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Some Vector Sequence Spaces-II

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Abstract

In this paper, we present some theorems involved in determining sets and Schauder basis. Matrix transformation of some vector sequence spaces are also studied.

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1 Introduction

Let X denote a commutative Banach algebra with the identity e. Then $ae = ea = a \; \forall \; a \in X$ and $\|e\| = 1$.

Let Φ be the set of all finite sequences in X. Let $x_k \in X$ and we write $x = \{x_k\}$ and it is a vector sequence. If S is a vector sequence space then $D = \{x \in S : \|\|x\|\| \leq 1\} \cap \Phi$. If $E \subset \Phi$ then the absolutely convex hull of E is denoted by A. The set E is called a determining set for S if $A = D$.

If S and V are sequence spaces, the set of all infinite matrices A with entries in X is denoted by $(S : V)$. This paper is devoted to characterization of $(l : l_\infty)$ and $(b_{v_0} : l_\infty)$ by using determining sets.
$\ell = \{x : \sum \|x_k\| < \infty\}$.
$\ell_\infty = \{x : sup\|x_k\| < \infty\}$.
$b_{v_0} = \{x : \sum \|x_k - x_{k+1}\| < \infty\}$
$c_0 = \{x : \lim_{k\to\infty} \|x_k\| < \varepsilon, \;\; \forall \varepsilon > 0\}$. Also the topological duals ℓ^* and C_0^* are also find out.

2 Results

Theorem 1: $E = (\delta^1, \delta^2, \dots)$ is determining set for ℓ.

Proof: $\delta^n = (0, \dots, 0, 1, 0, \dots)$ 1 in the n^{th} place and zeros elsewhere.

$A =$ absolutely convex hull of E where $E = (\delta^1, \delta^2, \dots)$

$D = ($ the closed unit ball in $\ell) \cap \Phi$. Let $x \in A$.

$\Rightarrow x = t_1\delta^1 + t_2\delta^2 + \dots + t_m\delta^m$

With $\|t_1\| + \|t_2\| + \dots + \|t_m\| \leq 1$ and $t_i \in X$.

$\Rightarrow x = t_1(e, 0, 0, \dots) + t_2(0, e, \dots) + \dots t_m(0, \dots e, 0, \dots)$

$\Rightarrow x = (t_1, t_2, \dots t_m, 0, 0, \dots)$

$\Rightarrow x \in \Phi.$ (1)

Also,

$$\begin{aligned} |||x||| &= |||t_1\delta^1 + t_2\delta^2 + \dots + t_m\delta^m||| \\ &\leq |||t_1\delta^1||| + |||t_2\delta^2||| + \dots + |||t_m\delta^m||| \\ &= |||t_1|||\quad |||\delta^1||| + |||t_2|||\quad |||\delta^2||| + \dots + |||t_m|||\quad |||\delta^m||| \end{aligned}$$

Note that $|||\delta^1||| = \|(1, 0, \dots)\| = 1 + 0 + \dots = 1,$

$|||\delta^2||| = 1, \dots, |||\delta^m||| = 1$

$|||x||| = |t_1| + |t_2| + \dots |t_m| \leq 1$

$|||x||| \quad \leq 1.$ (2)

$\Rightarrow x \in D \qquad \Rightarrow A \subset D$ (3)

On the other hand let $x \in D$.

$\Rightarrow |||x||| \leq 1$ and $x \in \Phi$. But $x \in \Phi \Rightarrow x$ is a finite sequence.

$$\begin{aligned} \Rightarrow x &= (x_1, x_2, \dots x_m, 0, 0, \dots) \\ &= (x_1, 0, \dots) + (0, x_2, \dots) + \dots + (0, 0, \dots, x_m, 0, 0, \dots) \\ &= x_1(1, 0, 0, \dots) + x_2(0, 1, 0 \dots) + \dots + x_m(0, \dots, 1, 0, \dots) \\ &= x_1\delta^1 + x_2\delta^2 + \dots + x_m\delta^m. \end{aligned}$$

where $|||x_1||| + \dots + |||x_m||| \leq |||x||| \leq 1$.

Hence $|||x_1||| + \dots + |||x_m||| \leq 1$. Thus, $x = x_1\delta^1 + x_2\delta^2 \dots + x_m\delta^m$ with $||x_1|| + \dots ||x_m|| \leq 1 \Rightarrow x \in A$

$\Rightarrow D \subset A$ (4)

From (3) and (4), $A = D$.

$\Rightarrow E$ is a determining set of ℓ.

Theorem 2: $\{\delta^k\}$ is a Schauder basis for ℓ.

Proof:

$$\begin{aligned}
\sum_{k=1}^{n} x_k\delta^k &= x_1(e,0,0,\dots)+x_2(0,e,0,\dots)+\dots+x_n(0,\dots,e,0,\dots)\\
&= (x_1,x_2,\dots,x_n,0,\dots)\\
&= x^{[n]} \qquad (5)\\
\Rightarrow |||x-x^{[n]}||| &= |||(0,0,\dots,0,x_{n+1},x_{n+2},0,\dots)|||\\
&= \sum_{k\geq n+1}^{\infty} \|x_k\| \quad \to 0 \text{ as } n\to\infty\\
\Rightarrow |||x-\sum_{k=1}^{n} x_k\delta^k||| &\to 0\text{as } n\to\infty. \qquad (6)\\
\text{Also if } |||x-\sum_{k=1}^{n} y_k\delta^k||| &\to 0 \text{ as } n\to\infty\\
x &= \sum x_k\delta^k = \sum y_k\delta^k\\
\Rightarrow (x_1,x_2,\dots) &= (y_1,y_2,\dots)\\
\Rightarrow x_k = y_k\ \forall k &\\
\Rightarrow \{x_k\} \text{ is a unique.} & \qquad (7)
\end{aligned}$$

From (6) and (7) it follows that $\{\delta^k\}$ is a Schauder basis for ℓ.

Theorem 3: Let $A=(a_{nk})$, $(n,k=1,2,\dots)$ be an infinite matrix. Then

$$A\in(\ell:\ell_\infty)\Leftrightarrow \sup_{(n,k)}||a_{nk}||<\infty$$

Proof: $E=(\delta^1,\delta^2,\dots)$ is a determining set for ℓ. Also $E=(\delta^1,\delta^2,\dots)$ is a Schauder basis for ℓ, Hence ℓ has AK. We know that ℓ_∞ is a BK space with the norm $|||x|||=\sup_{(k)}||x_k||$.

$A\in(\ell:\ell_\infty)\Leftrightarrow$

1. The columns of A are in ℓ_∞

2. $A(E)$ is a bounded set in ℓ_∞

$$\begin{aligned}
We\ shall\ calculate\ A(E). We\ have\ A(E) &= A\{(\delta^1,\delta^2,\dots)\}\\
A_n(x) &= \sum_{k=1}^{\infty} a_{nk}x_k \text{ with } x_k=1, x_i=0\ \ \&\ \ x\neq k\\
&= (a_{nk})_{k=1}^{\infty}\ \ \forall n
\end{aligned}$$

But $A(E)$ is bounded $\Leftrightarrow |||(a_{nk}|||$ is bounded in ℓ_∞

$$\Leftrightarrow \sup_{(n,k)} ||a_{nk}|| < \infty$$

Theorem 4:$A \in (\ell : \ell) \Leftrightarrow \sup_{(n)} \sum_{k=1}^{\infty} ||a_{nk}|| < \infty.$
Proof :

$$\begin{aligned} A(E) \text{ is bounded in } \ell &\Leftrightarrow \sum_{k=1}^{\infty} ||a_{nk}|| \text{ is bounded unifomly in } \ell \\ &\Leftrightarrow \sup_{(n)} \sum_{k=1}^{\infty} ||a_{nk}|| < \infty. \end{aligned}$$

Theorem 5: $\ell^* = \ell_\infty$
Proof : $x \in \ell \Rightarrow x = \sum x_k \delta^k$
$\Rightarrow f(x) = \sum x_k y_k, \;\; y_k = f(\delta^k)$
Also $||y_n|| = ||f(\delta^n)|| \leq ||f|| \;\; ||\delta^n|| = ||f||$
$\Rightarrow y \in \ell_\infty$
Also

$$|||y||| = \sup_{(n)} ||y_n|| \leq ||f|| \tag{8}$$

Define $T : \ell^* \to \ell_\infty$ by $T(f) = y$
From (8)

$$|||T(f)||| \leq ||f|| \tag{9}$$

Give $y \in \ell_\infty$, define f on ℓ by $f(x) = \sum x_n y_n \Rightarrow f$ is linear. Also

$$\begin{aligned} ||f(x)|| &\leq \sum ||x_n y_n|| \\ &\leq \sup_{(n)} ||y_n|| \left(\sum ||x_n|| \right) \\ &\leq \left(\sup_{(n)} |y_n| \right) |||x||| \end{aligned}$$

$\Rightarrow f$ is bounded and hence continuous
$\Rightarrow f \in \ell^*$ with $T(f) = y$
$\Rightarrow T$ is subjective
Also $|f(x)| \leq ||f|| \;\; |||x|||$ where

$$||f|| = \sup_{||x|| \leq 1} |f(x)| \leq |||y(f)||| \;\; |||T(f)||| \tag{10}$$

From (9) and (10)

$$|||T(f)||| = ||f||$$

$\Rightarrow T$ is a linear Isometry of ℓ^* on to $\ell_\infty \Rightarrow \ell^* = \ell_\infty$.

Theorem 6: $c_0^* = \ell$

Proof : $x = \sum x_n \delta^n$

$$\Rightarrow f(x) = \sum x_n y_n \tag{11}$$

$y_n = f(\delta^n)\ \forall f \in c_0^*$.

Let $x^{(n)} = \sum_{k=1}^{n} z_k \delta^k$

Where $z_k = \frac{y_k}{||y_k||}, 1 \leq k \leq n$

But then $||x^{(n)}|| \leq 1$ From (11) $f(x^{(n)}) = \sum_{k=1}^{n} ||y_k||$

$\Rightarrow \sum_{k=1}^{n} ||y_k|| = |f(x^{(n)})| \leq ||f||\ \ ||x^{(n)}|| \leq ||f||$

Hence $(s_n), s_n = \sum_{k=1}^{n} ||y_k||$ is a Monotonically increasing sequence bounded above

$\Rightarrow (s_n)$ converges to its supremum, letting $n \to \infty$

$$\sum_{n=1}^{\infty} \|y_n\| \leq \|f\| \text{ and so } y \in l \Rightarrow ||y|| \leq ||f|| \tag{12}$$

Define $T : c_0^* \to \ell$ by $T(f) = y = y_n$

From(12)

$$||T(f)|| \leq ||f|| \tag{13}$$

Now for any given $y \in \ell$

$\sum ||x_n y_n||$ Converges $\forall x \in c_0$

Define f on c_0 by (11).Then

$|f(x)| \leq \sum ||x_n y_n|| \leq (\sum ||y_n||)||x||$

$\Rightarrow f \in c_n^*$ with $T(f) = y$

$\Rightarrow T$ is a Adjective. Also

$$||f|| \leq \sum ||y_n|| = ||y|| = ||T(f)|| \tag{14}$$

By(13) and (14) T is a linear bijective such that

$||T(f)|| = ||f|| \Rightarrow c_0^* = l$.

Definition: $bv_0 = \{x : \sum_{k=1}^{\infty} ||x_k - x_{k+1}|| < \infty\}$. bv_0 is a BK-Space with the norm $|||x||| = \sum_{k=1}^{\infty} ||x_k - x_{k+1}||$

Notation we write

$$\begin{aligned} e^{(1)} &= (e, 0, 0, \dots) \\ e^{(2)} &= (e, e, 0, 0, \dots) \\ e^{(k)} &= (e, e, \dots, e, 0, 0, \dots)(\text{k times}) \end{aligned}$$

And so on. Note that $e^{(1)} = \delta^1, \;\; e^{(2)} = \delta^1 + \delta^2, \;\; e^{(3)} = \delta^1 + \delta^2 + \delta^3$

Theorem 7:

Let $E = (e^{(1)}, e^{(2)}, \dots)$.Then E is a determining set for bv_0.

Proof : A = The absolute convex hull of E.

$D =$ (The closed unit ball in bv_0) $\cap \Phi$

Let $X \in A$

$$\Rightarrow x = (t_1 e^{(1)} + t_2 e^{(2)} + \dots + t_m e^{(m)})$$

with $||t_1|| + ||t_2|| + \dots + ||t_m|| \leq 1$, and also $t_i \in X$

$x = (t_1 + t_2 + \dots + t_m, t_2 + t_3 \dots + t_m, \dots , t_m, 0, 0, \dots)$

$$\Rightarrow x \in \phi \tag{15}$$

And

$$|||x||| \leq ||T_1|| \;\; ||e^{(1)}|| + ||t_2|| \;\; ||e^2|| + \dots + ||t_m|| \;\; ||e^m||$$

where

$$\begin{aligned} ||e^{(1)}|| &= ||e - 0|| + ||0 - 0|| + \dots \\ &= ||e|| \\ &= 1 \\ ||e^{(2)} &= 1, \dots , ||e^{(m)}|| = 1 \end{aligned}$$

$$\Rightarrow |||x||| \leq ||t_1|| + \dots + \;\; ||t_m|| \leq 1 \tag{16}$$

From(15) and (16)$X \in D$. Hence $A \subset D$.

Let $X \in D$

$$\begin{aligned}
&\Rightarrow && |||x||| \leq 1 and x \in \phi \\
&\Rightarrow && x = (x_1, x_2, \dots, x_m, 0, 0, \dots) \\
&= && (x_1, 0, \dots) + (0, x_2, 0, \dots) + (0, \dots, x_m, 0, \dots) \\
&= && x_1(e, 0, 0, \dots) + x_2(0, e, 0, \dots) + \dots + x_m(0, 0, \dots, 0, e, 0, \dots) \\
&= && x_1 e^{(1)} + x_2(e^{(2)} - e^{(1)}) + x_3(e^{(3)} - e^{(2)}) + \dots + x_m(e^{(m)} - e^{(m-1)}) \\
&= && \sum_{k=1}^{m} (x_k - x_{k+1}) e^{(k)}
\end{aligned}$$

By Abel's inequality
where $\sum_{k=1}^{m} \|x_k - x_{k+1}\| \leq |||x||| \leq 1$
$\Rightarrow x \in A$
$\Rightarrow D \subset A$
Thus $D = A$
$\Rightarrow E$ is a determining set for bv_0
Theorem 8: Let $A = (a_{nk})$ $n, k = 1, 2, \dots$ be an infinite matrix with $a_{nk} \in X$ then $A \in (bv_0 : \ell_\infty) \Leftrightarrow \sup_{(n,k)} ||a_{(n1)} + a_{(n2)} + \dots + a_{(nk)}|| < \infty$
Proof : We know that $E = (e^{(1)}, e^{(2)}, \dots)$ is a determining set for bv_0. By the lemma $A \in (bv_0 : \ell_\infty) \Leftrightarrow$

1. The columns of A are in ℓ_∞,and
2. $A(E)$ is a bounded set in l_∞

Then $A(E) = (a_{n_1} + a_{n_2} + \cdots + a_{n_k})_{n,k=1}^{\infty}$
Since $A(E)$ is bounded in ℓ_∞
We have $\Rightarrow \sup_{(n,k)} ||a_{(n1)} + a_{(n2)} + \dots + a_{(nk)}|| < \infty$
Thus $A \in (bv_0 : \ell_\infty) \Leftrightarrow \sup ||a_{n1} + a_{n2} + \dots + a_{nk}|| < \infty$

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