Int. Journal of Math. Analysis, Vol. 3, 2009, no. 35, 1745 - 1759

Approximating Common Fixed Points for a Finite Family of Non-Lipschitzian Mappings in Banach Spaces

S. Imnang and S. Suantai

Department of Mathematics, Faculty of Science
Chiang Mai University, Chiang Mai, 50200, Thailand
suwicha.n@hotmail.com (S. Imnang)
scmti005@chiangmai.ac.th (S. Suantai)

Abstract

In this paper, we introduce multi-step iterative scheme with errors for a finite family of asymptotically nonexpansive mappings in the intermediate sense. Mann-type, Ishikawa-type and Noor-type iterations are covered by the new iterative scheme. Weak and strong convergence theorems for the iterative scheme in a uniformly convex Banach space are established under some conditions which are weaker than demicompactness or completely continuous. Our results improve and generalize the recent known results in the literature.

Mathematics Subject Classification: 47H10, 47H09, 46B20

Keywords: Common fixed point; Uniformly convex Banach space; Strong and weak convergence; Asymptotically nonexpansive mapping in the intermediate sense; Opial's property

1 Introduction

Let C be a subset of a real Banach space X. Let T be a self-mapping of C and $F(T)$ denote the fixed points set of T, *i.e*, $F(T) := \{x \in C : Tx = x\}$. Recall that a mapping T is said to be *asymptotically nonexpansive* on C if there exists a sequence $\{b_n\}$ in $[0, \infty)$ with $\lim_{n\to\infty} b_n = 0$ such that for each x, $y \in C$,

$$\|T^n x - T^n y\| \leq (1 + b_n)\|x - y\|,\ \forall n \geq 1.$$

If $b_n = 0$ for all $n \geq 1$, then T is known as a *nonexpansive mapping.* T is called *asymptotically nonexpansive mapping in the intermediate sense* [2] provided T is uniformly continuous and

$$\limsup_{n\to\infty} \sup_{x,y\in C} (\|T^n x - T^n y\| - \|x-y\|) \leq 0.$$

From the above definitions, it follows that an asymptotically nonexpansive mapping must be asymptotically nonexpansive mapping in the intermediate sense. It is known [10] that if X is a uniformly convex Banach space and T is self-mapping of bounded closed convex subset C of X which is an asymptotically nonexpansive mapping in the intermediate sense, then $F(T) \neq \emptyset$.

A mapping T is called *semi-compact* if any bounded sequence $\{x_n\}$ in C with $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\{x_{n_i}\}$ converges strongly to some x^* in C.

Recall that a mapping $T : C \to C$ with $F(T) \neq \emptyset$ is said to satisfy *condition* (I) [17] if there is a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(t) > 0$ for all $t \in (0,\infty)$ such that $\|x - Tx\| \geq f(d(x, F(T)))$ for all $x \in C$, where $d(x, F(T)) = \inf \{\|x - p\| : p \in F(T)\}$.

In 1974, Senter and Dotson [17] studied the convergence of the Mann iteration scheme defined by $x_1 \in C$,

$$x_{n+1} = \alpha_n T x_n + (1-\alpha_n) x_n, \ \forall n \geq 1, \tag{1.1}$$

in a uniformly convex Banach space, where $\{\alpha_n\}$ is a sequence satisfying $0 < a \leq \alpha_n \leq b < 1 \ \forall n \geq 1$ and T is a nonexpansive (or a quasi-nonexpansive) mapping. They established a relation between *condition* (I) and *demicompactness*.

A mapping T is said to be *demicompact* if for every bounded sequence $\{x_n\}$ in C such that $\{x_n - Tx_n\}$ converges, there exists a subsequence say $\{x_{n_j}\}$ of $\{x_n\}$ such that converges strongly to some y in C. Every compact and semicompact mapping are demicompact. They actually showed that *condition* (I) is weaker than demicompactness for a nonexpansive mapping defined on bounded set.

Xu and Noor [19], in 2002, introduced a three-step iterative scheme as follows:

$$\begin{aligned} z_n &= a_n T^n x_n + (1-a_n) x_n, \\ y_n &= b_n T^n z_n + (1-b_n) x_n, \\ x_{n+1} &= \alpha_n T^n y_n + (1-\alpha_n) x_n, \ n \geq 1, \end{aligned} \tag{1.2}$$

where $\{a_n\}$, $\{b_n\}$, $\{\alpha_n\}$ are appropriate sequences in $[0,1]$. The theory of three-step iterative scheme is very rich, and this scheme, in the context of one or more mappings, has been extensively studied (for example, see Khan et at. [7], Plubtieng and Wangkeeree [15], Fukhar-ud-din and Khan [5], Petrot [13] and Suantai [18]). It has been shown in [1], that three-step method performs better than two-step and one-step methods for solving variational inequalities.

Very recently, in 2009, Yang et al. [20] proved the weak and strong convergence theorems for the modified three step iterations with errors for asymptotically nonexpansive mappings in the intermediate sense in a uniformly convex Banach space. Their results generalized and improved the recent ones announced by Osilike and Aniagbosor [12], Nammanee and Suantai [11], Kim and Kim [9], Xu and Noor [19] and some others.

Inspired and motivated by these facts, we introduce a new iteration process for a finite family of $\{T_i : i = 1, 2, \dots, k\}$ of asymptotically nonexpansive mappings in the intermediate sense as follows:

Let $T_i : C \to C$ $(i = 1, 2, \dots, k)$ be mappings and $F := \bigcap_{i=1}^{k} F(T_i)$. For a given $x_1 \in C$, and a fixed $k \in \mathbb{N}$ ($\mathbb{N}$ denote the set of all positive integers), compute the iterative sequences $\{x_n\}$ and $\{y_{in}\}$ by

$$\begin{aligned}
x_{n+1} = y_{kn} &= \alpha_{kn} T_k^n y_{(k-1)n} + \beta_{kn} x_n + \gamma_{kn} u_{kn}, \\
y_{(k-1)n} &= \alpha_{(k-1)n} T_{k-1}^n y_{(k-2)n} + \beta_{(k-1)n} x_n + \gamma_{(k-1)n} u_{(k-1)n}, \\
&\vdots \\
y_{3n} &= \alpha_{3n} T_3^n y_{2n} + \beta_{3n} x_n + \gamma_{3n} u_{3n}, \\
y_{2n} &= \alpha_{2n} T_2^n y_{1n} + \beta_{2n} x_n + \gamma_{2n} u_{2n}, \\
y_{1n} &= \alpha_{1n} T_1^n y_{0n} + \beta_{1n} x_n + \gamma_{1n} u_{1n},
\end{aligned} \tag{1.3}$$

where $y_{0n} = x_n$ and $\{u_{1n}\}, \{u_{2n}\}, \dots, \{u_{kn}\}$ are bounded sequences in C with $\{\alpha_{in}\}, \{\beta_{in}\}$ and $\{\gamma_{in}\}$ are appropriate real sequences in $[0, 1]$ such that $\alpha_{in} + \beta_{in} + \gamma_{in} = 1$ for all $i = 1, 2, \dots, k$ and all n. Our iteration includes and extends the Mann iteration (1.1), three-step iteration by Xu and Noor (1.2), the multi-step Noor iterations with errors by Plubtieng and Wangkeeree [15] and the iteration defined by Khan et al. [7], simultaneously.

The purpose of this paper is to establish strong convergence theorems of the iterative scheme (1.3) for a finite family of asymptotically nonexpansive mappings in the intermediate sense when one mapping T_i satisfies a condition which is weaker than demicompactness and we also prove weak convergence theorem for a finite family of asymptotically nonexpansive mappings in the intermediate sense in a uniformly convex Banach space satisfying Opial's property. Our results generalize and improve the corresponding ones announced by H. Fukhar-ud-din and A. R. Khan [4], J. U. Jeong and S. H. Kim [6], and many others.

2 Preliminaries

In the sequel, the following lemmas are needed to prove our main results.

A mapping T with domain $D(T)$ and range $R(T)$ in X is said to be *demiclosed* at 0 if whenever $\{x_n\}$ is a sequence in $D(T)$ such that $\{x_n\}$ converges weakly to $x \in D(T)$ and $\{Tx_n\}$ converging strongly to 0, we have $Tx = 0$.

A Banach space X is said to satisfy *Opial's property* if for each x in X and each sequence $\{x_n\}$ weakly convergent to x, the following condition holds for $x \neq y$:

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|.$$

It is well know that all Hilbert spaces and l_p $(1 < p < \infty)$ spaces have Opial's property while L_p spaces $(p \neq 2)$ have not. A family $\{T_i : i = 1, 2, \ldots, k\}$ of self-mappings of C with $F := \bigcap_{i=1}^{k} F(T_i) \neq \emptyset$ is said to satisfy

1. *condition* $(\overline{A})$ [3] if there is a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(t) > 0$ for all $t \in (0, \infty)$ such that $\frac{1}{k}\sum_{i=1}^{k} \|x - T_i x\| \geq f(d(x, F))$ for all $x \in C$, where $d(x, F) = \inf\{\|x - p\| : p \in F\}$;

2. *condition* $(\overline{B})$ [3] if there is a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(t) > 0$ for all $t \in (0, \infty)$ such that $\max_{1\leq i\leq k}\{\|x - T_i x\|\} \geq f(d(x, F))$ for all $x \in C$;

3. *condition* $(\overline{C})$ [3] if there is a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(t) > 0$ for all $t \in (0, \infty)$ such that $\|x - T_l x\| \geq f(d(x, F))$ for all $x \in C$ and for at least one $T_l, l = 1, 2, \ldots, k$.

Note that conditions $(\overline{B})$ and $(\overline{C})$ are equivalent, condition $(\overline{B})$ can reduces to condition (I) when all but one of T_i's are identities

It is well known that every continuous and demicompact mapping must satisfy condition (I) (see [17]). Since every completely continuous is continuous and demicompact so that it satisfies condition (I). Thus we shall use the condition $(\overline{C})$ instead of demicompactness and complete continuity of a family $\{T_i : i = 1, 2, \ldots, k\}$.

Lemma 2.1. [16, Lemma 1] *Let $\{a_n\}, \{b_n\}$ and $\{\delta_n\}$ be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq (1 + \delta_n)a_n + b_n, \quad \forall n = 1, 2, \ldots.$$

If $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then

(i) $\lim_{n\to\infty} a_n$ *exists;*

(ii) $\lim_{n\to\infty} a_n = 0$ *whenever* $\liminf_{n\to\infty} a_n = 0$.

Lemma 2.2. [15, Lemma 3.1] *Let X be a uniformly convex Banach space, $\{x_n\}$, $\{y_n\} \subset X$, real numbers $a \geq 0$, $\alpha, \beta \in (0,1)$ and $\{\alpha_n\}$ be a real sequence number which satisfies*

(i) $0 < \alpha \leq \alpha_n \leq \beta < 1$, $\forall n \geq n_0$ *and for some* $n_0 \in \mathbb{N}$;

(ii) $\limsup_{n\to\infty} \|x_n\| \leq a$ *and* $\limsup_{n\to\infty} \|y_n\| \leq a$;

(iii) $\lim_{n\to\infty} \|\alpha_n x_n + (1-\alpha_n)y_n\| = a$.

Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

Lemma 2.3. [18, Lemma 2.7] *Let X be a Banach space which satisfies Opial's property and let $\{x_n\}$ be a sequence in X. Let u, $v \in X$ be such that $\lim_{n\to\infty} \|x_n - u\|$ and $\lim_{n\to\infty} \|x_n - v\|$ exist. If $\{x_{n_k}\}$ and $\{x_{m_k}\}$ are subsequences of $\{x_n\}$ which converge weakly to u and v, respectively, then $u = v$.*

Lemma 2.4. [20, Lemma 1.6] *Let X be a real uniformly convex Banach space and C a nonempty closed convex subset of X. Let $T : C \to C$ be a mapping which is asymptotically nonexpansive in the intermediate sense. If $\{x_n\}$ is a sequence in C converging weakly to x^* and if*

$$\lim_{j\to\infty} (\limsup_{n} \|x_n - T^j x_n\|) = 0,$$

then $I - T$ is demiclosed at 0, i.e., for each sequence $\{x_n\}$ in C, if the sequence $\{x_n\}$ converges weakly to $x^* \in C$ and $\{(I-T)x_n\}$ converges strongly to 0, then $(I-T)x^* = 0$.

3 Convergence theorems in Banach spaces

We first prove a strong convergence theorem of the iterative scheme (1.3) for a finite family of asymptotically nonexpansive mappings in the intermediate sense in a Banach space. In order to prove this, the following lemma is needed.

Lemma 3.1. *Let C be a nonempty closed convex subset of a Banach space X. Let $T_1, T_2, \ldots, T_k : C \to C$ be asymptotically nonexpansive mappings in the intermediate sense. Put*

$$G_{in} = \sup_{x,y \in C} (\|T_i^n x - T_i^n y\| - \|x-y\|) \vee 0, \qquad \forall n \geq 1,$$

so that $\sum_{n=1}^{\infty} G_{in} < \infty$ for all $i = 1, 2, \ldots, k$. Assume that $F \neq \emptyset$ and $\sum_{n=1}^{\infty} \gamma_{in} < \infty$ for each $i = 1, 2, \ldots, k$. For a given $x_1 \in C$, let the sequences $\{x_n\}$ and $\{y_{in}\}$ be defined by (1.3). Then

(a) *there exist sequences* $\{d_{in}\}$ *in* $[0,\infty)$ *such that* $\sum_{n=1}^{\infty} d_{in} < \infty$ *and* $\|y_{in} - p\| \leq \|x_n - p\| + d_{in}$*, for all* $i = 1, 2, \ldots, k$ *and all* $p \in F$*;*

(b) $\lim\limits_{n\to\infty} \|x_n - p\|$ *exists for all* $p \in F$*.*

Proof. Let $p \in F$ and $G_n = \max\limits_{1\leq i\leq k}\{G_{in}\}$, for all $n \in \mathbb{N}$. Then $\sum_{n=1}^{\infty} G_n < \infty$.

(a) For each $n \geq 1$, we note that

$$\begin{aligned}
\|y_{1n} - p\| &= \|\alpha_{1n}T_1^n x_n + \beta_{1n}x_n + \gamma_{1n}u_{1n} - p\| \\
&\leq \alpha_{1n}\|T_1^n x_n - p\| + \beta_{1n}\|x_n - p\| + \gamma_{1n}\|u_{1n} - p\| \\
&\leq \alpha_{1n}\|x_n - p\| + \alpha_{1n}G_{1n} + \beta_{1n}\|x_n - p\| + \gamma_{1n}\|u_{1n} - p\| \\
&\leq (\alpha_{1n} + \beta_{1n})\|x_n - p\| + \alpha_{1n}G_n + \gamma_{1n}\|u_{1n} - p\| \\
&\leq \|x_n - p\| + d_{1n}, \qquad (3.1)
\end{aligned}$$

where $d_{1n} = \alpha_{1n}G_n + \gamma_{1n}\|u_{1n} - p\|$. Since $\{u_{1n}\}$ is bounded, $\sum_{n=1}^{\infty} G_n < \infty$ and $\sum_{n=1}^{\infty} \gamma_{1n} < \infty$, wee obtain that $\sum_{n=1}^{\infty} d_{1n} < \infty$. It follows from (3.1) that

$$\begin{aligned}
\|y_{2n} - p\| &= \|\alpha_{2n}T_2^n y_{1n} + \beta_{2n}x_n + \gamma_{2n}u_{2n} - p\| \\
&\leq \alpha_{2n}\|T_2^n y_{1n} - p\| + \beta_{2n}\|x_n - p\| + \gamma_{2n}\|u_{2n} - p\| \\
&\leq \alpha_{2n}\|y_{1n} - p\| + \alpha_{2n}G_n + \beta_{2n}\|x_n - p\| + \gamma_{2n}\|u_{2n} - p\| \\
&\leq \alpha_{2n}(\|x_n - p\| + d_{1n}) + \alpha_{2n}G_n + \beta_{2n}\|x_n - p\| + \gamma_{2n}\|u_{2n} - p\| \\
&= (\alpha_{2n} + \beta_{2n})\|x_n - p\| + \alpha_{2n}d_{1n} + \alpha_{2n}G_n + \gamma_{2n}\|u_{2n} - p\| \\
&\leq \|x_n - p\| + d_{2n},
\end{aligned}$$

where $d_{2n} = \alpha_{2n}d_{1n} + \alpha_{2n}G_n + \gamma_{2n}\|u_{2n} - p\|$. Since $\{u_{2n}\}$ is bounded, $\sum_{n=1}^{\infty} G_n < \infty$, $\sum_{n=1}^{\infty} d_{1n} < \infty$ and $\sum_{n=1}^{\infty} \gamma_{2n} < \infty$, it follows that $\sum_{n=1}^{\infty} d_{2n} < \infty$. By continuing the above method we can show that, there are nonnegative real sequences $\{d_{in}\}$ such that $\sum_{n=1}^{\infty} d_{in} < \infty$ and

$$\|y_{in} - p\| \leq \|x_n - p\| + d_{in}, \qquad (3.2)$$

for all $i = 1, 2, \ldots, k$.

(b) From part (a), for the case $i = k$, we have

$$\|x_{n+1} - p\| \leq \|x_n - p\| + d_{kn}, \qquad (3.3)$$

for all n and all $p \in F$. It follows by Lemma 2.1 (i) that $\lim\limits_{n\to\infty} \|x_n - p\|$ exists for all $p \in F$. ∎

Theorem 3.2. *Let C be a nonempty closed convex subset of a Banach space X. Let $T_1, T_2, \dots, T_k : C \to C$ be asymptotically nonexpansive mappings in the intermediate sense. Put*

$$G_{in} = \sup_{x,y \in C} (\|T_i^n x - T_i^n y\| - \|x - y\|) \vee 0, \qquad \forall n \geq 1,$$

so that $\sum_{n=1}^{\infty} G_{in} < \infty$ for all $i = 1, 2, \dots, k$. Assume that $F \neq \emptyset$ and $\sum_{n=1}^{\infty} \gamma_{in} < \infty$ for each $i = 1, 2, \dots, k$. Then the sequence $\{x_n\}$, defined by (1.3)*, converges strongly to a common fixed point of the family of mappings if and only if $\liminf_{n\to\infty} d(x_n, F) = 0$.*

Proof. We prove only the sufficiency because the necessity is obvious. From (3.3), we have $\|x_{n+1} - p\| \leq \|x_n - p\| + d_{kn}$, for all n and all $p \in F$. This implies that

$$d(x_{n+1}, F) \leq d(x_n, F) + d_{kn},$$

for all $n \in \mathbb{N}$. Since $\sum_{n=1}^{\infty} d_{kn} < \infty$ and $\liminf_{n\to\infty} d(x_n, F) = 0$, it follows from Lemma 2.1 (ii) that $\lim_{n\to\infty} d(x_n, F) = 0$.

Next, we prove that $\{x_n\}$ is a Cauchy sequence. From (3.3), we have

$$\|x_{n+m} - p\| \leq \|x_n - p\| + \sum_{j=n}^{\infty} d_{kj}, \tag{3.4}$$

for all $p \in F$ and $n, m \in \mathbb{N}$. Since $\lim_{n\to\infty} d(x_n, F) = 0$ and $\sum_{n=1}^{\infty} d_{kn} < \infty$, therefore for $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$d(x_n, F) < \frac{\epsilon}{4}, \text{ and } \sum_{j=n_0}^{\infty} d_{kj} < \frac{\epsilon}{4}, \tag{3.5}$$

for all $n \geq n_0$. Therefore, there exists z_1 in F such that

$$\|x_{n_0} - z_1\| < \frac{\epsilon}{4}. \tag{3.6}$$

From (3.4)-(3.6), for $n \geq n_0$ and $m \geq 1$, we have

$$\begin{aligned}
\|x_{n+m} - x_n\| &\leq \|x_{n+m} - z_1\| + \|x_n - z_1\| \\
&\leq \|x_{n_0} - z_1\| + \sum_{j=n_0}^{\infty} d_{kj} + \|x_{n_0} - z_1\| + \sum_{j=n_0}^{\infty} d_{kj} \\
&< \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{4} = \epsilon.
\end{aligned}$$

This shows that $\{x_n\}$ is a Cauchy sequence, hence $x_n \to z \in C$. It remains to show that $z \in F$. Since $|d(z, F) - d(x_n, F)| \leq \|z - x_n\|$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} d(x_n, F) = 0$, we can conclude that $z \in F$. $\square$

The following corollary follows from Theorem 3.2 directly.

Corollary 3.3. *Let C be a nonempty closed convex subset of a Banach space X. Let $T_1, T_2, \dots, T_k : C \to C$ be asymptotically nonexpansive mappings in the intermediate sense. Put*

$$G_{in} = \sup_{x,y \in C} (\|T_i^n x - T_i^n y\| - \|x - y\|) \vee 0, \qquad \forall n \geq 1,$$

so that $\sum_{n=1}^{\infty} G_{in} < \infty$ for all $i = 1, 2, \dots, k$. Assume that $F \neq \emptyset$ and $\sum_{n=1}^{\infty} \gamma_{in} < \infty$ for each $i = 1, 2, \dots, k$. Then the sequence $\{x_n\}$, defined by (1.3), converges strongly to a point $p \in F$ if and only if there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ converging to p.

Since an asymptotically nonexpansive mapping is asymptotically nonexpansive mapping in the intermediate sense, so the following corollary is obtained.

Corollary 3.4. *Let X be a Banach space and C a nonempty closed and convex subset of X and $\{T_i : i = 1, 2, \dots, k\}$ a family of asymptotically nonexpansive self-mappings of C with the sequences $\{b_{in}\} \subset [0, \infty)$ such that $\sum_{n=1}^{\infty} b_{in} < \infty$ for all $i = 1, 2, \dots, k$. Assume that $F \neq \emptyset$ and $\sum_{n=1}^{\infty} \gamma_{in} < \infty$ for each $i = 1, 2, \dots, k$. Then the sequence $\{x_n\}$, defined by (1.3), converges strongly to a common fixed point of the family of mappings if and only if $\liminf_{n \to \infty} d(x_n, F) = 0$.*

Theorem 3.5. *Let C be a nonempty closed convex subset of a Banach space X. Let $T_1, T_2, \dots, T_k : C \to C$ be asymptotically nonexpansive mappings in the intermediate sense. Put*

$$G_{in} = \sup_{x,y \in C} (\|T_i^n x - T_i^n y\| - \|x - y\|) \vee 0, \qquad \forall n \geq 1,$$

so that $\sum_{n=1}^{\infty} G_{in} < \infty$ for all $i = 1, 2, \dots, k$. Assume that $F \neq \emptyset$. Let $x_1 \in C$ and $\{x_n\}$ be the sequence defined by (1.3). If $\sum_{n=1}^{\infty} \gamma_{in} < \infty$, $\lim_{n \to \infty} \|x_n - T_i x_n\| = 0$ for all $i = 1, 2, \dots, k$ and a family $\{T_i : i = 1, 2, \dots, k\}$ satisfies condition $(\overline{C})$, then $\{x_n\}$ converges strongly to a common fixed point of the family of mappings.

Proof. From $\lim_{n \to \infty} \|x_n - T_i x_n\| = 0$ for each $i \in \{1, 2, \dots, k\}$ and $\{T_i : i = 1, 2, \dots, k\}$ satisfies the condition $(\overline{C})$, there is a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(t) > 0$ for all $t \in (0, \infty)$ such that $\|x_n - T_{i_0} x_n\| \geq f(d(x_n, F))$ for some $i_0 \in \{1, 2, \dots, k\}$, it follows that $\lim_{n \to \infty} d(x_n, F) = 0$. By Theorem 3.2, we can conclude that $\{x_n\}$ converges strongly to a common fixed point of the family of mappings. □

4 Convergence theorems in uniformly convex Banach spaces

In this section, we establish weak and strong convergence theorems of the iterative scheme (1.3) for a finite family of asymptotically nonexpansive mappings in the intermediate sense in a uniformly convex Banach space. In order to prove our main results, we need the following lemma:

Lemma 4.1. *Let C be a nonempty closed and convex subset of a uniformly convex Banach space X. Let $T_1, T_2, \dots, T_k : C \to C$ be asymptotically nonexpansive mappings in the intermediate sense. Put*

$$G_{in} = \sup_{x,y\in C} (\|T_i^n x - T_i^n y\| - \|x - y\|) \vee 0, \qquad \forall n \geq 1,$$

so that $\sum_{n=1}^{\infty} G_{in} < \infty$ for all $i = 1, 2, \dots, k$. Assume that $F \neq \emptyset$ and $\sum_{n=1}^{\infty} \gamma_{in} < \infty$ for all $i = 1, 2, \dots, k$. For a given $x_1 \in C$ let $\{x_n\}$ and $\{y_{in}\}$ be the sequences defined by (1.3) with $0 < \eta \leq \alpha_{in} \leq \rho < 1$, for all $i = 1, 2, \dots, k$ and all $n \geq n_0$ and for some $n_0 \in \mathbb{N}$. Then

(i) $\lim_{n\to\infty} \|T_j^n y_{(j-1)n} - x_n\| = 0$ *for all* $j = 1, 2, \dots, k$;

(ii) $\lim_{n\to\infty} \|T_j x_n - x_n\| = 0$ *for all* $j = 1, 2, \dots, k$;

(iii) $\lim_{n\to\infty} \|y_{jn} - x_n\| = 0$ *for all* $j = 1, 2, \dots, k$.

Proof. Let $p \in F$ and $G_n = \max_{1\leq i\leq k} \{G_{in}\}$, for all n.

(i) From Lemma 3.1 (b), we have $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in F$. Suppose that

$$\lim_{n\to\infty} \|x_n - p\| = a. \tag{4.1}$$

From (3.2) and (4.1), we get that

$$\limsup_{n\to\infty} \|y_{jn} - p\| \leq a, \text{ for } 1 \leq j \leq k-1. \tag{4.2}$$

For each $j \in \{1, 2, \dots, k-1\}$ and $n \in \mathbb{N}$, we have

$$\begin{aligned} \|y_{jn} - p\| &\leq \alpha_{jn} \|T_j^n y_{(j-1)n} - p\| + \beta_{jn} \|x_n - p\| + \gamma_{jn} \|u_{jn} - p\| \\ &\leq \alpha_{jn} \|y_{(j-1)n} - p\| + \alpha_{jn} G_n + (1 - \alpha_{jn}) \|x_n - p\| \\ &\quad + \gamma_{jn} \|u_{jn} - p\|. \end{aligned} \tag{4.3}$$

By using (4.3) and (1.3), for each $j = 1, 2, \dots, k-1$, we have

$$\begin{aligned}\|x_{n+1} - p\| &= \|\alpha_{kn}(T_k^n y_{(k-1)n} - p) + \beta_{kn}(x_n - p) + \gamma_{kn}(u_{kn} - p)\| \\ &\le \alpha_{kn}\|y_{(k-1)n} - p\| + (1 - \alpha_{kn})\|x_n - p\| \\ &\quad + \alpha_{kn}G_n + \gamma_{kn}\|u_{kn} - p\| \\ &\le (\alpha_{kn}\alpha_{(k-1)n})\|y_{(k-2)n} - p\| \\ &\quad + (1 - \alpha_{kn}\alpha_{(k-1)n})\|x_n - p\| \\ &\quad + (\alpha_{kn} + \alpha_{(k-1)n})G_n \\ &\quad + (\gamma_{kn}\|u_{kn} - p\| + \gamma_{(k-1)n}\|u_{(k-1)n} - p\|) \\ &\;\;\vdots \\ &\le (\alpha_{kn}\alpha_{(k-1)n}\cdots\alpha_{(j+1)n})\|y_{jn} - p\| \\ &\quad + (1 - \alpha_{kn}\alpha_{(k-1)n}\cdots\alpha_{(j+1)n})\|x_n - p\| \\ &\quad + (\alpha_{kn} + \alpha_{(k-1)n} + \cdots + \alpha_{(j+1)n})G_n \\ &\quad + (\gamma_{kn}\|u_{kn} - p\| + \gamma_{(k-1)n}\|u_{(k-1)n} - p\| + \cdots \\ &\quad + \gamma_{(j+1)n}\|u_{(j+1)n} - p\|).\end{aligned}$$

Since $0 < \eta \le \alpha_{in} \le \rho < 1$, for all $i = 1, 2, \dots, k$ and all $n \ge n_0$, we have that for all $n \ge n_0$ and all $j = 1, 2, \dots, k-1$,

$$\|x_n - p\| \le \frac{\|x_n - p\|}{\eta^{k-j}} - \frac{\|x_{n+1} - p\|}{\eta^{k-j}} + \|y_{jn} - p\| + \frac{\xi_{jn}}{\eta^{k-j}}G_n + \frac{\vartheta_{jn}}{\eta^{k-j}},$$

where $\xi_{jn} = \alpha_{kn} + \alpha_{(k-1)n} + \cdots + \alpha_{(j+1)n}$ and $\vartheta_{jn} = \gamma_{kn}\|u_{kn} - p\| + \gamma_{(k-1)n}\|u_{(k-1)n} - p\| + \cdots + \gamma_{(j+1)n}\|u_{(j+1)n} - p\|$. Since $\lim_{n\to\infty}\|x_n - p\| = a$ and $\lim_{n\to\infty}\vartheta_{jn} = \lim_{n\to\infty}G_n = 0$, it follows that

$$a \le \liminf_{n\to\infty}\|y_{jn} - p\|, \tag{4.4}$$

for all $j = 1, 2, \dots, k-1$.
From (4.2) and (4.4), we have

$$\lim_{n\to\infty}\|y_{jn} - p\| = a = \lim_{n\to\infty}\|x_n - p\|, \tag{4.5}$$

for all $j = 1, 2, \dots, k-1$.
That is, for each $j = 1, 2, \dots, k$, we have

$$\lim_{n\to\infty}\|\alpha_{jn}\big(T_j^n y_{(j-1)n} - p + \gamma_{jn}(u_{jn} - x_n)\big) + (1 - \alpha_{jn})\big(x_n - p + \gamma_{jn}(u_{jn} - x_n)\big)\| = a. \tag{4.6}$$

Since

$$\begin{aligned}\|T_j^n y_{(j-1)n} - p + \gamma_{jn}(u_{jn} - x_n)\| &\le \|T_j^n y_{(j-1)n} - p\| + \gamma_{jn}\|u_{jn} - x_n\| \\ &\le \|y_{(j-1)n} - p\| + G_n + \gamma_{jn}\|u_{jn} - x_n\|\end{aligned}$$

and

$$\|x_n - p + \gamma_{jn}(u_{jn} - x_n)\| \le \|x_n - p\| + \gamma_{jn}\|u_{jn} - x_n\|,$$

it follows that

$$\limsup_{n\to\infty} \|T_j^n y_{(j-1)n} - p + \gamma_{jn}(u_{jn} - x_n)\| \le a \tag{4.7}$$

and

$$\limsup_{n\to\infty} \|x_n - p + \gamma_{jn}(u_{jn} - x_n)\| \le a, \tag{4.8}$$

for all $j = 1, 2, \dots, k$.
From (4.6)-(4.8), we can conclude from Lemma 2.2 that

$$\lim_{n\to\infty} \|T_j^n y_{(j-1)n} - x_n\| = 0, \text{ for all } \ j = 1, 2, \dots, k.$$

(ii) It follows from part (i) in the case $j = 1$ that $\lim_{n\to\infty} \|T_1^n x_n - x_n\| = 0$. For $j = 2, 3, \dots, k$, we obtain from part (i) that

$$\|x_{n+1} - x_n\| \le \alpha_{kn}\|T_k^n y_{(k-1)n} - x_n\| + \gamma_{kn}\|u_{kn} - x_n\| \to 0 \text{ as } n \to \infty, \tag{4.9}$$

and

$$\begin{aligned}\|T_j^n x_n - x_n\| &\le \|T_j^n x_n - T_j^n y_{(j-1)n}\| + \|T_j^n y_{(j-1)n} - x_n\| \\ &\le \|x_n - y_{(j-1)n}\| + G_n + \|T_j^n y_{(j-1)n} - x_n\| \\ &\le \alpha_{(j-1)n}\|T_{j-1}^n y_{(j-2)n} - x_n\| + \gamma_{(j-1)n}\|u_{(j-1)n} - x_n\| \\ &\quad + G_n + \|T_j^n y_{(j-1)n} - x_n\| \to 0 \ \text{ as } n \to \infty.\end{aligned}$$

Therefore,

$$\lim_{n\to\infty} \|T_j^n x_n - x_n\| = 0, \tag{4.10}$$

for all $j = 1, 2, \dots, k$.
Since

$$\begin{aligned}\|x_n - T_j x_n\| \ &\le \ \|x_{n+1} - x_n\| + \|x_{n+1} - T_j^{n+1} x_{n+1}\| \\ &\quad + \|T_j^{n+1} x_{n+1} - T_j^{n+1} x_n\| + \|T_j^{n+1} x_n - T_j x_n\|,\end{aligned}$$

it follows from (4.9), (4.10) and uniformly continuity of T_i that

$$\lim_{n\to\infty} \|T_j x_n - x_n\| = 0 \text{ for all } j \in \{1, 2, \dots, k\}. \tag{4.11}$$

(iii) Since $\lim\limits_{n\to\infty} \gamma_{jn} = 0$ and $\|y_{jn} - x_n\| \le \alpha_{jn}\|T_j^n y_{(j-1)n} - x_n\| + \gamma_{jn}\|u_{jn} - x_n\|$ for all $j = 1, 2, \dots, k$, (iii) is directly obtained by (i). □

Theorem 4.2. *Let C be a nonempty closed and convex subset of a uniformly convex Banach space X satisfying the Opial's property. Let $T_1, T_2, \dots, T_k : C \to C$ be asymptotically nonexpansive mappings in the intermediate sense. Put*

$$G_{in} = \sup_{x,y\in C} (\|T_i^n x - T_i^n y\| - \|x-y\|) \vee 0, \qquad \forall n \geq 1,$$

so that $\sum_{n=1}^{\infty} G_{in} < \infty$ for all $i = 1, 2, \dots, k$. For a given $x_1 \in C$ let $\{x_n\}$ be the sequence defined by (1.3) with $0 < \eta \leq \alpha_{in} \leq \rho < 1$, for all $i = 1, 2, \dots, k$ and all $n \geq n_0$ and for some $n_0 \in \mathbb{N}$. If $F \neq \emptyset$ and $\sum_{n=1}^{\infty} \gamma_{in} < \infty$ for each $i = 1, 2, \dots, k$, then $\{x_n\}$ converges weakly to a common fixed point of the family $\{T_i : i = 1, 2, \dots, k\}$.

Proof. By Lemma 4.1 (ii), we have $\lim_{n\to\infty} \|T_i x_n - x_n\| = 0$, for all $i = 1, 2, \dots, k$. Since X is uniformly convex and $\{x_n\}$ is bounded, without loss of generality we may assume that $x_n \to u$ weakly as $n \to \infty$ for some $u \in C$. For all $i = 1, 2, \dots, k$ and all $j \in \mathbb{N}$, we have $\|x_n - T_i^j x_n\| \leq \|x_n - T_i x_n\| + \|T_i x_n - T_i^2 x_n\| + \cdots + \|T_i^{j-1} x_n - T_i^j x_n\|$. Since $\lim_{n\to\infty} \|T_i x_n - x_n\| = 0$ and T_i are uniformly continuous, it follows that $\lim_{n\to\infty} \|x_n - T_i^j x_n\| = 0$ for all $i = 1, 2, \dots, k$ and all $j \in \mathbb{N}$. By Lemma 2.4, we obtain $u \in F$. Suppose that there are subsequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$ converge weakly to u and v, respectively. Again, as above, we can prove that $u, v \in F$. By Lemma 3.1 (b), $\lim_{n\to\infty} \|x_n - u\|$ and $\lim_{n\to\infty} \|x_n - v\|$ exist. It follows from Lemma 2.3 that $u = v$. Therefore $\{x_n\}$ converges weakly to a common fixed point of $\{T_i : i = 1, 2, \dots, k\}$. □

Theorem 4.3. *Under the hypotheses of Lemma 4.1, assume that the family $\{T_i : i = 1, 2, \dots, k\}$ satisfies condition $(\overline{C})$. Then $\{x_n\}$ and $\{y_{jn}\}$ converge strongly to a common fixed point of the family of mappings for all $j = 1, 2, \dots, k$.*

Proof. From (3.3), we have

$$\|x_{n+1} - p\| \leq \|x_n - p\| + d_{kn}, \text{ for all } n \text{ and all } p \in F.$$

Therefore, $d(x_{n+1}, F) \leq d(x_n, F) + d_{kn}$. Since $\sum_{n=1}^{\infty} d_{kn} < \infty$, it follows from Lemma 2.1 (i), that $\lim_{n\to\infty} d(x_n, F)$ exists. By Lemma 4.1 (ii), we have $\lim_{n\to\infty} \|T_i x_n - x_n\| = 0$ for all $i = 1, 2, \dots, k$. Since $\{T_i : i = 1, 2, \dots, k\}$ satisfies the condition $(\overline{C})$, there is a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(t) > 0$ for all $t \in (0, \infty)$ such that $\|x_n - T_{i_0} x_n\| \geq f(d(x_n, F))$ for some $i_0 \in \{1, 2, \dots, k\}$, it follows that $\lim_{n\to\infty} d(x_n, F) = 0$. By Theorem 3.2, we can conclude that $\{x_n\}$ converges strongly to a common fixed point

q of the family $\{T_i : i = 1, 2, \ldots, k\}$. From Theorem 4.1 (iii), we have $\lim_{n\to\infty} \|y_{jn} - x_n\| = 0$ for all $j = 1, 2, \ldots, k$, it follows that $\lim_{n\to\infty} y_{jn} = q$ for all $j = 1, 2, \ldots, k$. □

Remark 4.4. The family of asymptotically nonexpansive mappings in the intermediate sense in Lemma 3.1-Theorem 4.3 can be replaced by a family of asymptotically nonexpansive mappings. Lemma 3.1 and 4.1 generalize and extend Lemma 3.2 and Lemma 3.3 of [15], Lemma 3.1, Lemma 3.4 and Lemma 3.5 of [4], Lemma 3.3 of [6] and Lemma 2.3 of [14] to any finite family of asymptotically nonexpansive mappings in the intermediate sense. Theorem 4.2 generalizes and extends Theorem 3.1 of [6], Theorem 2.9 of [14], Theorem 1 of [8] and Theorem 1 of [12] to any finite family of asymptotically nonexpansive mappings in the intermediate sense. Theorem 4.3 generalizes and improves Theorem 4.2 of [4], Theorem 3.4 of [15], Theorem 3.2 of [6], Theorem 2.4 of [14], Theorem 2 of [8], Theorem 1 of [9] and Theorem 2 of [12] by using condition $(\overline{C})$ instead of condition $(\overline{A})$ or (I) or semicompactness or completely continuous or compactness to the more general class of a finite family of asymptotically nonexpansive mappings in the intermediate sense.

Acknowledgement. The authors would like to thank the Commission on Higher Education, the Thailand Research Fund, the Thaksin university and the Graduate School of Chiang Mai University, Thailand for their financial support.

References

[1] A. Bnouhachem, M. A. Noor, Th. M. Rassias, Three-steps iterative algorithms for mixed variational inequalities, Appl. Math. Comput., 183 (2006), 436-446.

[2] R. E. Bruck, T. Kuczumow, S. Reich, Convergence of iterates of asymptotically nonexpansive mappings in Banach spaces with the uniform Opial property, Colloq. Math., 65 (1993), 169-179.

[3] C. E. Chidume, B. Ali Weak and strong convergence theorems for finite families of asymptotically nonexpansive mappings in Banach spaces, J. Math. Anal. Appl., 330 (2007), 377-387.

[4] H. Fukhar-ud-din, A. R. Khan, Approximating common fixed points of asymptotically nonexpansive maps in uniformly convex Banach spaces, Computers and Mathematics with Applications, 53 (2007), 1349-1360.

[5] H. Fukhar-ud-din, S. H. Khan, Convergence of iterates with errors of asymptotically quasi-nonexpansive mappings and applications, J. Math. Anal. Appl., 328 (2007), 821-829.

[6] J. U. Jeong, S. H. Kim, Weak and strong convergence of Ishikawa iteration process with errors for two asymptotically nonexpansive mappings, Applied Mathematics and Computation, 181 (2006), 1394-1401.

[7] A. R. Khan, A. Aziz Domlo, H. Fukhar-ud-din, Common fixed points Noor iteration for a finite family of asymptotically quasi-nonexpansive mappings in Banach spaces, J. Math. Anal. Appl., 341 (2008), 1-11.

[8] S. H. Khan, H. Fukhar-ud-din, Weak and strong convergence of a scheme with errors for two nonexpansive mappings, Nonlinear Analysis, 61 (2005), 1295-1301.

[9] G. Kim, T. Kim, Mann and Ishikawa iterations with errors for non-Lipschitzian mappings in Banach spaces, Computers and Mathematics with Applications., 42 (2001), 1565-1570.

[10] W. A. Kirk, Fixed point theorems for non-Lipschitzian mappings of asymptotically nonexpansive type, Israel J. Math., 17 (1974), 339-346.

[11] K. Nammanee, S. Suantai, The modified Noor iterations with errors for non-Lipschitzian mappings in Banach spaces, Appl. Math. Comput., 187 (2007), 669-679.

[12] M. O. Osilike, S. C. Aniagbosor, Weak and strong convergence theorems for fixed points of asymptotically nonexpansive mappings, Mathematical and Computer Modelling, 32 (2000), 1181-1191.

[13] N. Petrot, Modified Noor iterative process by non-Lipschitzian mappings for nonlinear equations in Banach spaces, J. Math. Anal. Appl., in press

[14] S. Plubtieng, R. Wangkeeree, R. Punpaeng, On the convergence of modified Noor iterations with errors for asymptotically nonexpansive mappings, J. Math. Anal. Appl., 322 (2006), 1018-1029.

[15] S. Plubtieng, R. Wangkeeree, Strong convergence theorems for multi-step Noor iterations with errors in Banach spaces, J. Math. Anal. Appl., 321 (2006), 10-23.

[16] L. Qihou, Iterative sequences for asymptotically quasi-nonexpansive mappings with error member, J. Math. Anal. Appl., 259 (2001), 18-24.

[17] H. F. Senter, W. G. Dotson, Approximating fixed points of nonexpansive mappings, Proc. Amer. Math. Soc., 44 (2) (1974), 375-380.

[18] S. Suantai, Weak and strong convergence Criteria of Noor Iterations for asymptotically nonexpansive mappings, J. Math. Anal. Appl., 311 (2005), 506-517.

[19] B. L. Xu, M. A. Noor, Fixed point iterations for asymptotically nonexpansive mappings in Banach spaces, J. Math. Anal. Appl., 267 (2002), 444-453.

[20] L. Yang, X. Xie, G. Hu, Demi-closed principle and convergence for modified three step iterative process with errors of non-Lipschitzian mappings, Journal of Computational and Applied Mathematics (2009), doi:10.1016/j.cam.2009.01.022.

Received: March, 2009