares and DDDD Shares are: 0.28210.19490.16270.2598.

(3) Estimate the standard error of different securities. The sample standard errors of AAAA Shares, BBBB Shares, CCCC Shares and DDDD Shares are: 1.23371.54491.65631.7406.

(4) Compute the covariance matrix of the 4 securities:

$$V = \begin{pmatrix} 1.5520 & 0.3909 & 0.3369 & 1.0352 \\ 0.3909 & 2.3868 & 1.3109 & 1.1206 \\ 0.3369 & 1.3109 & 2.7433 & 1.1712 \\ 1.0352 & 1.1206 & 1.1712 & 3.0296 \end{pmatrix}$$

(5) Solve the constants $A = I^T V{-}1\bar{R} = 0.2314$, $B = \bar{R}^T V^{-1}\bar{R} = 0.0596$, $C = I^T V^{-1} I = 0.9719$, $D = BC - A^2 = 0.0043 > 0$.

(6) The expected returns yield from 0.01 to 0.5 by increasing 0.005 every time, 99 points in total. For each expected return, we can calculate a minimum standard error and get a point in the mean- standard variance coordinate system by applying the efficient boundary equation

$$\frac{\sigma^2(r_x)}{1/C} - \frac{[E(r_x) - A/C]^2}{D/C^2} = 1.$$

As a result, we can have 99 point from the 99 expected yields which constitute the minimum standard variance curve.

(7) Solve the minimum variance portfolio point, whose standard variance $1/\sqrt{C} = 1.0144$ and the expected yield is $A/C = 0.2381$.

(8) Solve the expected yield and variance corresponding to the minimum spectral risk measure, where the confidence level is 95%, i.e. $\alpha = 0.05$. For convenience, let $f(\alpha, p) = 1/\alpha$, then $T(\alpha) = \frac{1}{\alpha}\phi(\Phi^{-1}(\alpha)) = 2.0793$. Then, we have

$$E(r_{x^*}) = A/C + \sqrt{\frac{D}{C}[\frac{T^2(\alpha)}{CT^2(\alpha) - D} - \frac{1}{C}]} = 0.2402, \sigma(r_{x^*}) = \sqrt{\frac{T^2(\alpha)}{CT^2(\alpha) - D}} = 1.0149.$$

According to these steps above, one can draw the graph for mean-spectral risk measures by using the similar argument as drawing the graph for mean-variance portfolio selection problem. Here we omit it.

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