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Double Matrix Summability

of Double Fourier Series

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Abstract

In this paper, a new theorem on double matrix summability of double Fourier series has been established.

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1. Introduction. Chow ([1]), for the first time, studied Cesàro summability of double Fourier series. Sharma ([5]) and Mishra ([4]) extended the result of Chow for (H,1,1) and (N, p_m, q_n) summabilities respectively. But nothing seems to have been done so far to study double Fourier series by a double factorable summability method which ,as known, includes as special cases, the methods of (C, 1,1), (H,1,1) and (N, p_m, q_n). In this paper a more general result than those of Chow ([1]), Sharma ([5]) and Mishra ([4]) has been established. Our theorem includes their results as particular cases.

2.Definitions. Let f(u,v) be a function of (u,v), periodic with respect to u and with respect to v, in each case with period 2π and summable in the square $Q(-\pi,-\pi;\ \pi,\pi)$. The double Fourier series of a function f(u,v) is given by

$$f(u,v) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \lambda_{m,n}[\alpha_{m,n}\cos mu\cos nv + \beta_{m,n}\sin mu\cos nv + \gamma_{m,n}\cos mu\sin nv + \delta_{m,n}\sin mu\sin nv]$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \lambda_{m,n} A_{m,n}(u,v) \qquad (1)$$

where
$$\lambda_{m,n} = \begin{cases} 1/4, & \text{for } m=0, n=0, \\ 1/2, & \text{for } m>0, n=0 \text{ and } m=0, n>0, \\ 1, & \text{for } m>0, n>0 \end{cases}$$

and $\alpha_{m,n} = \frac{1}{\pi^2}\iint_Q f(u,v)\cos mu\cos nv\, du\, dv$, with three similar expressions for $\beta_{m,,n}$, $\gamma_{m,n}$ and $\delta_{m,n}$,

where Q denotes the fundamental square $(-\pi,\pi).(-\pi,\pi)$.

Let T = $(a_{m,j})$ and S = $(b_{n,k})$ be two infinite lower triangular matrices. Let $\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} u_{m,n}$ be a double series with $s_{m,n} = \sum_{j=0}^{m}\sum_{k=0}^{n} u_{j,k}$ as its $(m,n)^{th}$ partial sums. The double matrix mean $t_{m,n}$ is given by $t_{m,n} = \sum_{j=0}^{m}\sum_{k=0}^{n} a_{m,j}\, b_{n,k}\, s_{j,k}$.

The double series $\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} u_{m,n}$ with the sequence of (m, n)th partial sums $(s_{m,n})$ is said to summable by double matrix summability method or summable (T, S) if $t_{m,n}$ tends to a limit s as $m \to \infty$ and $n \to \infty$.

The regularity conditions of double matrix summability means are given by

$\sum_{j=0}^{m}\sum_{k=0}^{n} a_{m,j}\, b_{n,k} \to 1$ as $m \to \infty$ and $n \to \infty$, $\lim_{m,n} \sum_{k=0}^{n} |a_{m,j}\, b_{n,k}| = 0$, for each j = 1,2,3,…

$\lim_{m,n} \sum_{j=0}^{m} |a_{m,j}\, b_{n,k}| = 0$, for each k = 1,2,3,… .

We write, $\phi(u,v) = \phi(x,y;u,v)$

$$= \frac{1}{4}[f(x+u,y+v)+f(x+u,y-v)+f(x-u,y+v)+f(x-u,y-v)-4f(x,y)],$$

$$\Phi(u,v) = \int_0^u\int_0^v |\phi(s,t)|\,ds\,dt\,,\ \Phi_1(u,t) = \int_0^u |\phi(s,t)|\,ds\ ,\Phi_2(s,v) = \int_0^v |\phi(s,t)|\,dt$$

$$\tau = (1/t) = \text{integral part of } 1/t\,,\ \sigma = (1/s) = \text{integral part of } 1/s\ ,$$

$$K_m(s) = \frac{1}{2\pi}\sum_{j=0}^{m} a_{m,j}\frac{\sin(j+1/2)s}{\sin s/2}\,, \qquad K_n(t) = \frac{1}{2\pi}\sum_{k=0}^{n} b_{n,k}\frac{\sin(k+1/2)t}{\sin t/2}\,.$$

Important particular cases of the double matrix summability method are

(i) (C,1,1) summability mean ([1]) if $a_{m,j} = \frac{1}{m+1}\ \forall\, m$ and $b_{n,k} = \frac{1}{n+1}\ \forall\, n$.

(ii) (H,1,1) summability mean ([5]) if $a_{m,j} = \frac{1}{(m-j+1)\log m}$ and

$$b_{n,k} = \frac{1}{(n-k+1)\log n}\ .$$

(iii) (N,p_m,q_n) summability mean ([2]) if $a_{m,j} = \frac{p_{m-j}}{P_m}$ and $b_{n,k} = \frac{q_{n-k}}{Q_n}$,

provided $P_m = \sum_{j=0}^{m} p_j \neq 0$ and $Q_n = \sum_{k=0}^{n} q_k \neq 0$.

Double matrix summability method (T,S) is assumed as regular throughout this paper.

3. Theorem.We prove the following theorem :

Theorem. Let $(a_{m,j})_{j=0}^{m}$ and $(b_{n,k})_{k=0}^{n}$ be two real non-negative and non-decreasing sequences with $j \le m$ and $k \le n$ respectively. Let T = $(a_{m,j})$ and S = $(b_{n,k})$ be two infinite triangular matrices with $a_{m,j} \ge 0$, $a_{m,j} = 0$, $j > m$; $b_{n,k} \ge 0$, $b_{n,k} = 0$, $k > n$, $A_{m,\sigma} = \sum_{j=0}^{\sigma} a_{m,j}$,

$B_{n,\tau} = \sum_{k=0}^{\tau} b_{n,k}$, $A_{m,m} = 1 \ \forall\, m \ge 0$, $B_{n,n} = 1 \ \forall\, n \ge 0$.

If the conditions ,
$$\Phi(u,v) = \int_0^u \int_0^v |\phi(s,t)|\, ds\, dt = o\left(\frac{u}{\alpha(1/u)} \frac{v}{\beta(1/v)}\right) \qquad (2)$$

$$\int_0^{\pi} \Phi_1(u,t)dt = O\left(\frac{u}{\alpha(1/u)}\right), \quad \int_0^{\pi} \Phi_2(s,v)ds = O\left(\frac{v}{\beta(1/v)}\right)$$

hold then the double Fourier series (1) is double matrix (T,S) summable to f(x,y), at the point (u,v) = (x,y), provided $\alpha(u)$ and $\beta(v)$ are two positive monotonic increasing functions of u and v such that $\alpha(m) \to \infty$, as $m \to \infty$ and $\beta(n) \to \infty$, as $n \to \infty$, $\int_1^m \frac{A_{m,s}}{s\,\alpha(s)} ds = O(1)$,

$m \to \infty$, $\int_1^n \frac{B_{n,t}}{t\,\beta(t)} dt = O(1)$, $n \to \infty$,

4.Lemmas :For the proof of our theorem, we need the following lemmas.

Lemma 1. $K_m(s) = O(m)$, for $0 \le s \le m^{-1}$

Proof. For $0 < s \le 1/(m+1)$, using $\sin(m+1)s \le (m+1)s, \sin(t/2) \ge (t/\pi)$, we have

$$K_m(s) \le \frac{1}{2} \sum_{j=0}^{m} a_{m,j}(2j+1)\frac{s/2}{s} \le \frac{(2m+1)}{4} \sum_{j=0}^{m} a_{m,j} = O(m).$$

Lemma 2. $K_n(t) = O(n)$. for $0 \le t \le n^{-1}$.

Lemma 3. ([3]) (i) If $(a_{m,\mu})$ is non-negative and non-decreasing with μ then for $0 \le a \le b \le \infty$ and $0 \le s \le \pi$ and any m, $$\left|\sum_{\mu=a}^{b} a_{m,m-\mu}\, e^{i(m-\mu)s}\right| = O(A_{m,\sigma}).$$

Lemma 4. If $(b_{n,\nu})$ is non-negative and non-decreasing with ν then for $0 \le a \le b \le \infty$, $0 \le t \le \pi$ and any n, $$\left|\sum_{\nu=a}^{b} b_{n,n-\nu}\, e^{i(n-\nu)t}\right| = O(B_{n,\tau}).$$

Lemma 5. $K_m(s) = O\left(\frac{A_{m,\sigma}}{s}\right)$, for $0 < m^{-1} < s \le \pi$.

Proof. Since $m^{-1} < s \le \pi$, $\sin(s/2) \ge (s/\pi)$, we have

$$|K_m(s)| = \left|\frac{1}{2\pi}\sum_{j=1}^{m} a_{m,j}\frac{\sin(j+1/2)s}{\sin s/2}\right| = \frac{1}{2\pi}\left|\text{Imaginary part of } \sum_{j=0}^{m} a_{m,m-j}\frac{e^{i(m-j+1/2)s}}{\sin s/2}\right|$$

$$= O\left(\frac{1}{s}\right)\left|\sum_{j=0}^{m} a_{m,m-j} = e^{i(m-j)s}\right|\left|e^{is/2}\right| = O\left(\frac{1}{s}\right)\left|\sum_{j=0}^{m} a_{m,m-j}\, e^{i(m-j)s}\right| = O\left(\frac{A_{m,\sigma}}{s}\right),$$ by lemma 3.

Lemma 6. $K_n(t) = O\left(\frac{B_{n,\tau}}{t}\right)$, for $0 < n^{-1} < t \le \pi$.

Proof. It can be proved similar to lemma 5.

5. Proof of the Theorem: $(j,k)^{th}$ partial sums $s_{j,k}(x,y)$ of the series (1) at (u,v) = (x,y) is given by $$s_{j,k}(x,y) - f(x,y) = \frac{1}{4\pi^2}\int_0^{\pi}\int_0^{\pi}\phi(s,t)\,\frac{\sin(j+1/2)s}{\sin s/2}\,\frac{\sin(k+1/2)t}{\sin t/2}\,ds\,dt\,.$$

Then

$$\sum_{j=0}^{m}\sum_{k=0}^{n} a_{m,j}\, b_{n,k}\{s_{j,k}(x,y) - f(x,y)\}$$

$$= \frac{1}{4\pi^2}\int_0^{\pi}\int_0^{\pi}\phi(s,t)\sum_{j=1}^{m}\sum_{k=0}^{n} a_{m,j}\, b_{n,k}\,\frac{\sin(j+1/2)s\ \ \sin(k+1/2)t}{\sin s/2\ \ \sin t/2}\,ds\ \ dt$$

$$\text{or, } t_{m,n}(x,y) - f(x,y) = \int_0^{\pi}\int_0^{\pi}\phi(s,t)\,K_m(s)\,K_n(t)\,ds\,dt$$

$$= \left(\int_0^{\delta}\int_0^{\xi} + \int_0^{\delta}\int_{\xi}^{\pi} + \int_{\delta}^{\pi}\int_0^{\xi} + \int_{\delta}^{\pi}\int_{\xi}^{\pi}\right)\phi(s,t)\,K_m(s)\,K_n(t)\,ds\,dt = I_1 + I_2 + I_3 + I_4,\quad \text{say.} \quad (3)$$

$$I_1 = \left(\int_0^{1/m}\int_0^{1/n} + \int_{1/m}^{\delta}\int_0^{1/n} + \int_0^{1/m}\int_{1/n}^{\xi} + \int_{1/m}^{\delta}\int_{1/n}^{\xi}\right)\phi(s,t)\,K_m(s)\,K_n(t)\,ds\,dt$$

$$= I_{1.1} + I_{1.2} + I_{1.3} + I_{1.4},\quad \text{say, where, for } s \le \delta,\ \ t \le \xi,\ (2) \text{ holds.}$$

Using lemma 1 and 2, we have

$$I_{1.1} = O(mn)\int_0^{1/m}\int_0^{1/n}|\phi(s,t)|\,ds\,dt = O(mn)\,\Phi\left(\frac{1}{m},\frac{1}{n}\right) = O(mn)\ o\left(\frac{1}{m\,n\,\alpha(m)\,\beta(n)}\right) = o(1),$$

as $m \to \infty$ $n \to \infty$.

$$\text{Next, } |I_{1.2}| \le \int_{1/m}^{\delta}\int_0^{1/n}|\phi(s,t)|\ |K_m(s)|\,|K_n(t)|\,ds\,dt$$

$$\le \int_0^{1/n}|K_n(t)|\,dt\int_{1/m}^{\delta}|\phi(s,t)|\,\frac{A_{m,[\frac{1}{s}]}}{s}\,ds\quad \text{, by Lemma 5}$$

$$= \left\{\int_0^{1/n} O(n)\ dt\int_{1/m}^{\delta}|\phi(s,t)|\,\frac{A_{m,[\frac{1}{s}]}}{s}\,ds\right\},\ \text{by Lemma 2}$$

$$= O(n)\left(\int_0^{1/n}\frac{A_{m,[\frac{1}{\delta}]}}{\delta}\,\Phi_1(\delta,t)\right)dt \;+\; O(mn\,A_{m,m})\left(\int_0^{1/n}\Phi_1(\frac{1}{m},t)\,dt\right)$$

$$+\;O(n)\left(\int_0^{1/n}dt\int_{1/m}^{\delta}\frac{d}{ds}\left(\frac{A_{m,[\frac{1}{s}]}}{s}\right)\left|\Phi_1(s,t)\right|ds\right)$$

$$= I_{1.2.1} + I_{1.2.2} + I_{1.2.3}\,,\quad \text{say.}$$

Now,

$$I_{1.2.1}+I_{1.2.2} \;=\; O(n)\int_0^{1/n}\phi_1(\delta,t)\,dt \;+\; O(mn)\int_0^{1/n}\Phi_1(\tfrac{1}{m},t)\,dt$$

$$= O(n)\;\Phi(\delta,\tfrac{1}{n}) \;+\; O(mn)\;\Phi(\tfrac{1}{m},\tfrac{1}{n})$$

$$= O(n)\;o\left(\frac{\delta}{\alpha(\frac{1}{\delta})}\;\frac{\frac{1}{n}}{\beta(n)}\right)+\;O(mn)\;o\left(\frac{1/mn}{\alpha(m)\beta(n)}\right)$$

$$= o(1),\text{ as } m\to\infty,\;\; n\to\infty\,.$$

$$\text{Lastly, } I_{1.2.3} \;=\; O(n)\int_0^{1/n}dt\int_{1/m}^{\delta}\Phi_1(s,t)\;\frac{A_{m,[\frac{1}{s}]}}{s^2}\;ds+O(n)\int_0^{1/n}dt\int_{1/m}^{\delta}\frac{\Phi_1(s,t)}{s}\,\frac{d}{ds}\left(A_{m,[\frac{1}{s}]}\right)ds$$

$$\le O(n)\left(\int_{1/m}^{\delta}\left(\int_0^{1/n}\Phi_1(s,t)\,dt\right)\frac{A_{m,[\frac{1}{s}]}}{s^2}\,ds\right)+\;O(n)\int_{1/m}^{\delta}\left(\int_0^{1/n}\Phi_1(\;s,t)\,dt\right)\frac{1}{s}\;\frac{d}{ds}\;(A_{m,[\frac{1}{s}]})\,ds$$

$$=O(n)\int_{1/m}^{\delta}\Phi\;\;(s,\tfrac{1}{n})\;\frac{A_{m,[\frac{1}{s}]}}{s^2}\,ds+O(n)\int_{1/m}^{\delta}\Phi(s,\tfrac{1}{n})\;\frac{1}{s}\;\frac{d}{ds}\;(A_{m,[\frac{1}{s}]})\,ds$$

$$=\;O(n)\int_{1/m}^{\delta}o\left(\frac{s}{\alpha(\frac{1}{s})}\;\frac{1}{n\,\beta(n)}\right)\frac{A_{m,[\frac{1}{s}]}}{s^2}\,ds+O(n)\left(\int_{1/m}^{\delta}o\left(\frac{s}{\alpha(\frac{1}{s})}\frac{1}{n\,\beta(n)}\right)\;\frac{1}{s}\frac{d}{ds}(A_{m,[\frac{1}{s}]})\,ds\right)$$

$$=o\left(\frac{1}{\beta(n)}\right)\;\left(\int_{1/\delta}^{m}\frac{A_{m,s}}{s\alpha(s)}\;ds\right)+o\left(\frac{1}{\beta(n)}\right)\left(\int_{1/\delta}^{m}\frac{1}{\alpha(s)}\;\frac{d}{ds}(A_{m,s})\,ds\right)$$

$$=\;\;o\left(\frac{1}{\beta(n)}\right)O(1)\;+\;o\left(\frac{1}{\alpha(m)\,\beta(n)}\right)(A_{m,m})\quad=o(1)\,,\text{ as } m\to\infty, n\to\infty.$$

Thus, $I_{1.2} = o(1)$ as $m \to \infty$ and $n \to \infty$.

Similarly, $I_{1.3} = o(1)$ as $m \to \infty$ and $n \to \infty$.

$$|I_{1.4}| = O\left(\int_{1/m}^{\delta}\int_{1/n}^{\xi} |\phi(s,t)| \frac{A_{m,[\frac{1}{s}]}}{s}\frac{B_{n,[\frac{1}{t}]}}{t} dt\, ds\right)$$

$$= O\left(\frac{B_{n,[\frac{1}{\xi}]} A_{m,[\frac{1}{\delta}]} \Phi(\delta,\xi)}{\delta\,\xi} - \frac{m B_{n,[\frac{1}{\xi}]} A_{m,m} \Phi(\frac{1}{m},\xi)}{\xi}\right.$$

$$- \frac{B_{n,[\frac{1}{\xi}]}}{\xi}\int_{1/m}^{\delta}\phi(s,\xi)\frac{d}{ds}\left(\frac{A_{m,[\frac{1}{s}]}}{s}\right)ds - \frac{n\, B_{n,n}\, A_{m,[\frac{1}{\delta}]}\Phi(\delta,\frac{1}{n})}{\delta}$$

$$+ m\,n\, B_{n,n}\, A_{m,m}\,\Phi(\frac{1}{m},\frac{1}{n}) + n B_{nn}\int_{1/m}^{\delta}\Phi(s,\frac{1}{n})\frac{d}{ds}\left(\frac{A_{m,[\frac{1}{s}]}}{s}\right)ds$$

$$+ \frac{A_{m,[\frac{1}{\delta}]}}{\delta}\int_{1/m}^{\delta}\Phi_2(s,t)\frac{d}{dt}\left(\frac{B_{n,[\frac{1}{t}]}}{t}\right)dt - m A_{mm}\int_{1/n}^{\xi}\Phi_2(\frac{1}{m},t)\frac{d}{dt}\left(\frac{B_{n,[\frac{1}{t}]}}{t}\right)dt$$

$$\left. - \left(\int_{1/m}^{\delta}\int_{1/n}^{\xi}\Phi\ (s,t)\frac{d}{ds}\left(\frac{A_{m,[\frac{1}{s}]}}{s}\right)\frac{d}{dt}\left(\frac{B_{n,[\frac{1}{t}]}}{t}\right)dt\, ds\right)\right)$$

$= I_{1.4.1} + I_{1.4.2} + I_{1.4.3} + I_{1.4.4} + I_{1.4.5} + I_{1.4.6} + I_{1.4.7} + I_{1.4.8} + I_{1.4.9}$, say.

$$I_{1.4.1} + I_{1.4.2} = O\left(\frac{B_{n,[\frac{1}{\xi}]} A_{m,[\frac{1}{\delta}]}\ \Phi(\delta,\xi)}{\delta\ \ \xi} + \frac{m B_{n,[\frac{1}{\xi}]} A_{m,m}\Phi(\frac{1}{m},\xi)}{\xi}\right)$$

$$= o\left(\frac{B_{n,[\frac{1}{\xi}]}\ A_{m,[\frac{1}{\delta}]}}{\alpha(\frac{1}{\delta})\,\beta(\frac{1}{\xi})}\right) + o\left(m\ B_{n,[\frac{1}{\xi}]}\left(\frac{1}{m\,\alpha(m)}\cdot\frac{\xi}{\beta(\frac{1}{\xi})}\right)\right)$$

$$= o\left(\frac{B_{n,[\frac{1}{\xi}]}\ A_{m,[\frac{1}{\delta}]}}{\alpha(\frac{1}{\delta})\,\beta(\frac{1}{\xi})}\right) + o\left(B_{n,[\frac{1}{\xi}]}\frac{1}{\alpha(m)}\cdot\frac{\xi}{\beta(\frac{1}{\xi})}\right)$$

$$= o(1) + o(1)\left(\frac{1}{\alpha(m)}\right) = o(1), \text{ as } m \to \infty \text{ and } n \to \infty, \text{ by the regularity of } (T, S).$$

$$I_{1.4.3} = -\frac{B_{n,[\frac{1}{\xi}]}}{\xi}\int_{1/m}^{\delta} \Phi(s,\xi)\frac{d}{ds}\left(\frac{A_{m,[\frac{1}{s}]}}{s}\right)ds$$

$$= o\left(\frac{B_{n,[\frac{1}{\xi}]}}{\xi}\right)\int_{1/m}^{\delta}\left(\frac{s}{\alpha(\frac{1}{s})}\cdot\frac{\xi}{\beta(\frac{1}{\xi})}\right)\left(\frac{A_{m,[\frac{1}{s}]}}{s^2}\right)ds + o\left(\frac{B_{n,[\frac{1}{\xi}]}}{\xi}\right)\int_{1/m}^{\delta} o\left(\frac{s}{\alpha(\frac{1}{s})}\cdot\frac{\xi}{\beta(\frac{1}{\xi})}\right)\frac{1}{s}\frac{d}{ds}\left(A_{m,[\frac{1}{s}]}\right)ds$$

$$= o\left(\frac{B_{n,[\frac{1}{\xi}]}}{\beta(\frac{1}{\xi})}\right)\int_{1/m}^{\delta}\frac{A_{m,[\frac{1}{s}]}}{s\,\alpha(\frac{1}{s})}\,ds + o\left(\frac{B_{n,[\frac{1}{\xi}]}}{\beta(\frac{1}{\xi})}\right)\int_{1/m}^{\delta}\frac{1}{\alpha(\frac{1}{s})}\frac{d}{ds}\left(A_{m,[\frac{1}{s}]}\right)ds$$

$$= o\left(\frac{B_{n,[\frac{1}{\xi}]}}{\beta(\frac{1}{\xi})}\right)\int_{1/\delta}^{m}\frac{A_{m,s}}{s\alpha(s)}ds + o\left(\frac{B_{n,s/\xi}}{\beta(1/\xi)}\right)\int_{1/\delta}^{m}\frac{1}{\alpha(s)}\quad d(A_{m,s})$$

$$= o\left(\frac{B_{n,1/\xi}}{\beta(1/\xi)}\right)O(1) + o\left(\frac{B_{n,1/\xi}}{\beta(1/\xi)}\right)O(A_{m,m}) = o(1) \text{ as } m \to \infty,\ n \to \infty.$$

Thus we get, $I_{1.4.3} = o(1)$, as $m \to \infty$ and $n \to \infty$.

Similarly, $I_{1.4.4} = o(1)$, as $m \to \infty$ and $n \to \infty$.

Further, $I_{1.4.5} = mn\ B_{n,n}\ A_{m,m}\ \Phi(\frac{1}{m},\frac{1}{n}) = o\left(mn\ B_{n,n}\ A_{m,m}\ \frac{1}{mn\ \alpha(m)\beta(n)}\right)$

$= o(1)$, as $m \to \infty$ and $n \to \infty$.

Let us consider , $I_{1.4.6} = nB_{n,n}\int_{1/m}^{\delta}\Phi(s,\frac{1}{n})\frac{d}{ds}\left(\frac{A_{m,[\frac{1}{s}]}}{s}\right)ds$

$$= nB_{n,n}\int_{1/m}^{\delta}\Phi(s,\frac{1}{n})\frac{A_{m,[\frac{1}{s}]}}{s^2}ds + n\,B_{n,n}\int_{1/m}^{\delta}\Phi(s,\frac{1}{n})\frac{1}{s}\frac{d}{ds}\left(A_{m,[\frac{1}{s}]}\right)ds$$

$$= nB_{n,n}\int_{1/m}^{\delta}\Phi(s,\tfrac{1}{n})\frac{A_{m,[\frac{1}{s}]}}{s^2}ds + n\,B_{n,n}\int_{1/m}^{\delta}\Phi(s,\tfrac{1}{n})\tfrac{1}{s}\tfrac{d}{ds}\left(A_{m,[\frac{1}{s}]}\right)ds$$

$$= nB_{n,n}\int_{1/m}^{\delta} o\left(\frac{s}{\alpha(\frac{1}{s})}\frac{1}{n\beta(n)}\right)\frac{A_{m,[\frac{1}{s}]}}{s^2}ds + n\,B_{n,n}\int_{1/m}^{\delta} o\left(\frac{s}{\alpha(\frac{1}{s})}\frac{1}{n\beta(n)}\right)\tfrac{1}{s}\tfrac{d}{ds}\left(A_{m,[\frac{1}{s}]}\right)ds$$

$$= o\left(\frac{nB_{n,n}}{n\beta(n)}\right)\int_{1/m}^{\delta}\frac{A_{m,\left[\frac{1}{s}\right]}}{s\alpha\left(\frac{1}{s}\right)}ds + o\left(\tfrac{1}{\beta(n)}\right)\int_{1/m}^{\alpha}\tfrac{1}{\alpha(\frac{1}{s})}\frac{d}{ds}(A_{m,1/s})ds$$

$$= o(1)\ \text{as}\ m\to\infty,\ \ n\to\infty.$$

Similar to $I_{1.4.3}$ $\quad I_{1.4.7} = o(1)$, as $m\to\infty$ and $n\to\infty$.

Similar to $I_{1.4.6}$ $\quad I_{1.4.8} = o(1)$, as $m\to\infty$ and $n\to\infty$.

Lastly , $\quad I_{1.4.9} = O\left\{\left(\int_{1/m}^{\delta}\int_{1/n}^{\xi}\Phi\ (s,t)\frac{A_{m,[\frac{1}{s}]}}{s^2}+\frac{1}{s}\frac{d}{ds}\left(A_{m,[\frac{1}{s}]}\right)\right)\left(\frac{B_{n,[\frac{1}{t}]}}{t^2}+\tfrac{1}{t}\tfrac{d}{dt}B_{n,[\frac{1}{t}]}\right)dt\,ds\right\}$

$$= I_{1.4.9.1} + I_{1.4.9.2} + I_{1.4.9.3} + I_{1.4.9.4}$$

$$I_{1.4.9.1} + I_{1.4.9.2} = O\left(\int_{1/m}^{\delta}\int_{1/n}^{\xi}\Phi\ (s,t)\left(\frac{A_{m,[\frac{1}{s}]}}{s^2}\frac{B_{n,[\frac{1}{t}]}}{t^2}\right)dt\,ds\right) + O\left(\int_{1/m}^{\delta}\int_{1/n}^{\xi}\Phi\ (s,t)\frac{A_{m,[\frac{1}{s}]}}{s^2}\tfrac{1}{t}\tfrac{d}{dt}\left(B_{n,[\frac{1}{t}]}\right)dt\,ds\right)$$

$$= o\left(\int_{1/m}^{\delta}\int_{1/n}^{\xi}\left(\frac{s}{\alpha(\frac{1}{s})}\frac{t}{\beta(\frac{1}{t})}\right)\frac{A_{m,[\frac{1}{s}]}}{s^2}\frac{B_{n,[\frac{1}{t}]}}{t^2}dt\,ds\right) + O\left(\int_{1/m}^{\delta}\int_{1/n}^{\xi} o\left(\frac{s}{\alpha(\frac{1}{s})}\frac{t}{\beta(\frac{1}{t})}\right)\frac{A_{m,[\frac{1}{s}]}}{s^2}\tfrac{1}{t}\tfrac{d}{dt}\left(B_{n,[\frac{1}{t}]}\right)dt\,ds\right)$$

$$= o\left(\int_{1/m}^{\delta}\int_{1/n}^{\xi}\frac{A_{m,[\frac{1}{s}]}B_{n,[\frac{1}{t}]}}{\alpha(\frac{1}{s})s\ \beta(\frac{1}{t})\ t}dt\,ds\right) + o\left(\int_{1/m}^{\delta}\int_{1/n}^{\xi}\frac{A_{m,[\frac{1}{s}]}}{\alpha(\frac{1}{s})s\ \beta(\frac{1}{t})}\tfrac{d}{dt}\left(B_{n,[\frac{1}{t}]}\right)dt\,ds\right)$$

$$= o\left(\int_{1/\delta}^{m}\frac{A_{m,s}}{s\,\alpha(s)}ds\int_{1/\xi}^{n}\frac{B_{n,t}}{t\beta(t)}dt\right) + o\left(\int_{1/\delta}^{m}\frac{A_{m,s}}{s\alpha(s)}ds\int_{1/n}^{\xi}\frac{1}{\beta(\frac{1}{t})}\frac{d}{dt}\left(B_{n,[\frac{1}{t}]}\right)dt\right)$$

$$= o(1) + o(1)\int_{1/n}^{\xi}O(1)\tfrac{d}{dt}\left(B_{n,[\frac{1}{t}]}\right)dt$$

$$= o(1),\ \text{as}\ m\to\infty\ \text{and}\ n\to\infty.$$

Similarly, $I_{1.4.9.3} = o(1)$.

$$I_{1.4.9.4} = O\left(\int_{1/m}^{\delta}\int_{1/n}^{\xi} \Phi\ (s,t)\frac{1}{s}\frac{d}{ds}\left(A_{m,[\frac{1}{s}]}\right)\frac{1}{t}\frac{d}{dt}\left(B_{n,[\frac{1}{t}]}\right)dt\,ds\right)$$

$$= o\left(\int_{1/m}^{\delta}\int_{1/n}^{\xi}\left(\frac{s}{\alpha(\frac{1}{s})}\frac{t}{\beta(\frac{1}{t})}\right)\frac{1}{s}\frac{d}{ds}\left(A_{m,[\frac{1}{s}]}\right)\frac{1}{t}\frac{d}{dt}\left(B_{n,[\frac{1}{t}]}\right)dt\,ds\right)$$

$$= o\left(\int_{1/m}^{\delta}\int_{1/n}^{\xi}\left(\frac{1}{\alpha(\frac{1}{s})\beta(\frac{1}{t})}\right)\frac{d}{ds}\left(A_{m,[\frac{1}{s}]}\right)\frac{d}{dt}\left(B_{n,[\frac{1}{t}]}\right)dt\,ds\right)$$

$$= o\left(\int_{1/m}^{\delta}\frac{1}{\alpha(\frac{1}{s})}\frac{d}{ds}\left(A_{m,[\frac{1}{s}]}\right)ds\int_{1/n}^{\xi}\frac{1}{\beta(\frac{1}{t})}\frac{d}{dt}\left(B_{n,[\frac{1}{t}]}\right)dt\right)$$

$$= o\left(\int_{1/m}^{\delta}O(1)\frac{d}{ds}\left(A_{m,[\frac{1}{s}]}\right)ds\int_{1/n}^{\xi}O(1)\frac{d}{dt}\left(B_{n,[\frac{1}{t}]}\right)dt\right) = o(1),\ \text{ as } m\to\infty,\ n\to\infty.$$

Thus $I_{1.4} = o(1)$, as $m \to \infty$ and $n \to \infty$.

With the above estimations, we get $I_1 = o(1)$, as $m \to \infty$ and $n \to \infty$.

Now, $m^{-1} < \delta < \pi$, $n^{-1} < \xi < \pi$, thus we obtain

$$|I_3| \le \int_{\delta}^{\pi} |K_m(s)|\,ds \int_{0}^{1/n} |\phi(s,t)|\,|K_n(t)|\,dt + \int_{\delta}^{\pi} |K_m(s)|\,ds \int_{1/n}^{\xi} |\phi(s,t)|\,|K_n(t)|\,dt$$

$$= I_{3.1} + I_{3.2}.$$

Using lemma 2 and 5 we have

$$I_{3.1} = O(n)\left(\int_{\delta}^{\pi}\frac{A_{m,\sigma}}{s}\,ds\int_{0}^{1/n}|\phi(s,t)|\,dt\right)$$

$$= O(n)\int_{0}^{\pi}\Phi_2\left(s,\tfrac{1}{n}\right)ds = o\left(\frac{1}{\beta(n)}\right) = o(1),\ \text{as } n\to\infty.$$

Next. $I_{3.2} = O\left[\int_{\delta}^{\pi}\frac{A_{m,\sigma}}{s}\,ds\int_{1/n}^{\xi}|\phi(s,t)|\,\frac{B_{n,[\tau]}}{t}\,dt\right]$ by lemmas (5) and (6)

$$= O\left[\int_{\delta}^{\pi} ds \left\{ \left[\Phi_2(s,t) \frac{B_{n,[\tau]}}{t} \right]_{1/n}^{\xi} - \int_{1/n}^{\xi} \Phi_2(s,t) \frac{d}{dt}\left(\frac{B_{n,[\tau]}}{t} \right) dt \right\}\right]$$

$$= O\left[\int_{\delta}^{\pi} ds \left\{ \Phi_2(s,\xi) \frac{B_{n,[\frac{1}{\xi}]}}{\xi} - \Phi_2(s,\tfrac{1}{n})\, n\, B_{n,\,n} \right\}\right] + \left[\int_{\delta}^{\pi} ds \int_{1/n}^{\xi} \Phi_2(s,t) \frac{d}{dt}\left(\frac{B_{n,[\tau]}}{t} \right) dt \right]$$

$$= O\left[\int_{\delta}^{\pi} \Phi_2(s,\xi) \frac{B_{n,[\frac{1}{\xi}]}}{\xi} ds \right] + O(n)\int_{0}^{\pi} \Phi_2(s,\tfrac{1}{n})\, ds + O\left[\int_{\delta}^{\pi} ds \int_{1/n}^{\xi} \Phi_2(s,t) \frac{d}{dt}\left(\frac{B_{n,[\tau]}}{t} \right) dt \right]$$

$$= O\left(\frac{B_{n,[\frac{1}{\xi}]}}{\xi} \right) \int_{\delta}^{\pi} \Phi_2(s,\xi)\, ds + O(n)\, o\left(\frac{1}{n\beta(n)} \right) + O\left[\int_{\delta}^{\pi} ds \int_{1/n}^{\xi} \Phi_2(s,t) \frac{d}{dt}\left(\frac{B_{n,[\tau]}}{t} \right) dt \right]$$

$= o(1)+o(1)+o(1)$, similar to $I_{1.4.9}$

$= o(1)$, as $m \to \infty$ and $n \to \infty$.

Hence $I_3 = o(1)$, as $m \to \infty$ and $n \to \infty$.

Similarly , we get $I_2 = o(1)$, as $m \to \infty$ and $n \to \infty$.

By the regularity conditions of matrix summability and the Riemann-Lebesgue theorem, we have

$I_4 = o(1)$, as $m \to \infty$ and $n \to \infty$.

Therefore by the above estimations, our theorem is completely established.

6. Particular Cases.

1. The theorem of Chow ([1]) becomes a particular case of our theorem

if $a_{m,j} = \dfrac{1}{m+1} \;\; \forall\, j$, $b_{n,k} = \dfrac{1}{n+1} \;\; \forall\, k$, $\alpha(u) = \log u$ and $\beta(v) = \log v$.

2. If $a_{m,j} = \dfrac{1}{(m-j+1)\log m}$, $b_{n,k} = \dfrac{1}{(n-k+1)\log n}$, $\alpha(u) = \log u$ and $\beta(v) = \log v$

   then the theorem of Sharma ([5]) becomes a particular case of our theorem.

3. The theorem of Mishra ([4]) becomes a particular case of our theorem if

$a_{m,j} = \dfrac{p^{(1)}_{m-j}}{P^{(1)}_{m}}$, $b_{n,k} = \dfrac{p^{(2)}_{n-k}}{P^{(2)}_{n}}$, $\alpha(u) = \psi^{(1)}(u)$ and $\beta(v) = \psi^{(2)}(v)$.

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