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# A Generalized Definition of Joint Hyponormality

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**Abstract**

In this paper, we generalise the definition of joint hyponormality [1]. For $T = (T_1, \ldots, T_m)$ and $S = (S_1, \ldots, S_n)$ with $T_j, S_i \in B(\mathcal{H})$, we define the *joint commutator* of $T$ and $S$, denoted by $[T, S]$, as the $n \times m$ operator matrix $([T_j, S_i])$. Next we propose the definitions of strong and weak hyponormality of $((T_1, T_2), T_3)$. Using the results and properties established from these definitions we find conditions under which the strong and weak hyponormality of $(T_1, \ldots, T_n)$ become equivalent.

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## 1 Introduction

Here $\mathcal{H}$ denotes a separable complex Hilbert space and $B(\mathcal{H})$ denotes the space of all bounded linear operators on $\mathcal{H}$.
We recall here a few definitions which will be frequently refered to in the paper.

**Definition 1.1.** [3] A *commutator* is an operator of the form $PQ - QP$, where $P$ and $Q$ are operators on a Hilbert space. It is usually denoted by $[P, Q]$. Moreover, $[A^*, A] = A^*A - AA^*$ is refered to as the self commutator of $A$.

**Definition 1.2.** [1] For $T_1, \ldots, T_m \in B(\mathcal{H})$, the tuple $T = (T_1, \ldots, T_m)$ is *hyponormal*, or the operators $T_1, \ldots, T_m$ are *jointly hyponormal* if $\mathcal{C}(T) = ([T_j^*, T_i]) \geq 0$ on $\mathcal{H}^{(m)}$. Moreover, if $\{S_\gamma\}_{\gamma \in \Gamma}$ is a family of operators in $B(\mathcal{H})$, then $\{S_\gamma\}_{\gamma \in \Gamma}$ is said to be jointly hyponormal if $(S_{\gamma_1}, \ldots, S_{\gamma_m})$ is hyponormal for every finite index set $\{\gamma_1, \ldots, \gamma_m\} \subset \Gamma$.

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**Definition 1.3.** [4] For any positive integer $k$, $T \in B(\mathcal{H})$ is k-hypononormal if $(T, T^2, \ldots, T^k)$ is hypononormal .

**Definition 1.4.** [4] For $T_1, \ldots, T_n \in B(\mathcal{H})$, the n-tuple $T = (T_1, \ldots, T_n)$ is *weakly hyponormal* if $\{\sum_{i=1}^{n} \lambda_i T_i : \lambda_1, \ldots, \lambda_n \in \mathbb{C}\}$ consists entirely of hyponormal operators.

**Definition 1.5.** [4] For any positive integer $k$, $T \in B(\mathcal{H})$ is *weakly k-hypononormal* if $(T, T^2, \ldots, T^k)$ is weakly hypononormal.

**Definition 1.6.** [7] $T \in B(\mathcal{H})$ is *polynomially hypononormal* if it is weakly k-hypononormal for all $k \geq 1$.

# 2 Motivation

In [4] the concept of joint hyponormality has been used extensively to study the relative position of the class of subnormals inside the class of hyponormals. It has been shown in proposition1 [1] and also in corollary 2.3 [4] that $T = (T_1, \ldots, T_m)$ hyponormal implies $T$ is weakly hyponormal. However, the converse need not be true as shown by several examples [5], [6]. Hence the question arises that "under what conditions will the converse also hold ?" In this context we have the following result:

**Proposition 2.1.** *[1] : If $T_1$ is hyponormal and $T_2$ is normal, then $(T_1, T_2)$ is hyponormal if and only if $(T_1, T_2)$ is weakly hypononormal.*

Our aim in this paper is to extend this result to any n-tuple of operators $T = (T_1, \ldots, T_n)$. This gives us the motivation to work in the following direction :
(a) Propose the definition of joint commutator of $T = (T_1, \ldots, T_m)$ and $S = (S_1, \ldots, S_n)$.
(b) Using (a), give suitable definitions for strong and weak hyponormality of $((T_1, T_2), T_3)$.

# 3 Definition and properties of joint commutator

**Definition 3.1.** For $T_1, \ldots, T_n$ in $B(\mathcal{H})$ and $T = (T_1, \ldots, T_n)$, we define the adjoint of $T$ as $T^* = (T_1^*, \ldots, T_n^*)$.

**Definition 3.2.** For $T = (T_1, \ldots, T_m)$ and $S = (S_1, \ldots, S_n)$ with $T_j, S_i \in B(\mathcal{H})$, we define the joint commutator of $T$ and $S$, denoted by $[T, S]$, as the $n \times m$ operator matrix $([T_j, S_i])$. For $n = m$, we say $[T, S] \geq 0$ if $([T_j, S_i]) \geq 0$.

**Remark 3.1.** Though the joint commutator of $T$ and $S$ is denoted by $[T, S]$, it is not defined in the sense $TS - ST$, as will be seen from remark(3.2). However, we still use the notation $[T, S]$ as several basic properties of ordinary commutators continue to hold for joint commutators as will be seen from properties P1 to P5.

**Definition 3.3.** For $T = (T_1, \ldots, T_m)$, $P = (P_1, \ldots, P_r)$, $S = (S_1, \ldots, S_n)$,$Q = (Q_1, \ldots, Q_w)$ we define the joint commutator of $(T, P)$ and $(Q, S)$ as

$$[(T, P),(Q, S)] := \begin{pmatrix} [T, Q]_{w\times m} & [P, Q]_{w\times r} \\ [T, S]_{n\times m} & [P, S]_{n\times r} \end{pmatrix}_{(w+n)\times(m+r)}$$

**Definition 3.4.** Let $T = (T_1, \ldots, T_m)$ where $T_1, \ldots, T_m$ are in $B(\mathcal{H})$. Also for $A_{ij}$, $B_{\lambda\mu}$ in $B(\mathcal{H})$, let $A = (A_{ij})$ and $B = (B_{\lambda\mu})$ be two operator matrices of size $m \times n$ and $p \times m$ respectively. We define $BT := ((BT)_1, \ldots, (BT)_p)$ and $TA := ((TA)_1, \ldots, (TA)_n)$, where

$$(BT)_\lambda := \sum_{\mu=1}^{m} B_{\lambda\mu}T_\mu \;\; for\ 1 \le \lambda \le p \tag{1}$$

$$(TA)_j := \sum_{i=1}^{m} T_iA_{ij} \;\; for\ 1 \le j \le n \tag{2}$$

**Remark 3.2.** For $T = (T_1, \ldots, T_m)$ and $S = (S_1, \ldots, S_m)$ we get $TS = \sum_{i=1}^{m} T_iS_i$ and $ST = \sum_{i=1}^{m} S_iT_i$ (by (1) above). Thus $TS - ST = \sum_{i=1}^{m}[T_i, S_i]$. But taking $n = m$ in definition(3.2) we get $[T, S] = ([T_j, S_i])_{m\times m}$. Hence $[T, S]$ cannot be interpreted as $TS - ST$ as already mentioned in remark(3.1)

**Properties :** For $T = (T_1, \ldots, T_m)$ and $S = (S_1, \ldots, S_n)$, we list below a few properties of the commutator of $T$ and $S$.
P1. $[T, S] = -[S, T]^t$ where $t$ denotes transpose of a matrix.
P2. $[T, S] = [S^*, T^*]^*$.
P3. $[T, T] = 0$ if and only if $[T_j, T_i] = 0$, $\forall\ i, j$.
P4. $\alpha[T, S] = [\alpha T, S] = [T, \alpha S]$ for any scalar $\alpha$.
P5. $[T + T', S + S'] = [T, S] + [T, S'] + [T', S] + [T', S']$
where $T'$ and $S'$ are m tuples and n tuples respectively of operators in $B(\mathcal{H})$.
P6. $[T, S] \ge 0$ if and only if $\sum_{i,j=1}^{m} \left\langle S_ix_j, T_j^*x_i\right\rangle - \sum_{i,j=1}^{m} \left\langle T_jx_j, S_i^*x_i\right\rangle \ge 0$ for all $x_1, \ldots, x_m \in \mathcal{H}$, provided $m = n$.
P7. $[T, T] \ge 0$ if and only if $\sum_{i,j=1}^{m} \left\langle [T_j, T_i]x_j, x_i\right\rangle_{\mathcal{H}} \ge 0$ for all $x_1, \ldots, x_m \in \mathcal{H}$.
Also for $A_{ij}$, $B_{\lambda\mu}$ in $B(\mathcal{H})$, if $A = (A_{ij})$ and $B = (B_{\lambda\mu})$ are operator matrices of size $m \times p$ and $q \times n$ respectively then we have the following :
P8. $(TA)^* = A^*T^*$ and $(BS)^* = S^*B^*$.
P9. $[TA, BS] = B[T, S]A$ provided $[B_{\lambda\mu}, A_{ij}] = [A_{ij}, S_\mu] = [B_{\lambda\mu}, T_i] = 0$

for all $1 \leq i \leq m$, $1 \leq j \leq p$, $1 \leq \lambda \leq q$, $1 \leq \mu \leq n$.
P10. $[(BS)^*, BS] = B[S^*, S]B^*$ provided $[B_{\lambda\mu}, B^*_{ij}] = [B^*_{\lambda\mu}, S_\mu] = 0 \ \forall \ i, j, \lambda, \mu$.
Proof of P1:
$[T, S] = ([T_j, S_i])_{n\times m} = (-[S_i, T_j])_{n\times m} = -([S_j, T_i])^t_{m\times n} = -[S, T]^t$
Proof of P2 :

$$\begin{aligned}
[T, S] &= ([T_j, S_i])_{n\times m} \\
&= ([T_i, S_j]^*)^*_{m\times n}, \quad \textit{since} \ ([T_j, S_i])^*_{n\times m} = ([T_i, S_j]^*)_{m\times n} \\
&= ([S^*_j, T^*_i])^*_{m\times n}, \quad \textit{since} \ [T_i, S_j]^* = [S^*_j, T^*_i] \\
&= [S^*, T^*]^*, \qquad \textit{since} \ [S^*, T^*] = ([S^*_j, T^*_i])_{m\times n}
\end{aligned}$$

Proof of P3 :
From definition(3.2), $[T, T] = ([T_j, T_i])$ and so $T$ commutes with itself if and only if $T_i$ and $T_j$ commute for all $i$ and $j$.

Proof of P4 :
$\alpha[T, S] = \alpha([T_j, S_i]) = (\alpha[T_j, S_i]) = ([\alpha T_j, S_i]) = ([(\alpha T)_j, S_i]) = [\alpha T, S]$
Similarly, $\alpha[T, S] = [T, \alpha S]$

Proof of P5 :
We have $T = (T_1, \ldots, T_m)$, $T' = (T'_1, \ldots, T'_m)$, $S = (S_1, \ldots, S_n)$, $S' = (S'_1, \ldots, S'_n)$
Thus, $T + T' = (T_1 + T'_1, \ldots, T_m + T'_m)$, $S + S' = (S_1 + S'_1, \ldots, S_n + S'_n)$ and

$$\begin{aligned}
[T + T', S + S'] &= ([(T + T')_j, (S + S')_i]) \\
&= ([T_j + T'_j, S_i + S'_i]) \\
&= ([T_j, S_i] + [T_j, S'_i] + [T'_j, S_i] + [T'_j, S'_i]) \\
&= ([T_j, S_i]) + ([T_j, S'_i]) + ([T'_j, S_i]) + ([T'_j, S'_i]) \\
&= [T, S] + [T, S'] + [T', S] + [T', S']
\end{aligned}$$

Proof of P6 :
Let $x_1, \ldots, x_m$ be in $\mathcal{H}$ and $x$ be the transpose of the row vector $(x_1, \ldots, x_m)$.
For $m = n$,

$$\begin{aligned}
[T, S] \geq 0 &\Leftrightarrow ([T_j, S_i]) \geq 0 \\
&\Leftrightarrow \big\langle ([T_j, S_i])x, x \big\rangle_{\mathcal{H}^{(m)}} \geq 0 \\
&\Leftrightarrow \Sigma_i \big\langle \Sigma_j [T_j, S_i] x_j, x_i \big\rangle_{\mathcal{H}} \geq 0 \\
&\Leftrightarrow \Sigma_{ij} \big\langle (T_j S_i - S_i T_j) x_j, x_i \big\rangle \geq 0 \qquad (3) \\
&\Leftrightarrow \Sigma^m_{i,j=1} \big\langle S_i x_j, T^*_j x_i \big\rangle - \Sigma^m_{i,j=1} \big\langle T_j x_j, S^*_i x_i \big\rangle \geq 0
\end{aligned}$$

Proof of P7 :
$[T, T] \geq 0$ if and only if $\Sigma_{ij} \big\langle [T_j, T_i] x_j, x_i \big\rangle \geq 0$, for all $x_1, \ldots, x_m \in \mathcal{H}$ (by (3)

of proof P6).

Proof of P8 :
Here $T = (T_1, \ldots, T_m)$, $A = (A_{ij})_{m\times p}$ and so $TA := ((TA)_1, \ldots, (TA)_p)$ where $(TA)_j := \sum_{i=1}^{m} T_i A_{ij}$ Therefore, $(TA)_j^* = (\Sigma_i T_i A_{ij})^* = \Sigma_i A_{ij}^* T_i^* = \Sigma_i (A^*)_{ji} T_i^* = (A^*T^*)_j$
Hence, $(TA)^* = A^*T^*$ and similarly $(BS)^* = S^*B^*$

Proof of P9 :
$[TA, BS] = ([(TA)_j, (BS)_\lambda])_{q\times p}$
$= ([(TA)_j, (BS)_\lambda])_{q\times p}$
$= ([\Sigma_{i=1}^{m} T_i A_{ij}, \Sigma_{\mu=1}^{n} B_{\lambda\mu} S_\mu])$
$= (\Sigma_{i=1}^{m} \Sigma_{\mu=1}^{n} [T_i A_{ij}, B_{\lambda\mu} S_\mu])$
$=(\Sigma_{i=1}^{m} \Sigma_{\mu=1}^{n} B_{\lambda\mu} (T_i S_\mu - S_\mu T_i) A_{ij}),$ *as* $[A_{ij}, B_{\lambda\mu}] = [T_i, B_{\lambda\mu}] = [A_{ij}, S_\mu] = 0$
$= (\Sigma_{i=1}^{m} \Sigma_{\mu=1}^{n} B_{\lambda\mu} [T_i, S_\mu] A_{ij})$ (4)
Again, $B = (B_{\lambda\mu})_{q\times n}$ and $[T, S] = ([T_i, S_\mu])_{n\times m}$
This gives $B[T, S] = (\Sigma_{\mu=1}^{n} B_{\lambda\mu} [T_i, S_\mu])_{q\times m}$
Also, $A = (A_{ij})_{m\times p}$ and so,
$B[T, S]A = (\Sigma_{i=1}^{m} \Sigma_{\mu=1}^{n} B_{\lambda\mu} [T_i, S_\mu] A_{ij})_{q\times p} = [TA, BS]$ *(by* (4) *above)*

Proof of P10 :
$[(BS)^*, BS] = [S^*B^*, BS] = B[S^*, S]B^*$, (by property P8 and P9)

# 4 Strong and weak hyponormality of $((T_1, T_2), T_3)$

**Definition 4.1.** For $T_1, T_2, T_3$ in $B(\mathcal{H})$, $((T_1, T_2), T_3)$ is said to be (strongly) hyponormal if $[((T_1, T_2), T_3)^*, ((T_1, T_2), T_3)] \geq 0$ where the joint commutator is defined as in definition (3.3), and $((T_1, T_2), T_3)^*$ is defined as $((T_1, T_2)^*, T_3^*)$ in keeping with definition (3.1)

**Proposition 4.1.** *If* $T_1, T_2, T_3$ *are in* $B(\mathcal{H})$*, then* $(T_1, T_2, T_3)$ *is hyponormal if and only if* $((T_1, T_2), T_3)$ *is hyponormal.*

*Proof.* $(T_1, T_2, T_3)$ is hyponormal
$\Leftrightarrow ([T_j^*, T_i]) \geq 0$ *on* $\mathcal{H}^{(3)}$

$$\Leftrightarrow \left(\begin{array}{cc|c} [T_1^*, T_1] & [T_2^*, T_1] & [T_3^*, T_1] \\ [T_1^*, T_2] & [T_2^*, T_2] & [T_3^*, T_3] \\ \hline [T_1^*, T_3] & [T_2^*, T_3] & [T_3^*, T_3] \end{array}\right) \geq 0 \ on\, \mathcal{H}^{(3)}$$

$$\Leftrightarrow \begin{pmatrix} [(T_1^*, T_2^*), (T_1, T_2)] & [T_3^*, (T_1, T_2)] \\ [(T_1^*, T_2^*), T_3] & [T_3^*, T_3] \end{pmatrix} \geq 0 \ on\, \mathcal{H}^{(2)} \oplus \mathcal{H}^{(1)}$$

$$\Leftrightarrow \begin{pmatrix} [(T_1, T_2)^*, (T_1, T_2)] & [T_3^*, (T_1, T_2)] \\ [(T_1, T_2)^*, T_3] & [T_3^*, T_3] \end{pmatrix} \geq 0$$

$\Leftrightarrow [((T_1, T_2)^*, T_3^*), ((T_1, T_2), T_3)] \geq 0$ (*by definition* (3.3))
$\Leftrightarrow [((T_1, T_2), T_3)^*, ((T_1, T_2), T_3)] \geq 0)$
$\Leftrightarrow ((T_1, T_2), T_3)$ is hyponormal □

**Corollary 4.1.** *For any $p < m$, $(T_1, \ldots, T_m)$ is hyponormal if and only if $((T_1, \ldots, T_p), (T_{p+1}, \ldots, T_m))$ is hyponormal.*

**Corollary 4.2.** *By remark 2(b) [1], hyponormality of $(T_1, \ldots, T_m)$ is not affected by permuting the operators $T_i$ and so in view of Proposition(4.1) $((T_1, T_2), T_3)$ is hyponormal iff $((T_1, T_3), T_2)$ is hyponormal or iff $(T_1, (T_2, T_3))$ is hyponormal and so on.*

**Definition 4.2.** : $((T_1, T_2), T_3)$ is weakly hyponormal if $(T_1, T_2)$ is hyponormal and $(T_1, T_2, T_3)$ is weakly hyponormal. Similarly, $(T_1, (T_2, T_3))$ is weakly hyponormal if $(T_2, T_3)$ is hyponormal and $(T_1, T_2, T_3)$ is weakly hyponormal.

**Proposition 4.2.** *$((T_1, T_2), T_3)$ is hyponormal implies $((T_1, T_2), T_3)$ weakly hyponormal.*

*Proof.* $((T_1, T_2), T_3)$ is hyponormal iff the operator matrix
$\begin{pmatrix} [(T_1, T_2)^*, (T_1, T_2)] & [T_3^*, (T_1, T_2)] \\ [(T_1, T_2)^*, T_3] & [T_3^*, T_3] \end{pmatrix}$ is positive, and this implies that $[(T_1, T_2)^*, (T_1, T_2)] \geq 0$. That is, $(T_1, T_2)$ is hyponormal. Also $((T_1, T_2), T_3)$ hyponormal implies $(T_1, T_2, T_3)$ hyponormal (by Proposition(4.1)) and this in turn implies that $(T_1, T_2, T_3)$ is weakly hyponormal (by Proposition 1 [1]).
Thus, if $((T_1, T_2), T_3)$ is hyponormal then $(T_1, T_2)$ is hyponormal and $(T_1, T_2, T_3)$ is weakly hyponormal. That is $((T_1, T_2), T_3)$ is weakly hyponormal.
□

**Remark 4.1.** Proposition(4.1) cannot be extended to the case of weak hyponormality because weak hyponormality of $(T_1, T_2, T_3)$ does not necessarily imply weakly hyponormality of $((T_1, T_2), T_3)$ as shown by the following example.

**Example 4.1.** *Let $\alpha : \sqrt{\frac{141}{250}}, \sqrt{\frac{2}{3}}, \sqrt{\frac{3}{4}}, \sqrt{\frac{4}{5}}, \ldots$ be a weighted sequence with the Bergman tail and let $W_\alpha$ be the weighted shift operator with weight sequence $\alpha$. Then it is shown in Corollary 3.5 [2] that $W_\alpha$ is cubically hyponormal but not 2-hyponormal.*
*Let $T_1 = W_\alpha$, $T_2 = W_\alpha^2$, $T_3 = W_\alpha^3$. Then $W_\alpha$ cubically hyponormal implies $(T_1, T_2, T_3)$ is weakly hyponormal. Again, $W_\alpha$ not 2-hyponormal implies that $(T_1, T_2)$ is not hyponormal. Hence $((T_1, T_2), T_3)$ cannot be weakly hyponormal.*

**Remark 4.2.** Weak hyponormality of $((T_1, T_2), T_3)$ does not necessarily imply weak hyponormality of $(T_1, (T_2, T_3))$ as shown by the following example.

**Example 4.2.** *For $W_\alpha$ given in Example(4.1), we have $W_\alpha$ is quadratically hyponormal but not 2-hyponormal. Let $T_1 = I$ , $T_2 = W_\alpha$ , $T_3 = W_\alpha^2$. Then,* $[(T_1,T_2)^*,(T_1,T_2)] = \begin{pmatrix} [T_1^*,T_1] & [T_2^*,T_1] \\ [T_1^*,T_2] & [T_2^*,T_2] \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & [T_2^*,T_2] \end{pmatrix} \geq 0$ *and so $(T_1,T_2)$ is hyponormal. Also $W_\alpha$ quadratically hyponormal implies $(T_1,T_2,T_3)$ is weakly hyponormal. Hence, $((T_1,T_2),T_3)$ is weakly hyponormal. But since $W_\alpha$ is not 2-hyponormal so $(T_2,T_3)$ is not hyponormal and hence $(T_1,(T_2,T_3))$ is not weakly hyponormal.*

# 5 Main Results

**Lemma 5.1.** *If $(T_1,T_2)$ is weakly hyponormal and $T_2$ is normal then $[T_2^*,T_1] = 0$.*

*Proof.* Since $(T_1,T_2)$ is weakly hyponormal and $T_2$ is normal, so by Proposition (2.1) $(T_1,T_2)$ is hyponormal.
Therefore, $[(T_1,T_2)^*,(T_1,T_2)] = \begin{pmatrix} [T_1^*,T_1] & [T_2^*,T_1] \\ [T_1^*,T_2] & [T_2^*,T_2] \end{pmatrix} \geq 0$. Here $[T_2^*,T_2] = 0$ as $T_2$ is normal and so, by Lemma 2 [1], $[T_1^*,T_2] = [T_2^*,T_1] = 0$. □

**Proposition 5.1.** *If $(T_1,T_2)$ is hyponormal, $T_3$ is normal. Then $((T_1,T_2),T_3)$ is hyponomal if and only if $((T_1,T_2),T_3)$ is weakly hyponormal.*

*Proof.* In view of Proposition(4.2) we only need to show that if $((T_1,T_2),T_3)$ is weakly hyponomal then $((T_1,T_2),T_3)$ is hyponormal under the given condition. Now,

$$\begin{aligned} ((T_1,T_2),T_3)\, \textit{weakly hyponormal} &\Rightarrow (T_1,T_2,T_3)\, \textit{weakly hyponomal} \\ &\Rightarrow (T_1,T_3)\, \textit{weakly hyponomal} \\ &\Rightarrow (T_1,T_3)\, \textit{hyponomal}\ (\textit{by Proposition}(2.1)) \\ &\Rightarrow [T_3^*,T_1] = 0\ (\textit{by Lemma}(5.1)) \end{aligned}$$

Similarly, $[T_3^*,T_2] = 0$.
Thus, $[T_3^*,(T_1,T_2)] = \begin{pmatrix} [T_3^*,T_1] \\ [T_3^*,T_2] \end{pmatrix} = 0$
Hence, $[((T_1,T_2),T_3)^*,((T_1,T_2),T_3)]$

$$= \begin{pmatrix} [(T_1,T_2)^*,(T_1,T_2)] & [T_3^*,(T_1,T_2)] \\ [(T_1,T_2)^*,T_3] & [T_3^*,T_3] \end{pmatrix}$$

$$= \begin{pmatrix} [(T_1,T_2)^*,(T_1,T_2)] & 0 \\ 0 & 0 \end{pmatrix} \geq 0\ (\textit{since}\,(T_1,T_2)\,\text{is hyponormal})$$

Therefore, $((T_1,T_2),T_3)$ is hypopnormal. □

Applying Propositions (4.1) and (5.1) we can now extend proposition (2.1) as follows:

**Proposition 5.2.** *If $T_1$ is hyponormal and $T_2, \ldots, T_n$ are normal operators in $B(\mathcal{H})$, then $(T_1, \ldots, T_n)$ is hyponormal if and only if it is weakly hyponormal.*

*Proof.* As (strong) hyponormality always implies weak hyponormality, we only need to show that the converse holds under the given conditions.
As $(T_1, \ldots, T_n)$ is weakly hyponormal so $(T_1, T_2)$ is also weakly hyponormal which implies $(T_1, T_2)$ is hyponormal (by Proposition(4.1). Also $(T_1, T_2, T_3)$ is weakly hyponormal. Therefore by Definition (4.2) $((T_1, T_2), T_3)$ is weakly hyponormal and as $T_3$ is normal so $((T_1, T_2), T_3)$ is hyponormal (by Proposition (5.1)). This gives that $(T_1, T_2, T_3)$ is hyponormal (by Proposition (4.1)).
Repeatedly applying a similar argument we conclude that $(T_1, \ldots, T_n)$ is hyponormal. □

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