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Singular Behaviour of Bounded Radially Symmetric Solutions of p - Laplace Nonlinear Equation

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Abstract

We study the boundary singular behaviour of radially symmetric solutions $u(x)$ of a class of p - Laplace nonlinear equations: $-\Delta_p u = f(|x|, u, |\nabla u|)$ in a ball $B_R \subset \mathbf{R}^N$, where $u = 0$ on ∂B_R and $u \in W^{1,p}_{loc}(B_R) \cap L^\infty(B_R)$. If the nonlinear term $f(x, \eta, \xi)$ satisfies some suitable jumping and singular conditions near ∂B_R, we show that box (fractal)-dimension of the graph $G(u)$ of $u(x)$ takes a fractional value $s > N$. It numerically verifies that $G(u)$ is very high concentrated near ∂B_R. Next, a kind of singular behavior of $|\nabla u|$ near ∂B_R is established by giving the lower bound for the box-dimension of its graph $G(|\nabla u|)$ which in particular implies $u \notin W^{1,p}(B_R)$. It generalizes a study on the fractal dimension of the graph of solutions of the one-dimensional p - Laplace nonlinear equation presented in an early paper: Pašić [J. Differential Equations 190 (2003), 268-305].

1 The statement of the main problem

Let $B_R = B_R(0)$ be a ball in $\mathbf{R}^N$, $N > 1$, with radius $R > 0$ and centered at the origin, and let ∂B_R denote the boundary of B_R. We consider the following Dirichlet boundary value problem:

$$\begin{cases} -\Delta_p u = f(|x|, u, |\nabla u|) \quad \text{in } B_R, \\ u = 0 \quad \text{on } \partial B_R, \\ u \in W^{1,p}_{loc}(B_R) \cap C(\overline{B}_R), \end{cases} \tag{1}$$

where $p > 1$ and $|\,.\,|$ denotes as usual the Euclidean norm in $\mathbf{R}^N$. The nonlinear term $f(t, \eta, \xi)$ is a Carathéodory function that is $f(t, \eta, \xi)$ is measurable in t and continuous in (η, ξ). The condition $u \in W^{1,p}_{loc}(B_R)$ means that $u \in W^{1,p}(\Omega)$ for all $\Omega \subset\subset B_R$. Let $\theta = \theta(t)$ be increasing and $\omega = \omega(t)$ be decreasing real functions defined on $[0, R]$, $\theta(t) \leq 0 \leq \omega(t)$, and $\theta(R) = \omega(R) = 0$.

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Definition 1.1 *We say that a function* $u : \overline{B}_R \longrightarrow \mathbf{R}$ is jumping over θ and ω near ∂B_R *if there exists a sequence* (r_k) *of positive real numbers such that* $r_k \nearrow R$ *and a sequence* $\sigma_k \in (r_{k-1}, r_k)$ *such that*

$$\begin{cases} u(x) \geq \operatorname{ess\,sup}_{z \in B_{2k-1,2k}} \omega(|z|) & \textit{for all } x \in B_{2k-1,2k},\ |x| = \sigma_{2k}, \\ u(x) \leq \operatorname{ess\,inf}_{z \in B_{2k,2k+1}} \theta(|z|) & \textit{for all } x \in B_{2k,2k+1},\ |x| = \sigma_{2k+1}, \end{cases} \tag{2}$$

where the ring $B_{k,k+1}$ *is defined by* $B_{k,k+1} := \{x \in B_R : r_k < |x| < r_{k+1}\}$.

Although the condition (2) possesses a kind of radially symmetric structure, it is clear that a function $u(x)$ satisfying (2) may be non-radial on B_R. Before giving the motivation to consider it, we present two explicitly given classes of functions satisfying (2).

Example 1.2 The function $u(x) = -(R - |x|)^{\alpha} \sin(R - |x|)^{-\beta}$, $x \in B_R$, is jumping over θ and ω near ∂B_R, where $\omega(t) = -\theta(t) = (R - t)^{\alpha}$, $r_k = R - (k\pi)^{-\beta}$, and $0 < \alpha < \beta$. Required sequence σ_k in (2) can be taken as $\sigma_k = R - [(k + 1/2)\pi]^{-\beta}$. □

Example 1.3 Let $a(t)$ be a bounded function with $a(0+) = 0$, let $F(t)$ be a T- periodic bounded function with $F(T_0) = 0$ for some T_0, and let $b(t)$ be decreasing and positive function such that $b(0+) = \infty$. Then the function $u(x) = a(R-|x|)F(b(R-|x|))$ is jumping over θ and ω near ∂B_R, where $\theta(t) = -|a(t)|$, $\omega(t) = |a(t)|$, and $r_k = R - b^{-1}(T_0 + kT)$ and $b^{-1}(t)$ denotes the inverse function of $b(t)$. For instance, the function $u(x) = (R - |x|)^{\alpha} \sin(\ln(R - |x|))$, $x \in B_R$, is jumping over θ and ω near ∂B_R, where $-\theta(t) = \omega(t) = (R - t)^{\alpha}$, $\alpha > 0$, and $r_k = R - e^{-k\pi}$. □

A motivation to introduce (2)**.** We consider the linear differential equation (P): $y'' + f(t)y = 0$, where $f(t) > 0$ on $(0, R)$, $f(R-) = \infty$, and the Hartman-Wintner asymptotic condition is satisfied: $f^{-1/4}(f^{-1/4})'' \in L^1(0, R)$. On the first hand, we know that all solutions $y = y(t)$ of equation (P) satisfy the following a priori estimate (A): $|y'(t)| \leq f^{1/4}(t)$ near $t = R$, see for instance [1] and [3]. On the other hand, if $y_1(t)$ and $y_2(t)$ are fundamental system of solutions of equation (P), then the Wronskian of $y(t)$ and $y_i(t)$ satisfies: $|W(y(t), y_i(t))| = c > 0$ for $i = 0$ or $i = 1$ and for all $t \in I$. In particular for $t = s_k$, where $y'(s_k) = 0$, we obtain $|y(s_k)| = |c|/|y_i'(s_k)|$ which together with (A) implies the following jumping condition:

$$|y(s_k)| \geq \frac{c}{\sqrt[4]{f(s_k)}},\ s_k \in (a_k, a_{k+1}),\ \text{for sufficiently large } k, \tag{3}$$

where a_k is increasing sequence of consecutive zeros of $y(t)$ such that $a_k \to R$. In particular for obstacles $\theta(t) = -\omega(t)$, $\omega(t) = c/\sqrt[4]{f(t)}$, and sequences $r_k =$

a_k and $\sigma_k = s_k$ from (3) follows that all solution $y(t)$ of equation (P) satisfy the proposed condition (2). $\square$

For the equation (1) we are not able to use the technique of Wronskian and a priori estimate (A) mentioned above to get a jumping condition (2). Therefore, we will use a method of the control of essential infimum and supremum of solutions of quasilinear elliptic differential equations presented in [4]. In this sense, the following assumption on the sign-change of $f(t,\eta,\xi)$ in respect to η,

$$\begin{cases} f(t,\eta,\xi) < 0 \quad \text{a.e. } t \in (0,R),\, \eta > \tilde{\omega}_0,\, \xi \in \mathbf{R}\,, \\ f(t,\eta,\xi) > 0 \quad \text{a.e. } t \in (0,R),\, \eta < \tilde{\theta}_0,\, \xi \in \mathbf{R}\,, \end{cases} \tag{4}$$

will ensure that any solution $u(x)$ of equation (1) satisfies a priori estimate $\theta(|x|) \le u(x) \le \omega(|x|)$, $x \in B_R$. The following assumptions on the singular behaviour of $f(t,\eta,\xi)$ in respect to t near R:

$$\begin{cases} f(t,\eta,\xi) \ge 0 \quad \text{a.e. } t \in (r_{2k-1}, r_{2k}),\, \eta \in (\tilde{\theta}_0, \omega_{2k}),\, \xi \in \mathbf{R}\,, \\ f(t,\eta,\xi) \ge f_{2k}(t) \quad \text{a.e. } t \in (r_{2k-1}+\delta_{2k}, r_{2k}-\delta_{2k}),\, \eta \in (\tilde{\theta}_0, \omega_{2k}),\, \xi \in \mathbf{R}\,, \\ \int_{r_{2k-1}+\delta_{2k}}^{r_{2k}-\delta_{2k}} f_{2k}(t)dt > p^p \gamma_{2k} (\omega_{2k} - \tilde{\theta}_0)^{p-1}\,, \end{cases} \tag{5}$$

and

$$\begin{cases} f(t,\eta,\xi) \le 0 \quad \text{a.e. } t \in (r_{2k}, r_{2k+1}),\, \eta \in (\theta_{2k+1}, \tilde{\omega}_0),\, \xi \in \mathbf{R}\,, \\ f(t,\eta,\xi) \le f_{2k+1}(t), t \in (r_{2k}+\delta_{2k+1}, r_{2k+1}-\delta_{2k+1}),\, \eta \in (\theta_{2k+1}, \tilde{\omega}_0),\, \xi \in \mathbf{R}, \\ \int_{r_{2k}+\delta_{2k+1}}^{r_{2k+1}-\delta_{2k+1}} f_{2k+1}(t)(t)dt < -p^p \gamma_{2k+1} (\tilde{\omega}_0 - \theta_{2k+1})^{p-1}\,, \end{cases} \tag{6}$$

will ensure that any radial solution $u(x)$ of equation (1) satisfies the jumping condition (2), where:

$$\begin{cases} \omega_{2k} := \text{ess sup}_{(r_{2k-1}, r_{2k})} \omega \quad \text{and} \quad \theta_{2k+1} := \text{ess inf}_{(r_{2k}, r_{2k+1})} \theta\,, \\ \tilde{\theta}_0 \le \text{ess inf}_{(0,R)} \theta < \theta_{2k+1} \quad \text{and} \quad \omega_{2k} < \text{ess sup}_{(0,R)} \omega \le \tilde{\omega}_0, \\ \delta_k := \frac{r_k - r_{k-1}}{4} \quad \text{and} \quad \gamma_k := \frac{1}{N r_{k-1}^{N-1}} \frac{r_k^N - r_{k-1}^N}{(r_k - r_{k-1})^p}\,, \\ f_k \in L^1(r_{k-1}+\delta_k, r_k+\delta_k). \end{cases} \tag{7}$$

Here and in the sequel, a function $u(x)$ is said to be a radial solution of equation (1) if $u(x) = y(|x|)$, where $y(t)$ is a solution of the following second-order differential equation:

$$\begin{cases} -(t^{N-1}|y'|^{p-2}y')' = t^{N-1} f(t, y, |y'|) \quad \text{in } (0,R), \\ y(R) = 0, \\ y \in \mathrm{W}^{1,p}_{loc}([0,R)) \cap \mathrm{C}([0,R]). \end{cases} \tag{8}$$

The first main result of the paper is the following.

Theorem 1.4 *Let $f(t,\eta,\xi)$ satisfy the assumptions* (4), (5), *and* (6). *Then every radial solution $u(x)$ of equation* (1) *is jumping over θ and ω near ∂B_R.*

When $p=2$, in [6, Appendix] is presented the existence of at least one solution of a class of equations like (8) where $f(t,\eta,\xi)$ satisfies the assumptions (4), (5), and (6). Such a class of the functions $f(t,\eta,\xi)$ is explicitly given in the following example.

Example 1.5 Let $h_1(t,\xi)$ and $h_2(t,\xi)$ be two measurable functions in $t\in(a,b)$ and continuous in $\xi\in\mathbf{R}$ such that $h_i(t,\xi)>0$, $i=1,2$. The function $h(t,\xi,\eta)$ defined by

$$h(t,\eta,\xi)=-h_1(t,\xi)(\eta-\tilde{\omega}_0)^+ + h_2(t,\xi)(\eta-\tilde{\theta}_0)^- \ ,\ \ t\in(0,R)\ ,\eta\in\mathbf{R},\ \xi\in\mathbf{R}, \tag{9}$$

satisfies the condition (4). Also, we note that the function $g(t,\eta,\xi)$ defined by

$$\begin{aligned} g(t,\eta,\xi) \ &= \ c_0(p)(\eta-\tilde{\omega}_0)^-\sum_{k=1}^{\infty} g_{2k}(t,\xi)\chi_{[r_{2k-1},r_{2k}]}(t) \\ &- \ c_0(p)(\eta-\tilde{\theta}_0)^+\sum_{k=1}^{\infty} g_{2k+1}(t,\xi)\chi_{[r_{2k},r_{2k+1}]}(t), \end{aligned} \tag{10}$$

where

$$\begin{cases} c_0(p)=\frac{\pi}{2\sin\frac{\pi}{4}}p^p, \\ g_{2k}(t,\xi)=\frac{\gamma_{2k}}{r_{2k}-r_{2k-1}}\frac{(\omega_{2k}-\tilde{\theta}_0)^{p-1}}{\tilde{\omega}_0-\omega_{2k}}\sin(\pi\frac{r_{2k}-t}{r_{2k}-r_{2k-1}}), \\ g_{2k+1}(t,\eta)=\frac{\gamma_{2k+1}}{r_{2k+1}-r_{2k}}\frac{(\tilde{\omega}_0-\theta_{2k+1})^{p-1}}{\theta_{2k+1}-\tilde{\theta}_0}\sin(\pi\frac{r_{2k+1}-t}{r_{2k+1}-r_{2k}}), \\ \gamma_k=\frac{1}{Nr_{k-1}^{N-1}}\frac{r_k^N-r_{k-1}^N}{(r_k-r_{k-1})^p}. \end{cases}$$

Now, let $f(t,\eta,\xi)$ be defined by

$$f(t,\eta,\xi)=h(t,\eta,\xi)+g(t,\eta,\xi),$$

where $h(t,\eta,\xi)$ and $g(t,\eta,\xi)$ are defined as in (9) and (10) respectively. Since $g_k(t,\xi)>0$ for all $t\in(r_{k-1},r_k)$ and $\xi\in\mathbf{R}$, the function $f(t,\eta,\xi)$ satisfies the conditions (4), (5), and (6), where $\tilde{\theta}_0=\operatorname{ess\,inf}_{(0,R)}\theta$ and $\tilde{\omega}_0=\operatorname{ess\,sup}_{(0,R)}\omega$. $\square$

Next, let $G(u)$ denote the graph of $u(x)$ defined by

$$G(u)=\{(x,u(x)):x\in B_R\}\subseteq\mathbf{R}^{N+1}$$

and let $\dim_M G(u)$ denote the Minkowski-Bouligand dimension of the graph $G(u)$ defined by:

$$\dim_M G(u) = \limsup_{\varepsilon \to 0} \left(N + 1 - \frac{\log |G_\varepsilon(u)|}{\log \varepsilon}\right),$$

where $G_\varepsilon(u)$ denotes the $\varepsilon-$ neighborhood of the graph $G(u)$ and $|G_\varepsilon(u)|$ denotes the Lebesgue measure of $G_\varepsilon(u)$, see [2, Proposition 3.2]. For a function $u(x)$ which is jumping over θ and ω near ∂B_R, we are able to estimate the lower bounds for $|G_\varepsilon(u)|$ and $\dim_M G(u)$ in term of the obstacles θ and ω. It is numerically determine the order of growth for the concentration of the graph $G(u)$. It is subjected to the following theorems which will be proved in Section 3 below.

Theorem 1.6 *For some $\varepsilon_0 > 0$ and for all $\varepsilon \in (0, \varepsilon_0)$, let there be a natural number $k = k(\varepsilon)$ depending on $\varepsilon > 0$ such that:*

$$r_j - r_{j-1} \le \varepsilon/3 \text{ for each } j \ge k(\varepsilon). \tag{11}$$

Let $f(t, \eta, \xi)$ satisfy the assumptions (4), (5), *and* (6). *Then for every radial solution $u(x)$ of equation* (1) *we have:*

$$|G_\varepsilon(u)| \ge c \int_{r_{k(\varepsilon)}}^{R} (\omega(t) - \theta(t))dt, \tag{12}$$

$$\dim_M G(u) \ge \lim_{\varepsilon \to 0} \left(N + 1 - \frac{\log \int_{r_{k(\varepsilon)}}^{R} (\omega(t) - \theta(t))dt}{\log \varepsilon} \right). \tag{13}$$

Now, we are able to observe the following consequence.

Corollary 1.7 *Let $f(t, \eta, \xi)$ satisfy the assumptions* (4), (5), *and* (6) *in respect to $\theta(t)$, $\omega(t)$, and (r_k) given by*

$$\omega(t) = (R - t)^\alpha \ , \ \ \theta(t) = -(R - t)^\alpha \quad \text{and} \quad r_k = R - \frac{R}{2}\Big(\frac{1}{k}\Big)^{\frac{1}{\beta}} \ , \tag{14}$$

where $0 < \alpha < \beta$, $\alpha < 1$. Then for every radial solution $u(x)$ of equation (1) *we have:*

$$|G_\varepsilon(u)| \ge c \, \varepsilon^{\frac{\alpha+1}{\beta+1}} \quad \text{and} \quad \dim_M G(u) \ge N + 1 - \frac{\alpha + 1}{\beta + 1} > N. \tag{15}$$

If $u(x)$ is a smooth function in B_R and bounded on $\overline{B_R}$ (for instance: $u \in W^{1,p}(B_R) \cap C(\overline{B}_R)$ or $u \in C^1(\overline{B}_R)$), then by using an elementary geometric argumentation, it is not difficult to check that $|G_\varepsilon(u)| \sim \varepsilon^1$ for $\varepsilon \approx 0$. On the other hand, for the function $u(x)$ defined in Example 1.2, one can show that $|G_\varepsilon(u)| \sim \varepsilon^{\frac{\alpha+1}{\beta+1}}$ for $\varepsilon \approx 0$ as well as $u \in W^{1,p}_{loc}(B_R) \cap C^\infty(\overline{B}_R)$ and $u \notin W^{1,p}(B_R)$. Hence by (15), we can expect that the gradient ∇u of every radial solution of (1) is singular near ∂B_R in the sense of the following two results. The proof of the following result will be sketched in Section 4 below.

Theorem 1.8 *Let $f(t,\eta,\xi)$ satisfy the assumptions* (4), (5), *and* (6). *Then for every radial solution $u(x)$ of equation* (1), *we have*

$$\dim_M G(\|\nabla u\|) \geq \limsup_{\varepsilon \longrightarrow 0} \left(N + 1 - \frac{\log \sum_{k=k(\varepsilon/2)}^{+\infty} (\omega_{2k} - \theta_{2k-1}))}{\log \varepsilon} \right), \tag{16}$$

where (ω_{2k}) and (θ_{2k-1}) are defined by (7). *In particular, if $\theta(t)$, $\omega(t)$, and (r_k) are given by* (14), *then there holds*

$$\dim_M G(\|\nabla u\|) \geq N + 1 - \frac{\alpha}{1+\beta} > N + \frac{1}{2}\,. \tag{17}$$

Under the hypotheses (14) from Theorem 1.8 especially follows that $\nabla u \notin L^p(B_R)$ and so

$$\lim_{\varepsilon \to 0} \int_{B_{R-\varepsilon}} |\nabla u|^p dx = \infty, \tag{18}$$

where $B_{R-\varepsilon}$ is a ball with radius $R - \varepsilon$ and centered at the origin. The order of growth for the divergence in (18) will be given in the following theorem.

Theorem 1.9 *Let $f(t,\eta,\xi)$ satisfy the assumptions* (4), (5), *and* (6) *in respect to $\theta(t)$, $\omega(t)$ and (r_k) given by* (14). *Then then for every radial solution $u(x)$ of equation* (1) *we have*

$$\limsup_{\varepsilon \longrightarrow 0} \frac{\log \|\nabla u\|_{\mathrm{L}^p(B_{R-\varepsilon})}}{\log \frac{1}{\varepsilon}} \geq s - N \geq 0\,. \tag{19}$$

The proof will be sketched in Section 4 below.

In Theorem 1.9, a lower bound for the order of growth of singular behaviour of $||\nabla u||_{L^p}$ is determined by the singular boundary behaviour of the nonlinear term $f(t,\eta,\xi)$. It could be compared with some known facts about the local behaviour of $||\nabla u||_{L^p}$, where it was shown that the order of growth of regular asymptotic behaviours of both $||\nabla u||_{L^p}$ and coefficients in the equation are equivalent each to other, in some way, see for instance [9], [10], [11], and references therein.

2 The proof of Theorem 1.4

At the first, we establish some lemmas which state sufficient conditions on $f(t,\eta,\xi)$ so that any radial solution $u(x)$ of equation (1) is jumping over θ and ω near ∂B_R.

Lemma 2.1 *Let $0 < a < b < R$ and let $\delta = (b-a)/4$. Then for every $c_0 > 1$ there exists $\Phi \in C_0^\infty(a,b)$ with properties*

$$\begin{cases} 0 \le \Phi(t) \le 1,\ t \in \mathbf{R}\ , \\ \Phi(t) = 1,\ t \in (a+\delta, b-\delta) \text{ and } \Phi(t) = 0,\ t \in \mathbf{R}/(a,b), \\ \text{and } |\Phi'(t)| \le \frac{c_0}{b-a},\ t \in \mathbf{R}. \end{cases}$$

Proof. The claim follows immediately from Lemma 5, p. 267, [4], with $A := (a+\delta, b-\delta)$ and $A_r = (a,b)$, where $r := \delta$. □

Lemma 2.2 *Let $\tilde{\theta}_0$, ω_2 and $\tilde{\omega}_0$ be real numbers such that $\tilde{\theta}_0 < \omega_2 < \tilde{\omega}_0$. Let $0 < r_1 < r_2 < R$ and let $\delta_2 = (r_2 - r_1)/4$. Let the function $f(t,\eta,\xi)$ satisfy*

$$f(t,\eta,\xi) \ge 0 \quad a.e.\ t \in (r_1, r_2),\ \eta \in (\tilde{\theta}_0, \omega_2),\ \xi \in \mathbf{R}\ , \tag{20}$$

$$f(t,\eta,\xi) \ge f_2(t) \quad a.e.\ t \in (r_1+\delta_2, r_2-\delta_2),\ \eta \in (\tilde{\theta}_0, \omega_2),\ \xi \in \mathbf{R}\ ,$$

$$\int_{r_1+\delta_2}^{r_2-\delta_2} f_2(t)dt > (p-1)^{p-1}\gamma_2 \inf_{s>0} \frac{1}{s}((\omega_2 - \tilde{\theta}_0) + s)^p\ , \tag{21}$$

where

$$f_2 \in L^1(r_1+\delta_2, r_2-\delta_2) \quad \text{and} \quad \gamma_2 := \frac{1}{N r_1^{N-1}} \frac{r_2^N - r_1^N}{(r_2-r_1)^p}.$$

Then every solution $y \in W^{1,p}_{loc}([0,R)) \cap C([0,R])$ of equation (8) satisfies: if there holds $\operatorname{ess\,inf}_{(r_1,r_2)} y \ge \tilde{\theta}_0$, then there holds $\operatorname{ess\,sup}_{(r_1,r_2)} y \ge \omega_2$.

Proof. Let $y(t)$ be a solution of equation (8). Let us assume the opposite, i.e., let for every $t \in (r_1, r_2)$ there holds $y(t) \ge \tilde{\theta}_0$ and $y(t) \le \omega_2$. Testing equation (8) by any $\varphi \in C_0^\infty(r_1, r_2)$ we obtain

$$\int_{r_1}^{r_2} t^{N-1}|y'|^{p-2}y'\varphi' dt = \int_{r_1}^{r_2} t^{N-1} f(t,y,|y'|)\varphi dt\ .$$

In particular for $\varphi(t) = -(y(t)-\tau)^- \Phi^p(t)$, where $\eta^- := \max\{-\eta, 0\}$ and $\Phi(t)$ is from Lemma 2.1, we obtain

$$\begin{aligned} &\int_{(r_1,r_2)\cap\{y<\tau\}} t^{N-1}|y'|^{p-2}y'(y'\Phi^p + (y-\tau)p\Phi^{p-1}\Phi')dt \\ &= \int_{(r_1,r_2)\cap\{y<\tau\}} t^{N-1} f(t,y,|y'|)(y-\tau)\Phi^p dt\ , \end{aligned}$$

and

$$\begin{aligned} \int_{(r_1,r_2)\cap\{y<\tau\}} t^{N-1}|y'|^p\Phi^p dt \ &\le\ p\int_{(r_1,r_2)\cap\{y<\tau\}} t^{N-1}|y'|^{p-1}(\tau-y)\Phi^{p-1}|\Phi'|dt \\ &-\ \int_{(r_1,r_2)\cap\{y<\tau\}} t^{N-1} f(t,y,|y'|)(\tau-y)\Phi^p dt\ . \end{aligned}$$

We apply the Young inequality $c_1(pc_2) \le \frac{d}{p'} c_1^{p'} + (\frac{p}{d})^{p-1} c_2^p$ for any $d > 0$ and

$$c_1 := |y'|^{p-1}(\tau - y)^{\frac{1}{p'}} \Phi^{p-1} \ , \ \ c_2 := (\tau - y)^{\frac{1}{p}} |\Phi'| \ ,$$

to get

$$\begin{aligned} \int_{(r_1,r_2)\cap\{y<\tau\}} t^{N-1}|y'|^p \Phi^p dt \ &\le \ \frac{d}{p'} \int_{(r_1,r_2)\cap\{y<\tau\}} t^{N-1}|y'|^p (\tau - y) \Phi^p dt \\ &+ \ (\frac{p}{d})^{p-1} \int_{(r_1,r_2)\cap\{y<\tau\}} t^{N-1}(\tau - y)|\Phi'|^p dt \\ &- \ \int_{(r_1,r_2)\cap\{y<\tau\}} t^{N-1} f(t, y, |y'|)(\tau - y)\Phi^p dt \ . \end{aligned}$$

Then $y(t) \ge \tilde{\theta}_0$ and $y(t) \le \omega_2$, $t \in (r_1, r_2)$, gives

$$\begin{aligned} \int_{(r_1,r_2)\cap\{y<\tau\}} t^{N-1}|y'|^p \Phi^p dt \ &\le \ (\tau - \tilde{\theta}_0)\frac{d}{p'} \int_{(r_1,r_2)\cap\{y<\tau\}} t^{N-1}|y'|^p \Phi^p dt \\ &+ \ (\tau - \tilde{\theta}_0)(\frac{p}{d})^{p-1} \int_{(r_1,r_2)\cap\{y<\tau\}} t^{N-1}|\Phi'|^p dt \\ &- \ (\tau - \omega_2) \int_{(r_1,r_2)\cap\{y<\tau\}} t^{N-1} f(t, y, |y'|)\Phi^p dt \ , \end{aligned}$$

that is:

$$(1 - (\tau - \tilde{\theta}_0)\frac{d}{p'}) \int_{(r_1,r_2)\cap\{y<\tau\}} t^{N-1}|y'|^p \Phi^p dt \le$$

$$(\frac{p}{d})^{p-1}(\tau - \tilde{\theta}_0) \int_{r_1}^{r_2} t^{N-1}|\Phi'|^p dt - (\tau - \omega_2) \int_{(r_1,r_2)\cap\{y<\tau\}} t^{N-1} f(t, y, |y'|)\Phi^p dt \ ,$$

where we chose $\tau > \omega_2$. By properties of Φ and f it follows

$$(1 - (\tau - \tilde{\theta}_0)\frac{d}{p'}) \int_{(r_1,r_2)\cap\{y<\tau\}} t^{N-1}|y'|^p \Phi^p dt$$

$$\le (\frac{p}{d})^{p-1}(\tau - \tilde{\theta}_0)\frac{c_0^p}{(r_2 - r_1)^p} \int_{r_1}^{r_2} t^{N-1} dt - (\tau - \omega_2) r_1^{N-1} \int_{r_1+\delta_2}^{r_2-\delta_2} f_2(t) dt \ .$$

By putting $d := \frac{p'}{\tau - \tilde{\omega}_0}$, from the last inequality we get

$$0 \le (\frac{p}{p'})^{p-1}(\tau - \tilde{\theta}_0)^p \frac{c_0^p}{(r_2 - r_1)^p} \int_{r_1}^{r_2} t^{N-1} dt - (\tau - \omega_2) r_1^{N-1} \int_{r_1+\delta_2}^{r_2-\delta_2} f_2(t) dt \ .$$

Set $F_2 := \int_{r_1+\delta_2}^{r_2-\delta_2} f_2(t)dt$, $\tau = s + \omega_2$, $s > 0$. Then

$$(\tau - \omega_2) F_2 \le (p-1)^{p-1}(\tau - \tilde{\theta}_0)^p \frac{1}{r_1^{N-1}} \frac{c_0^p}{(r_2 - r_1)^p} \frac{1}{N}(r_2^N - r_1^N) \ .$$

Since $\gamma_2 = \frac{1}{Nr_1^{N-1}} \frac{r_2^N - r_1^N}{(r_2-r_1)^p}$, we have

$$sF_2 \le (p-1)^{p-1}(s + (\omega_2 - \tilde{\theta}_0))^p \gamma_2 c_0^p \ .$$

Since $c_0 > 1$ is arbitrarily given, we can pass to the limit as $c_0 \to 1$, so that

$$F_2 \le (p-1)^{p-1} \frac{1}{s}(s + (\omega_2 - \tilde{\theta}_0))^p \gamma_2 \ .$$

At last, we take infimum over all $s > 0$, and so

$$F_2 \le (p-1)^{p-1} \inf_{s>0} \frac{1}{s}(s + (\omega_2 - \tilde{\theta}_0))^p \gamma_2 \ .$$

By assumption $y(t) \le \omega_2$ we arrive at contradiction with (21), so that the claim of the proposition is true. □

Lemma 2.3 *Let $\tilde{\theta}_0$, θ_1 and $\tilde{\omega}_0$ be real numbers such that $\tilde{\theta}_0 < \theta_1 < \tilde{\omega}_0$. Let $0 < r_2 < r_3 < R$ and let $\delta_3 = (r_3 - r_2)/4$. Let the function $f(t, \eta, \xi)$ satisfy*

$$f(t, \eta, \xi) \le 0 \quad a.e. \ t \in (r_2, r_3), \ \eta \in (\theta_1, \tilde{\omega}_0), \ \xi \in \mathbf{R} \ , \tag{22}$$

$$f(t, \eta, \xi) \le f_3(t) \quad a.e. \ t \in (r_2 + \delta_3, r_3 - \delta_3), \ \eta \in (\theta_1, \tilde{\omega}_0), \ \xi \in \mathbf{R} \ ,$$

$$\int_{r_2+\delta_3}^{r_3-\delta_3} f_3(t)dt < -(p-1)^{p-1} \inf_{s>0} \frac{1}{s}((\tilde{\omega}_0 - \theta_1) + s)^p \gamma_3 \ , \tag{23}$$

where

$$f_3 \in L^1(r_2 + \delta_3, r_3 - \delta_3) \quad \textit{and} \quad \gamma_3 := \frac{1}{Nr_2^{N-1}} \frac{r_3^N - r_2^N}{(r_3 - r_2)^p} \ .$$

Then every solution $y \in W^{1,p}_{loc}([0,R)) \cap C([0,R])$ of equation (8) *satisfies: if there holds* $\operatorname{ess\,sup}_{(r_2,r_3)} y \le \tilde{\omega}_0$*, then there holds* $\operatorname{ess\,inf}_{(r_2,r_3)} y \le \theta_1$.

Proof. Similar to proof of Lemma 2.2. □

Remark 2.4 If $r_k \nearrow R$ as $k \longrightarrow +\infty$, we can set $r_k := \alpha_k R$, for instance $\alpha_k = 1 - \frac{1}{2^k}$. Then $r_1 = \frac{1}{2}R$, $r_2 = \frac{3}{4}R$, $r_3 = \frac{7}{8}R$, estimates are again of order $\frac{1}{R^{p-1}}$, as in the case $N = 1$. □

Remark 2.5 The right hand sides in (21) and (23) can be simplified because

$$\inf_{s>0} \frac{1}{s}(c+s)^p = \frac{p^p}{(p-1)^{p-1}} c^{p-1} \ ,$$

where $c > 0$. Therefore, the conditions (21) and (23) can be rewritten in the form

$$\int_{r_1+\delta_2}^{r_2-\delta_2} f_2(t)dt > p^p \gamma_2 (\omega_2 - \tilde{\theta}_0)^{p-1} \ , \tag{24}$$

$$\int_{r_2+\delta_3}^{r_3-\delta_3} f_3(t)dt < -p^p \gamma_3 (\tilde{\omega}_0 - \theta_1)^{p-1} \ . \tag{25}$$

Proof of Theorem 1.4. Directly from Lemma 2.2, Lemma 2.3, and Remark 2.5. □

3 The proof of Theorem 1.6 and Corollary 1.7

At the first, we give an useful lemma which generalizes [7, Lemma 2.1] to the multi-dimensional case.

Lemma 3.1 *Let $k(\varepsilon)$ be a natural number defined as in* (11). *Let $u(x)$ be a function with jumping over θ and ω near ∂B_R. Then for every $\varepsilon \in (0, \varepsilon_0)$ we have*

$$|G_\varepsilon(u)| \geq \int_{r_{k(\varepsilon)}}^{R} (\omega(t) - \theta(t))dt \, . \tag{26}$$

Proof. Let $u(x)$ be a function with jumping over θ and ω near ∂B_R and let σ_k be a sequence determined in (2). Let ε be a fixed number from the interval $(0, \varepsilon_0)$. Let S_k and T_k be two sequences of subsets in $\mathbf{R}^{N+1}$ defined as follows:

$$\begin{cases} S_k = \{(x,y) \in \mathbf{R}^N \times \mathbf{R} : \ \sigma_k \leq |x| \leq \sigma_{k+1} \text{ and } \theta(|x|) \leq y \leq \omega(|x|)\}, \\ T_{2k-1} = \{(x,y) \in \mathbf{R}^N \times \mathbf{R} : \ \sigma_{2k-1} \leq |x| \leq \sigma_{2k} \text{ and } \theta(\sigma_{2k-1}) \leq y \leq \omega(\sigma_{2k})\}, \\ T_{2k} = \{(x,y) \in \mathbf{R}^N \times \mathbf{R} : \ \sigma_{2k} \leq |x| \leq \sigma_{2k+1} \text{ and } \theta(\sigma_{2k+1}) \leq y \leq \omega(\sigma_{2k})\}. \end{cases}$$

A small difference in the definition of sets T_{2k-1} and T_{2k} appears because of (2), that is, since the function $u(x)$ is jumping over $\theta(|x|)$ and $\omega(|x|)$ on (r_{2k-2}, r_{2k-1}) and (r_{2k-1}, r_{2k}) respectively. Next, let S_ε be a set defined by:

$$S_\varepsilon = \{(x,y) \in \mathbf{R}^N \times \mathbf{R} : \ |x| \geq r_{k(\varepsilon)} \text{ and } \theta(|x|) \leq y \leq \omega(|x|)\},$$

where $k(\varepsilon)$ is a natural number defined in (11). It is clear that for all $k \geq k(\varepsilon)$ we have

$$T_k \supseteq S_{k+1}, \quad \bigcup_{k \geq k(\varepsilon)-1} S_{k+1} \supseteq S_\varepsilon, \ \text{ and } \ |S_\varepsilon| = \int_{r_{k(\varepsilon)}}^{R} (\omega(t) - \theta(t))dt. \tag{27}$$

Let us suppose for a moment that the following statement holds true:

$$G_\varepsilon(u|_{[\sigma_{k-1}, \sigma_k]^N}) \supseteq T_k, \ \ k \geq k(\varepsilon), \tag{28}$$

where $[a,b]^N$ is a classic cube in $\mathbf{R}^N$ that is $[a,b]^N = \{x \in \mathbf{R}^N : a \leq |x| \leq b\}$. Since $A_1 \supseteq A_2$ implies $A_1(\varepsilon) \supseteq A_2(\varepsilon)$ and $(A_3 \cup A_4)(\varepsilon) \supseteq A_3(\varepsilon) \cup A_4(\varepsilon)$ for any set A_i and its ε-neighborhood $A_i(\varepsilon)$, from (27) and (28) immediately follows that

$$G_\varepsilon(u) \supseteq G_\varepsilon(u|_{[r_{k(\varepsilon)-2}, R]^N}) \supseteq \bigcup_{k \geq k(\varepsilon)-1} T_k \supseteq \bigcup_{k \geq k(\varepsilon)-1} S_{k+1} \supseteq S_\varepsilon,$$

which proves the desired statement (26). Now, we need to show (28), i.e., that the assumption $(x_0, y_0) \in T_k$ implies $(x_0, y_0) \in G_\varepsilon(u|_{[\sigma_{k-1}, \sigma_k]^N})$. To show that, let $(x_0, y_0) \in T_k$ and let O_N denote the origin of $\mathbf{R}^N$. Let P_0 be a line in $\mathbf{R}^N$

generated by two points O_N and x_0. Let p_k and q_k be two sequences in $\mathbf{R}^N$ defined by:

$$p_k = P_0 \cap \{x \in \mathbf{R}^N : \ ||x|| = \sigma_k\} \ \text{ and } \ q_k = P_0 \cap \{x \in \mathbf{R}^N : \ |x| = r_k\}.$$

Obviously $x_0 \in P_{k,k+1} \subseteq [\sigma_k, \sigma_{k+1}]^N$, where $P_{k,k+1}$ denotes the segment in P_0 with the boundary points p_k and p_{k+1}. That is, the set $P_{k,k+1}$ is a projection of $[\sigma_k, \sigma_{k+1}]^N$ into P_0. By (2), it is clear that there is a point $p_0 \in P_{k-1,k}$ such that $y(p_0) = y_0$ obtained by

$$(p_0, y_0) = \{(x, y_0) \in \mathbf{R}^N \times \mathbf{R} : \ x \in P_{k-1,k}\} \cap G(u|_{P_{k-1,k}}).$$

Hence, we have

$$d((x_0, y_0), G(u|_{P_{k-1,k}})) \leq d((x_0, y_0), (p_0, y_0))). \tag{29}$$

Since $P_{k-1,k} \subseteq [\sigma_{k-1}, \sigma_k]^N$, we have $G(u|_{P_{k-1,k}}) \subseteq G(u|_{[\sigma_{k-1},\sigma_k]^N})$ and so,

$$d((x_0, y_0), G(u|_{[\sigma_{k-1},\sigma_k]^N})) \leq d((x_0, y_0), G(u|_{P_{k-1,k}})). \tag{30}$$

Now, as $x_0 \in P_{k,k+1}$ and $p_0 \in P_{k-1,k}$, by (11), (29), and (30), for all $k \geq k(\varepsilon)$ we obtain:

$$d((x_0, y_0), G(u|_{[\sigma_{k-1},\sigma_k]^N})) \leq d((x_0, y_0), G(u|_{P_{k-1,k}})) \leq d((x_0, y_0), (p_0, y_0)))$$

$$\leq |x_0 - p_0| \leq |p_{k+1} - p_{k-1}| = |\sigma_{k+1} - \sigma_{k-1}| \leq |r_{k+1} - r_{k-2}| \leq \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon,$$

which proves that $(x_0, y_0) \in G_\varepsilon(u|_{[\sigma_k,\sigma_{k+1}]^N})$. Hence, the statement (28) is shown. □

Proof of Theorem 1.6. By Theorem 1.4 every radial solution $u(x)$ of equation (1) is jumping over θ and ω near ∂B_R. Hence the desired statement (12) immediately follows from Lemma 3.1. Next, by the definition of the upper Minkowski-Bouligand dimension and by (12) we obtain that

$$\dim_M G(u) = \limsup_{\varepsilon \longrightarrow 0} \Big(N + 1 - \frac{\log |G_\varepsilon(u)|}{\log \varepsilon} \Big)$$

$$\geq \lim_{\varepsilon \to 0} \left(N + 1 - \frac{\log \int_{r_{k(\varepsilon)}}^R (\omega(t) - \theta(t)) dt}{\log \varepsilon} \right),$$

which proves (13). □

Proof of Corollary 1.7. At the first, it is clear that the functions $\theta(t)$ and $\omega(t)$ given in (14) are increasing and decreasing respectively and that $\theta(R) =$

$\omega(R) = 0$. Next, we claim that the sequence r_k given in (14) satisfies the required condition (11) in respect to $k(\varepsilon) \in \mathbf{N}$ determined by:

$$M_0\Big(\frac{1}{\varepsilon}\Big)^{\frac{\beta}{\beta+1}} \le k(\varepsilon) \le 2M_0\Big(\frac{1}{\varepsilon}\Big)^{\frac{\beta}{\beta+1}},\ \varepsilon \in (0,\varepsilon_0), \tag{31}$$

where $\varepsilon_0 = \frac{R}{\beta}$ and $M_0 = 2\varepsilon_0^{\frac{\beta}{\beta+1}}$. Indeed, the desired statement (11) follows immediately from (31) and from the following elementary inequalities:

$$\begin{aligned} \frac{1}{\beta}\Big(\frac{1}{k}\Big)^{1+\frac{1}{\beta}} &\le \Big(\frac{1}{k-1}\Big)^{\frac{1}{\beta}} - \Big(\frac{1}{k}\Big)^{\frac{1}{\beta}} \\ &\le \frac{1}{\beta}\Big(\frac{1}{k-1}\Big)^{1+\frac{1}{\beta}} \le \frac{2^{1+\frac{1}{\beta}}}{\beta}\Big(\frac{1}{k}\Big)^{1+\frac{1}{\beta}}. \end{aligned} \tag{32}$$

Then, by Theorem 1.6, we have:

$$|G_\varepsilon(u)| \ge c\int_{r_{k(\varepsilon)}}^R (\omega(t)-\theta(t))dt \ge \frac{2c}{\alpha+1}(R-r_{k(\varepsilon)})^{\alpha+1},$$

and

$$\begin{aligned} \dim_M G(u) &\ge \limsup_{\varepsilon\longrightarrow 0}\left(N+1-\frac{\log\int_{r_{k(\varepsilon)}}^R(\omega(t)-\theta(t))dt}{\log\varepsilon}\right) \\ &= \limsup_{\varepsilon\longrightarrow 0}\left(N+1-\frac{\log\frac{2}{\alpha+1}(R-r_{k(\varepsilon)})^{\alpha+1}}{\log\varepsilon}\right). \end{aligned}$$

By (14) and (31) follows that for every $\varepsilon \in (0,\varepsilon_0)$ and every $k \ge k(\varepsilon)$ there holds

$$R - r_{k(\varepsilon)} = \frac{R}{2}\Big(\frac{1}{k(\varepsilon)}\Big)^{\frac{1}{\beta}} \ge \frac{R}{2}\Big(\frac{1}{2M_0}\Big)^{\frac{1}{\beta}}\varepsilon^{\frac{1}{\beta+1}} = c_1\varepsilon^{\frac{1}{\beta+1}},$$

and hence

$$|G_\varepsilon(u)| \ge \frac{2c}{\alpha+1}(R-r_{k(\varepsilon)})^{\alpha+1} \ge c_2\varepsilon^{\frac{\alpha+1}{\beta+1}},$$

and

$$\begin{aligned} \dim_M G(u) &\ge \limsup_{\varepsilon\longrightarrow 0}\left(N+1-\frac{\log\frac{2}{\alpha+1}(c_1\varepsilon^{\frac{1}{\beta+1}})^{\alpha+1}}{\log\varepsilon}\right) \\ &\ge N+1-\frac{\alpha+1}{\beta+1}, \end{aligned}$$

which proves this corollary. □

4 The proofs of Theorem 1.8 and Theorem 1.9

In this section we sketch the proofs of Theorem 1.8 and Theorem 1.9.

Sketch of the proof of Theorem 1.8. Since for radial functions u there holds $|\nabla u| = |y'(t)|$ where $t = |x|$, $u(x) = y(t)$, we have $dist(x, G(|\nabla u|)) = dist(t, G(|y'|))$, $G_\varepsilon(|\nabla u|) = G_\varepsilon(|y'|)$, so that

$$\dim_M G(|\nabla u|) = \dim_M G(|y'|) \ .$$

By Lemma 3.1 and Lemma 3.2 in Pašić-Županović [7] there holds

$$|G_\varepsilon(y')| \geq \sum_{k=k(\varepsilon/2)}^{+\infty} (\omega_{2k} - \theta_{2k-1}) \ .$$

Then it follows

$$|G_\varepsilon(y')| \geq \sum_{k=k(\varepsilon/2)}^{+\infty} (\frac{R}{2})^\alpha \Big[\Big(\frac{1}{2k-1}\Big)^{\frac{\alpha}{\beta}} + \Big(\frac{1}{2k}\Big)^{\frac{\alpha}{\beta}}\Big]$$

$$|G_\varepsilon(y')| \geq \sum_{k=k(\varepsilon/2)}^{+\infty} \frac{R^\alpha}{2^{\alpha-1}}\Big(\frac{1}{2k}\Big)^{\frac{\alpha}{\beta}} \geq \frac{R^\alpha}{2^{\alpha-1}}\Big(\frac{1}{2k(\frac{\varepsilon}{2})}\Big)^{\frac{\alpha}{\beta}} \ .$$

By (31) for every $\varepsilon \in (0, \varepsilon_0)$ there holds $\frac{1}{k(\varepsilon/2)} \geq \frac{1}{2M_0}\Big(\frac{\varepsilon}{2}\Big)^{\frac{\beta}{\beta+1}}$, that is, $|G_\varepsilon(y')| \geq c_2 \varepsilon^{\frac{\alpha}{\beta+1}}$, where $c_2 := \frac{R^{\frac{\alpha\beta}{\beta+1}}}{2^{\alpha-1}8^{\frac{\alpha}{\beta}}}\frac{\beta}{2}^{\frac{\alpha}{\beta+1}}$. Thus $\dim_M G(y') \geq N + 1 - \frac{\alpha}{1+\beta}$. On the other hand, we can write $G(y') = A \cup B$, $G(|y'|) = A' \cup B$, where $A := \{(x, y'(x)) : y'(x) < 0\}$, $B := G(y')\backslash A$ and where A' is a set obtained by reflection of A with respect to x-axes. In particular, $\dim_M A = \dim_M A'$. By the finite stability of upper Minkowski-Bouligand dimension, there holds $\dim_M G(y') = \max\{\dim_M A, \dim_M B\}$, $\dim_M G(|y'|) = \max\{\dim_M A', \dim_M B\}$, and so we have $\dim_M G(y') = \dim_M G(|y'|)$. Therefore, (16) and (17) hold true. □

Proof of Theorem 1.9. Obviously there holds $|\nabla u|_{\mathrm{L}^p(B_{R-\varepsilon})} = |y'|_{\mathrm{L}^p(0,R-\varepsilon)}$. To prove (19), we adapt the proof of Theorem 8.1. in [6]. □

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