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On Hadmard-Type Inequalities for h-Convex Functions on the Co-ordinates

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Abstract. In this paper Hadmard-Type inequalities for h-convex functions on the co-ordinates on the rectangle from the plane are obtained.

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1. Introduction

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function and $a, b \in I$ with $a < b$. Then the following double inequality:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1.1}$$

is known as Hadmard's inequality for convex mapping. For particular choice of the function f in (1.1) yields some classical inequalities of means. Both inequalities in (1.1) hold in reversed if f is concave. In [9], S.S Dragomir and Fitzpatrick proved the following variant of Hadmard's inequality which holds for s-convex function in the first sense:

Theorem 1. *Suppose that $f : [0, \infty) \to [0, \infty)$ is an s-convex function in the first sense, where $s \in (0, 1)$ and let $a, b \in [0, \infty)$. If $f \in L_1([a, b])$ then the* *following inequalities hold:*

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+sf(b)}{s+1} \tag{1.2}$$

The above inequalities are sharp. Also in [9], Dragomir and Fitzpatrick proved another variant of Hadmard' inequality which holds for s-convex functions in the second sense:

Theorem 2. *Suppose that $f:[0,\infty)\to[0,\infty)$ is an s-convex function in the second sense, where $s\in(0,1)$ and let $a,b\in[0,\infty)$. If $f\in L_1([a,b])$ then the following inequalities hold:*

$$2^{s-1}f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{s+1} \tag{1.3}$$

The constant $k=\frac{1}{s+1}$ is the best possible in the second inequality in (1.3), that is the above inequalities are sharp.

In [18], M.Z Sarikaya, A. Saglam and H. Yildrim established the following Hadmard-type inequality for h-convex functions:

Theorem 3. *Let $f\in SX(h,I); a,b\in I$ with $a<b$ and $f\in L([a,b])$. Then*

$$\frac{1}{2h(\frac{1}{2})}f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le [f(a)+f(b)]\int_0^1 h(\alpha)d\alpha \tag{1.4}$$

In [8], Dragomir established the following similar inequality of Hadmard's type for convex functions on the co-ordinates on a rectangle from the plane $\mathbb{R}^2$.

Theorem 4. *Suppose $f:\Delta=[a,b]\times[c,d]\subseteq[0,\infty)\to\mathbb{R}$ is the co-ordinated convex functions on Δ. Then one has the inequalities:*

$$\begin{aligned} f\left(\frac{a+b}{2},\frac{c+d}{2}\right) &\le \frac{1}{(b-a))d-c)}\int_a^b\int_c^d f(x,y)dydx \\ &\le \frac{f(a,c)+f(b,c)+f(a,d)+f(b,d)}{4} \end{aligned} \tag{1.5}$$

The above inequalities are sharp. In [1] (see also [5]), Alomari and M. Darus proved the following similar Hadmard-type inequality for s-convex functions on the co-ordinates in the second sense on a rectangle from the plane $\mathbb{R}^2$.

Theorem 5. *Suppose $f:\Delta=[a,b]\times[c,d]\subseteq[0,\infty)\to\mathbb{R}$ is the co-ordinated s-convex function in the second sense on the co-ordinates on Δ. Then one has the inequalities:*

$$\begin{aligned} 4^{s-1}f\left(\frac{a+b}{2},\frac{c+d}{2}\right) &\le \frac{1}{(b-a))d-c)}\int_a^b\int_c^d f(x,y)dydx \\ &\le \frac{f(a,c)+f(b,c)+f(a,d)+f(b,d)}{(s+1)^2} \end{aligned} \tag{1.6}$$

Also in [4] (see also [5]), Alomari and M. Darus established the following similar inequality of Hadmard-type for s-convex functions on the co-ordinates in the first sense on a rectangle from the plane $\mathbb{R}^2$.

Theorem 6. *Suppose $f : \Delta = [a, b] \times [c, d] \subseteq [0, \infty) \to \mathbb{R}$ is the co-ordinated s-convex function in the first sense on the co-ordinates on Δ. Then one has the inequalities:*

$$(1.7) \quad f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \leq \frac{1}{(b-a))d-c)} \int_a^b \int_c^d f(x,y)dydx \leq \frac{f(a,c) + sf(b,c) + sf(a,d) + s^2 f(b,d)}{(s+1)^2}$$

For refinements, counterparts, generalizations and new Hadamard's–type inequalities see [1, 2, 3, 4, 5, 8, 9, 10, 11, 13].

Motivated by the results (1.4)-(1.7), the main purpose of the present paper is to establish Hadmard-type inequality for h-convex functions on the co-ordinates on a rectangle from the plane $\mathbb{R}^2$.

2. Preliminaries

Definition 1. *Let I be an interval of real numbers. The function $f : I \to \mathbb{R}$ is said to be convex on I if for all $x, y \in I$ and $\alpha \in [0, 1]$, one has the inequality:*

$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha)f(y)$$

if the above inequality reversed then f is said to be concave.

Definition 2. *A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to Godunova-Levin function or f is said to belong to the class $Q(I)$ if f is non-negative and for all $x, y \in I$ and for $\alpha \in (0, 1)$ we have the inequality:*

$$f(\alpha x + (1-\alpha)y) \leq \frac{f(x)}{\alpha} + \frac{f(y)}{1-\alpha}$$

The class $Q(I)$ was firstly described in [12] by Godunova-Levin. Some further properties of it can be found in [11], [15] and [16]. Among others, it is noted that non-negative monotone and non-negative convex functions belongs to this class of functions.

Definition 3. *Let $s \in (0, 1]$ be fixed real number. A function $f : [0, \infty) \to [0, \infty)$ is said to be s-convex in the second sense, or that f belongs to the class K_s^2, if*

$$f(\alpha x + (1-\alpha)y) \leq \alpha^s f(x) + (1-\alpha)^s f(y)$$

for all $x, y \in [0, \infty)$ and $\alpha \in [0, 1]$.

In [6], Breckner introduced s-convex functions as a generalization of convex functions. In [7], he proved the important fact that the set-valued map is s-convex only if associated support function is s-convex A number of properties and connections with s-convexity in the first sense are discussed in paper [13]. It is clear that s-convexity is merely convexity for $s = 1$.

Definition 4. *A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be P-function or that f is said to belong to the class $P(I)$ if f is non-negative and for all $x, y \in I$ and $\alpha \in [0, 1]$, if*

$$f(\alpha x + (1-\alpha)y) \leq f(x) + f(y)$$

Definition 5. *Let $h : (0, 1) \subseteq J \subseteq \mathbb{R} \to \mathbb{R}$ be a positive function. A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be h-convex or that f is said to belong to the class $SX(h, I)$, if f is non-negative and for all $x, y \in I$ and $\alpha \in (0, 1)$, we have*

$$f(\alpha x + (1-\alpha)y) \leq h(\alpha)f(x) + h(1-\alpha)f(y)$$

if the inequality is reversed then f is said to be h-concave and we say that f belongs to the class $SV(h, I)$.

The class of h-convex functions was introduced by S. Varosanec in [19] (see [19] for further properties of h-convex functions)

Remark 1. *Obviously, if $h(\alpha) = \alpha$, then all the non-negative convex functions belong to the class $SX(h, I)$ and all non-negative concave functions belong to the class $SV(h, I)$. Also note that if $h(\alpha) = \frac{1}{\alpha}$, then $SX(h, I) = Q(I)$; if $h(\alpha) = 1$, then $SX(h, I) \supseteq P(I)$; and if $h(\alpha) = \alpha^s$, where $s \in (0, 1)$, then $SX(h, I) \supseteq K_s^2$.*

Let us now consider a bidimensional interval $\Delta =: [a, b] \times [c, d]$ in $\mathbb{R}^2$ with $a < b$ and $c < d$. A mapping $f : \Delta \to \mathbb{R}$ is said to be convex on Δ if the following inequality:

$$f(\alpha x + (1-\alpha)z, \alpha y + (1-\alpha)w) \leq \alpha f(x, y) + (1-\alpha)f(z, w)$$

holds, for all $(x, y), (z, w) \in \Delta$ and $\alpha \in [0, 1]$. If the inequality reversed then f is said to be concave on Δ.

A modification for convex functions which is also known as co-ordinated convex functions was introduced by Dragomir in [8].

A function $f : \Delta \to \mathbb{R}$ is said to be convex on the co-ordinates on Δ if the partial mappings $f_y : [a, b] \to \mathbb{R}, f_y(u) = f(u, y)$ and $f_x : [c, d] \to \mathbb{R}, f_x(v) = f(x, v)$ are convex where defined for all $x \in [a, b], y \in [c, d]$.

A formal definition for co-ordinated convex functions may be stated as follow:

Definition 6. *A function $f : \Delta \to \mathbb{R}$ is said to be convex on the co-ordinates on Δ if the following inequality:*

$$\begin{aligned} &f(tx + (1-t)y, su + (1-s)w) \\ &\leq tsf(x, u) + t(1-s)f(x, w) + s(1-t)f(y, u) + (1-t)(1-s)f(y, w) \end{aligned}$$

holds for all $t, s \in [0, 1]$ and $(x, u), (x, w), (y, u), (y, w) \in \Delta$.

Clearly, every convex mapping $f : \Delta \to \mathbb{R}$ is convex on the co-ordinates. Furthermore, there exists co-ordinated convex function which is not convex, (see [10]).

The concept of s-convex functions and s-convex functions on the co-ordinates in both sense was introduced by Almoari and Darus in [1] and [4].

Definition 7. *Consider the bidimensional interval* $\Delta =: [a, b]\times[c, d]$ *in* $[0,\infty)^2$ *with* $a < b$ *and* $c < d$*. The mapping* $f : \Delta \to \mathbb{R}$ *is* s*-convex in the first sense (in the second sense) on* Δ *if*

$$f(\alpha x + \beta z, \alpha y + \beta w) \leq \alpha^s f(x, y) + \beta^s f(z, w)$$

, holds for all $(x, y), (z, w) \in \Delta$ *with* $\alpha, \beta \geq 0$ *with* $\alpha^s + \beta^s = 1(\alpha + \beta = 1)$ *and for some fixed* $s \in (0, 1]$*. We write* $f \in \Xi_s^i$ $(i = 1, 2)$ *which means that* f *is* s*-convex in the first sense when* $i = 1$*, (in the second sense when* $i = 2$*).*

Definition 8. *A function* $f : \Delta =: [a, b] \times [c, d] \subseteq [0,\infty)^2 \to \mathbb{R}$ *is called* s*-convex in first sense (in the second sense) on the co-ordinates on* Δ *if the partial mappings* $f_y : [a, b] \to \mathbb{R}, f_y(u) = f(u, y)$ *and* $f_x : [c, d] \to \mathbb{R}, f_x(v) = f(x, v)$*, are* s*-convex in the first sense (in the second sense) for all* $y \in [c, d]$*,* $x \in [a, b]$ *and* $s \in (0, 1]$*, i.e., the partial mappings* f_y *and* f_x *are* s*-convex in the first sense (second sense) with same fixed* $s \in (0, 1]$*.*

Lemma 1. [1] *Every* s*-convex mapping* $f : \Delta = [a, b] \times [c, d] \subseteq [0,\infty)^2 \to [0,\infty)$ *in the first sense (in the second sense) is* s*-convex on the co-ordinates on* Δ *in the first sense (in the second sense) but the converse is not true in general.*

Proof. Suppose that $f : \Delta = [a, b] \times [c, d] \subseteq [0,\infty)^2 \to [0,\infty)$ is s-convex in the first sense (in the second sense) on Δ. Consider the function $f_x : [c, d] \to [0,\infty), f_x(v) = f(x, v)$. Then for $\alpha, \beta \geq 0$ with $\alpha^s + \beta^s = 1(\alpha + \beta = 1)$ and for some fixed $s \in (0, 1]$ and $v_1, v_2 \in [c, d]$, one has:

$$\begin{aligned} f_x(\alpha v_1 + \beta v_2) &= f(x, \alpha v_1 + \beta v_2) \\ &= f(\alpha x + \beta x, \alpha v_1 + \beta v_2) \\ &\leq \alpha^s f(x, v_1) + \beta^s f(x, v_2) \\ &= \alpha^s f_x(v_1) + \beta^s f_x(v_2) \end{aligned}$$

here we have used the fact that $\beta = 1 - \alpha$ in the case of s- convexity in the second sense and $\beta = (1 - \alpha)^{\frac{1}{s}}$ in the case of s-convexity in the first sense. Therefore, $f_x(v) = f(x, v)$ is s-convex on $[c, d]$. The fact that $f_y : [a, b] \to [0,\infty), f_y(u) = f(u, y)$ is also s-convex on $[a, b]$ for all $y \in [c, d]$ goes likewise and we shall omit the details. □

s-convexity on the co-ordinates does not imply the s-convexity, that is there exist functions which are s-convex on the co-ordinates but are not s-convex (see [1]). In [3] (see also [5]), M. Alomari and M. Darus introduced a new class of s-convex functions and s-convex functions on the co-ordinates:

Definition 9. *Consider the bidimensional interval* $\Delta =: [a,b]\times[c,d]$ *in* $[0,\infty)^2$ *with* $a<b$ *and* $c<d$. *The mapping* $f:\Delta\to\mathbb{R}$ *is* s*-convex in the first sense on* Δ *if there exist* $s_1,s_2\in(0,1]$ *with* $s=\frac{s_1+s_2}{2}$, *such that*

$$f(\alpha x+\beta z,\alpha y+\beta w)\le\alpha^{s_1}f(x,y)+\beta^{s_2}f(z,w)$$

, holds for all $(x,y),(z,w)\in\Delta$ *with* $\alpha,\beta\ge 0$ *with* $\alpha^{s_1}+\beta^{s_2}=1$ *and for all fixed* $s_1,s_2\in(0,1]$. *We denote this class of functions by* $MWO^1_{s_1,s_2}$. *Let* $f:\Delta\to\mathbb{R}$, *then* f *is called co-ordinated* s*-convex of the first sense on* Δ *if the partial mappings* $f_y:[a,b]\to\mathbb{R}, f_y(u)=f(u,y)$ *and* $f_x:[c,d]\to\mathbb{R}, f_x(v)=f(x,v)$, *are* s_1*-,*s_2*-convex in the first sense for all* $s_1,s_2\in(0,1]$, $y\in[c,d]$ *and* $x\in[a,b]$*; respectively, with* $s=\frac{s_1+s_2}{2}\in(0,1]$.

Definition 10. *Consider the bidimensional interval* $\Delta =: [a,b]\times[c,d]$ *in* $[0,\infty)^2$ *with* $a<b$ *and* $c<d$. *The mapping* $f:\Delta\to\mathbb{R}$ *is* s*-convex in the second sense on* Δ *if there exist* $s_1,s_2\in(0,1]$ *with* $s=\frac{s_1+s_2}{2}$, *such that*

$$f(\alpha x+\beta z,\alpha y+\beta w)\le\alpha^{s_1}f(x,y)+\beta^{s_2}f(z,w)$$

, holds for all $(x,y),(z,w)\in\Delta$ *with* $\alpha,\beta\ge 0$ *with* $\alpha+\beta=1$ *and for all fixed* $s_1,s_2\in(0,1]$. *We denote this class of functions by* $MWO^2_{s_1,s_2}$. *Let* $f:\Delta\to\mathbb{R}$, *then* f *is called co-ordinated* s*-convex of the second sense on* Δ *if the partial mappings* $f_y:[a,b]\to\mathbb{R}, f_y(u)=f(u,y)$ *and* $f_x:[c,d]\to\mathbb{R}, f_x(v)=f(x,v)$, *are* s_1*-,*s_2*-convex in the second sense for all* $s_1,s_2\in(0,1]$, $y\in[c,d]$ *and* $x\in[a,b]$*; respectively, with* $s=\frac{s_1+s_2}{2}\in(0,1]$.

Remark 2. *Clearly the classes of functions as defined in Definition 9 and Definition 10 are contained in classes of functions as defined in Definition 7 and Definition 8 when* $s_1=s_2=s$.

Similarly one can give the notion of h-convexity of a function f on a rectangle from the plane $\mathbb{R}^2$ and h-convexity on the co-ordinates on a rectangle from the plane $\mathbb{R}^2$ as follows:

Definition 11. *Let us consider a bidimensional interval* $\Delta =: [a,b]\times[c,d]$ *in* $\mathbb{R}^2$ *with* $a<b$ *and* $c<d$. *Let* $h:J\subseteq\mathbb{R}\to\mathbb{R}$ *be a positive function. A mapping* $f:\Delta =: [a,b]\times[c,d]\to\mathbb{R}$ *is said to be* h*-convex on* Δ*, if* f *is non-negative and if the following inequality:*

$$f(\alpha x+(1-\alpha)z,\alpha y+(1-\alpha)w)\le h(\alpha)f(x,y)+h(1-\alpha)f(z,w)$$

holds, for all $(x,y),(z,w)\in\Delta$ *and* $\alpha\in(0,1)$ *. Let us denote this class of functions by* $SX(h,\Delta)$. *The function* f *is said to be* h*-concave if the inequality reversed. We denot this class of functions by* $SV(h,\Delta)$.

A function $f:\Delta\to\mathbb{R}$ is said to be h-convex on the co-ordinates on Δ if the partial mappings $f_y:[a,b]\to\mathbb{R}, f_y(u)=f(u,y)$ and $f_x:[c,d]\to\mathbb{R}, f_x(v)=f(x,v)$ are h-convex where defined for all $x\in[a,b], y\in[c,d]$. A formal definition of h-convex functions may also be stated as follows:

Definition 12. *A function $f : \Delta \to \mathbb{R}$ is said to be h-convex on the co-ordinates on Δ if the following inequality:*

$$\begin{aligned} f(tx + (1-t)y, su + (1-s)w) \\ &\leq h(t)h(s)f(x,u) + h(t)h(1-s)f(x,w) \\ &\quad + h(s)h(1-t)f(y,u) + h(1-t)h(1-s)f(y,w) \end{aligned}$$

holds for all $t, s \in [0,1]$ and $(x,u), (x,w), (y,u), (y,w) \in \Delta$.

It is clear that if $h(\alpha) = \alpha$, then the class of non-negative convex (concave) functions on Δ is contained in the class of h-convex (concave) functions on Δ and if $h(\alpha) = \alpha^s, s \in (0,1)$, then the class of s-convex functions on the Δ is contained in the class of h-convex functions on Δ. Similarly we can say that if $h(\alpha) = \alpha$, then the class of non-negative convex (concave) functions on the co-ordinates on Δ is contained in the class of h-convex (concave) functions on the co-ordinates on Δ and if $h(\alpha) = \alpha^s, s \in (0,1)$, then the class of s-convex functions on the the co-ordinates on Δ is contained in the class of h-convex functions on the co-ordinates on Δ.

Remark 3. *Let h be a non-negative function such that $h(\alpha) \leq \alpha$ for all $\alpha \in (0,1)$. For example, the function $h_k(x) = x^k$ where $k \leq 1$ and $x > 0$ has the property. If f is non-negative convex function on Δ, then for all $(x,y), (z,w) \in \Delta, \alpha \in (0,1)$ we have*

$$\begin{aligned} f(\alpha x + (1-\alpha)z, \alpha y + (1-\alpha)w) \\ &\leq \alpha f(x,y) + (1-\alpha)f(z,w) \\ &\leq h(\alpha)f(x,y) + h(1-\alpha)f(z,w) \end{aligned}$$

so $f \in SX(h, \Delta)$. Similarly, if the function h has the property: $h(\alpha) \geq \alpha$ for all $\alpha \in (0,1)$, then any non-negative concave function f belongs to the class $SV(h, \Delta)$.

Example 1. *Let $h_k, k < 0$, be a function defined as in Remark 3 and let the function f be defined as follows:*

$$f : [0,1]^2 \to \mathbb{R}, \; f(x,y) = \begin{cases} 1 & , x \neq \frac{1}{2}, y \neq \frac{1}{2} \\ 1 & , x = \frac{1}{2}, y \neq \frac{1}{2} \\ 1 & , x \neq \frac{1}{2}, y = \frac{1}{2} \\ 4^{1-k} & , x = \frac{1}{2}, y = \frac{1}{2} \end{cases}$$

Then f is a non-convex function, but it is h_k-convex.

Proof. Let $\alpha = \frac{1}{3}, x = y = \frac{1}{3}$ and $z = w = \frac{7}{12}$, then

$$f(\alpha x + (1-\alpha)z, \alpha y + (1-\alpha)w) = f\left(\frac{1}{2}, \frac{1}{2}\right) = 4^{1-k}$$

but on the other hand

$$\alpha f(x,y)+(1-\alpha)f(z,w)=\frac{1}{3}f\left(\frac{1}{3},\frac{1}{3}\right)+\frac{2}{3}f\left(\frac{7}{12},\frac{7}{12}\right)=1$$

Hence

$$f(\alpha x+(1-\alpha)z,\alpha y+(1-\alpha)w)>\alpha f(x,y)+(1-\alpha)f(z,w)$$

This shows that f is not convex on $[0,1]^2$. Now we prove that f is h_k-convex, to this end we have to prove that

$$f(\alpha x+(1-\alpha)z,\alpha y+(1-\alpha)w)\le h(\alpha)f(x,y)+h(1-\alpha)f(z,w)$$

That is we have to prove

$$\alpha^k f(x,y)+(1-\alpha)^k f(z,w)-f(\alpha x+(1-\alpha)z,\alpha y+(1-\alpha)w)\ge 0$$

for all $(x,y),(z,w)\in[0,1]^2$, $\alpha\in(0,1)$ and $k<0$. Let

$$H(\alpha)=\alpha^k f(x,y)+(1-\alpha)^k f(z,w)-f(\alpha x+(1-\alpha)z,\alpha y+(1-\alpha)w)$$

Obviously $H(0)=0$ and $H^{'}(\alpha)=k\alpha^{k-1}f(x,y)-k(1-\alpha)^{1-k}f(z,w)$. It can easily be seen that for all $(x,y),(z,w)\in[0,1]^2$, $\alpha\in(0,1)$ and $k<0$, $H^{'}(\alpha)>0$. It follows that $H(\alpha)\ge 0$.Hence it is proved that f is h_k convex for $k<0$. □

Lemma 2. *Every h-convex mapping $f:\Delta\to\mathbb{R}$ is h-convex on the co-ordinates, but the converse is not generally true.*

Proof. Suppose that $f:\Delta=[a,b]\times[c,d]\subseteq[0,\infty)^2\to[0,\infty)$ is h-convex on Δ. Consider the function $f_x:[c,d]\to[0,\infty), f_x(v)=f(x,v)$, then for all $v_1,v_2\in[c,d]$, one has:

$$\begin{aligned}f_x(\alpha v_1+(1-\alpha)v_2)&=f(x,\alpha v_1+(1-\alpha)v_2)\\&=f(\alpha x+(1-\alpha)x,\alpha v_1+\beta v_2)\\&\le h(\alpha)f(x,v_1)+h(1-\alpha)f(x,v_2)\\&=h(\alpha)f_x(v_1)+h(1-\alpha)f_x(v_2)\end{aligned}$$

Therefore, $f_x(v)=f(x,v)$ is h-convex on $[c,d]$. The fact that $f_y:[a,b]\to[0,\infty), f_y(u)=f(u,y)$ is also h-convex on $[a,b]$ for all $y\in[c,d]$ goes likewise and we shall omit the details. □

The converse of this Lemma is not true in general. To prove this fact we consider the same function as it was taken in [10], with $h(\alpha)=\alpha$.

3. Main Result

In this section we establish Hadmard's Inequality for h-convex functions on the co-ordinates on rectangle from the plane $\mathbb{R}^2$.

Theorem 7. *Let $f : \Delta = [a,b] \times [c,d] \subseteq \mathbb{R}^2 \to \mathbb{R}$ be an h-convex function on the co-ordinates on Δ and let $f \in L_2(\Delta)$ and $h \in L_1([0,1]$. Then one has the inequalities:*

$$\begin{aligned}(3.1)\quad \frac{1}{4\left(h(\frac{1}{2})\right)^2} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) &\le \frac{1}{(b-a))(d-c)} \int_a^b \int_c^d f(x,y)dydx \\ &\le [f(a,c)+f(b,c)+f(a,d)+f(b,d)] \left(\int_0^1 h(\alpha)d\alpha\right)^2\end{aligned}$$

Proof. Since $f : \Delta = [a,b] \times [c,d] \subseteq \mathbb{R}^2 \to \mathbb{R}$ is h-convex function on the co-ordinates on Δ, it follows that the mapping $g_x : [c,d] \to \mathbb{R}$, defined by $g_x(y) = f(x,y)$ is h-convex on $[c,d]$ for all $x \in [a,b]$. Therefore by Hadmard's inequality (1.4), we have

$$\frac{1}{2h(\frac{1}{2})} g_x\left(\frac{c+d}{2}\right) \le \frac{1}{d-c} \int_c^d g_x(y)dy \le [g_x(c)+g_x(d)] \int_0^1 h(\alpha)d\alpha$$

That is

$$\frac{1}{2h(\frac{1}{2})} f\left(x, \frac{c+d}{2}\right) \le \frac{1}{d-c} \int_c^d f(x,y)dy \le [f(x,c)+f(x,d)] \int_0^1 h(\alpha)d\alpha$$

Dividing both sides by $b-a$ and integrating this inequality on $[a,b]$, we have:

$$\begin{aligned}(3.2)\quad \frac{1}{2h(\frac{1}{2})(b-a)} \int_a^b f\left(x, \frac{c+d}{2}\right) dx &\le \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)dydx \\ &\le \left[\frac{1}{b-a} \int_a^b f(x,c)dx + \frac{1}{b-a} \int_a^b f(x,d)dx\right] \int_0^1 h(\alpha)d\alpha\end{aligned}$$

By applying similar arguments to the mapping $g_y : [a,b] \to \mathbb{R}$, defined by $g_y(x) = f(x,y)$, we get

$$\begin{aligned}(3.3)\quad \frac{1}{2h(\frac{1}{2})(d-c)} \int_c^d f\left(\frac{a+b}{2}, y\right) dy &\le \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)dydx \\ &\le \left[\frac{1}{d-c} \int_c^d f(a,y)dy + \frac{1}{d-c} \int_c^d f(b,y)dy\right] \int_0^1 h(\alpha)d\alpha\end{aligned}$$

Adding (3.2) and (3.3) we get

$$\begin{aligned}(3.4)\quad \frac{1}{4h(\frac{1}{2})}\left[\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)dx+\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)dy\right] \\ \le \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx \\ \le \frac{1}{2}[\frac{1}{b-a}\int_a^b f(x,c)dx+\frac{1}{b-a}\int_a^b f(x,d)dx \\ +\frac{1}{d-c}\int_c^d f(a,y)dy+\frac{1}{d-c}\int_c^d f(b,y)dy]\int_0^1 h(\alpha)d\alpha\end{aligned}$$

By Hadmard's inequality we also have

$$\frac{1}{2h(\frac{1}{2})}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\le\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)dx$$

$$\frac{1}{2h(\frac{1}{2})}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\le\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)dy$$

Adding these inequalities we get

$$\frac{1}{h(\frac{1}{2})}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\le\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)dx+\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)dy$$

Dividing both sides of last inequality by $4h(\frac{1}{2})$, we have

$$\begin{aligned}(3.5)\quad \frac{1}{4\left(h(\frac{1}{2})\right)^2}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\le \\ \frac{1}{4h(\frac{1}{2})}\left[\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)dx+\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)dy\right]\end{aligned}$$

Finally by the same inequality we get

$$\frac{1}{b-a}\int_a^b f(x,c)dx\le[f(a,c)+f(b,c)]\int_0^1 h(\alpha)d\alpha$$

$$\frac{1}{b-a}\int_a^b f(x,d)dx\le[f(a,d)+f(b,d)]\int_0^1 h(\alpha)d\alpha$$

$$\frac{1}{d-c}\int_c^d f(a,y)dy\le[f(a,c)+f(a,d)]\int_0^1 h(\alpha)d\alpha$$

$$\frac{1}{d-c}\int_c^d f(b,y)dy\le[f(b,c)+f(b,d)]\int_0^1 h(\alpha)d\alpha$$

Adding all these inequalities we get

$$\begin{aligned}(3.6)\quad &\frac{1}{2}[\frac{1}{b-a}\int_a^b f(x,c)dx + \frac{1}{b-a}\int_a^b f(x,d)dx \\ &+ \frac{1}{d-c}\int_c^d f(a,y)dy + \frac{1}{d-c}\int_c^d f(b,y)dy]\int_0^1 h(\alpha)d\alpha \\ &\leq [f(a,c)+f(b,c)+f(a,d)+f(b,d)]\left(\int_0^1 h(\alpha)d\alpha\right)^2\end{aligned}$$

From (3.2), (3.3), (3.4), (3.5), and (3.6) we get (3.1) which is the assertion of the theorem. This completes the proof. □

Remark 4. *if we take $h(t) = t$, then (3.1) reduces to (1.5) and if we take $h(t) = t^s$, then (3.1) reduces to (1.6).*

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