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q-Derivative of Basic Hypergeometric Series with Respect to Parameters

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Abstract

This article discusses the effect of the Difference operator D_q on the generalized hypergeometric series ${}_r\phi_s(a_1,...a_r;b_1,...,b_s;q,z)$ with respect to parameters $a_1,...a_r;b_1,...,b_s$ and gives some q-difference equations satisfied by ${}_r\phi_s$, u-exponential function and q-Appell's hypergeometric series. Moreover, I will prove that the basic hypergeometric functions ${}_r\Phi_s$ are basically completely monotonic with respect to parameters $a_i, \quad i=1,2,..,r$ if the parametr a_i is less than or equal to unity and the functions ${}_r\Phi_s$ have positive q-derivative of all orders. Finally, the basic hypergeometric functions ${}_r\Phi_s$ are totally basically completely monotonic if all parametrs are less than or equal to unity and the functions ${}_r\Phi_s$ have positive q-derivative of all orders.

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1 Preliminaries

In his fundamental paper [12] Jackson, introduced the q-difference operator

$$D_qf(x) = \frac{f(x) - f(qx)}{x - qx}, \quad q \neq 1 \tag{1.1}$$

The formulas for the q-difference D_q of a product and a quotient of functions are

$$D_q(f(x)g(x)) = f(qx)D_qg(x) + g(x)D_qf(x) \tag{1.2}$$

$$D_q(\frac{f(x)}{g(x)}) = \frac{g(x)D_qf(x) - f(x)D_qg(x)}{g(qx)g(x)}, \quad g(qx)g(x) \neq 0 \tag{1.3}$$

also, the general Leibniz rule for action of powers of the q-derivative operator on a product of functions is

$$D_q^{(n)}(fg)(x) = \sum_{k=0}^{n} \binom{n}{k}_q D_q^{(k)} f(xq^{n-k}) D_q^{(n-k)} g(x) \tag{1.4}$$

Here we use the q-binomial coefficients defined by

$$\binom{n}{k}_q = \frac{[n]_q!}{[k]_q![n-k]_q!} \tag{1.5}$$

for $k = 0, 1, 2, ..., n$, where

$$[n]_q = \sum_{k=1}^{n} q^{k-1}, \quad [0]_q = 0,$$

and

$$[n]_q! = \prod_{k=1}^{n} [k]_q, \quad [0]_q! = 1.$$

are the q-analogue of the natural numbers and the factorial function. The q-binomial coefficient $\binom{n}{k}_q$ are polynomials in q with integer coefficients. If $q = 1$, then $[n]_q = n$. If $q \neq 1$, then $[n]_q = \frac{1-q^n}{1-q}$. For more properties of the difference operator D_q see [6, 9, 14]. The generalized hypergeometric series is defined by [9]

$${}_r\Phi_s(a_1, ..., a_r, b_1, ..., b_s; q, z) = \sum_{n=0}^{\infty} \frac{(a_1; q)_n ... (a_r; q)_n}{(b_1; q)_n ... (b_s; q)_n} [(-1)^n q^{\binom{n}{2}}]^{1+s-r} \frac{z^n}{(q; q)_n} \tag{1.6}$$

where the q-shifted factorial $(a; q)_n$ is defined by

$$(a; q)_n = \prod_{m=0}^{n-1} (1 - aq^m)$$

The series ${}_r\Phi_s$ terminates if one of the numerator parameters is of the form q^{-m}, m=0,1,...and $q \neq 0$. When $0 < |q| < 1$, the series ${}_r\Phi_s$ converges absolutely for all z if $r \leqslant s$, and for $|z| < 1$ if $r = s + 1$. If $|q| > 1$ and $|z| < \frac{|b_1...b_s|}{|a_1...a_r|}$, then also ${}_r\Phi_s$ converges absolutely. The series ${}_r\Phi_s$ diverges for $z \neq 0$ when $0 < |q| < 1$and $r > s + 1$, and when $|q| > 1$ and $|z| > \frac{|b_1...b_s|}{|a_1...a_r|}$, unless it terminates. A function f is completely monotonic if for all n, $(-1)^n f^{(n)}(x) \geq 0$ on $(0, \infty)$; see Widder[18], Ismail et al[11], Feller [8], Choquet[5] and Berg[4] for properties of completely monotonic functions. Bernstien's theorem asserts that f is completely monotonic if and only if

$f(x) = \int_{\mathbb{R}} e^{-xt} d\mu(t)$ where μ is a positive measure supported on a subset of $[0, \infty)$. The work described in this article is intended to add to the known collection of q-difference formulas for the generalized hypergeometric functions${}_r\Phi_s(a_1, ..., a_r, b_1, ..., b_s; q, z)$, the q-analogues of Appell's series $\Phi^{(1)}, \Phi^{(2)}, \Phi^{(3)}$ and$\Phi^{(4)}$ and also for some functions related to the so called u-exponential function. This paper organized as follows: In section two we discuss the q-derivative of basic hypergeometric series with respect to parameters and prove some parametric q-difference equations satisfied by ${}_r\Phi_s$. In section three we introduce classes of basically completely monotonic functions (CM_q) on $\mathbb{R}$ and totally basically completely monotonic functions (TM_q) on $\mathbb{R}^m$ then we give some results satisfied by by the elements of these classes. In the fourth and fifth sections we verify some q-parametric difference equations satisfied by u-exponential functions[10] and the q-analogue of Appell's hypergeometric series in two variables[14].

2 Initial Computations

In this section, we shall prove some useful formulas related to q-parametric difference equations, by studying the effect of the operators $D_{a_i,q}, \quad i = 1, ..., r$ and $D_{b_i,q}, \quad i = 1, ..., s$ on the basic hypergeometric series ${}_r\phi_s$, where

$$D_{a_i,q}{}_r\Phi_s(a_1, ..., a_r, b_1, ..., b_s; q, z) = \frac{{}_r\Phi_s(a_1, ..., a_i, ..., a_r, b_1, ..., b_s; q, z) - {}_r\Phi_s(a_1, ..., qa_i, ..., a_r, b_1, ..., b_s; q, z)}{a_i - qa_i}$$

and

$$D_{b_i,q}{}_r\Phi_s(a_1, ..., a_r, b_1, ..., b_s; q, z) = \frac{{}_r\Phi_s(a_1, ..., a_r, b_1, ..., b_i, ..., b_s; q, z) - {}_r\Phi_s(a_1, ..., a_r, b_1, ..., qb_i, ..., b_s; q, z)}{b_i - qb_i}$$

For the aim of avoiding confusion we replace the notation of the q-difference operator D_q by $D_{z,q}$ to indicate that the operator D_q acts on the variable z.

Theorem 2.1.
The basic hypergeometric series ${}_r\Phi_s$ satisfies the following difference equations

$$D^{(n)}_{a_i,q}{}_r\Phi_s(a_1, ..., a_r, b_1, ..., b_s; q, z) = \frac{(-1)^n q^{\binom{n}{2}} z^n}{(a_i; q)_n} D^{(n)}_{z,q}{}_r\Phi_s(a_1, ..., a_r, b_1, ..., b_s; q, z), \quad n \geqslant 2 \tag{2.1}$$

and

$$D^{(n)}_{b_i,q}{}_r\Phi_s(a_1, ..., a_r, b_1, ..., b_s; q, z) = \frac{z}{(b_i; q)_n} D^{(n)}_{z,q}\{z^{n-1}{}_r\Phi_s(a_1, ..., a_r, b_1, ..., b_s; q, z)\}, \quad n \geqslant 2 \tag{2.2}$$

Proof
From the definition of the q-difference operator $D_{a_1,q}$and the properties of the q-shift factorial $(a_1; q)_n$ we have

$$D_{a_1,q}{}_r\Phi_s(a_1, ..., a_r, b_1, ..., b_s; q, z) = \sum_{n=0}^{\infty} \frac{(a_1; q)_n - (qa_1; q)_n}{a_1 - a_1 q} \times \frac{(a_2; q)_n ... (a_r; q)_n}{(b_1; q)_n ... (b_s; q)_n} [(-1)^n q^{\binom{n}{2}}]^{1+s-r} \frac{z^n}{(q; q)_n}$$

$$= \sum_{n=0}^{\infty} \frac{\prod_{m=0}^{n-1}(1-a_1q^m)[1-\frac{(1-a_1q^n)}{(1-a_1)}]}{a_1(1-q)} \times \frac{(a_2;q)_n...(a_r;q)_n}{(b_1;q)_n...(b_s;q)_n}[(-1)^n q^{\binom{n}{2}}]^{1+s-r}\frac{z^n}{(q;q)_n}$$

$$= -\frac{1}{1-a_1}\sum_{n=0}^{\infty}[n]_q\prod_{m=0}^{n-1}(1-a_1q^m) \times \frac{(a_2;q)_n...(a_r;q)_n}{(b_1;q)_n...(b_s;q)_n}[(-1)^n q^{\binom{n}{2}}]^{1+s-r}\frac{z^n}{(q;q)_n}$$

Hence

$$D_{a_1,q\,r}\Phi_s(a_1,...,a_r,b_1,...,b_s;q,z) = -\frac{z}{1-a_1}D_{z,q\,r}\Phi_s(a_1,...,a_r,b_1,...,b_s;q,z) \tag{2.3}$$

Suppose that

$$D^{(m-1)}_{a_1,q}{}_r\Phi_s(a_1,...,a_r,b_1,...,b_s;q,z) = \frac{(-1)^{m-1}q^{\binom{m-1}{2}}z^{m-1}}{(a_1;q)_{m-1}}D^{(m-1)}_{z,q}{}_r\Phi_s(a_1,...,a_r,b_1,...,b_s;q,z)$$

hence,

$$D^{(m)}_{a_1,q\,r}\Phi_s(a_1,...,a_r,b_1,...,b_s;q,z) = D_{a_1,q}(D^{(m-1)}_{a_1,q}{}_r\Phi_s(a_1,...,a_r,b_1,...,b_s;q,z))$$

$$= (-1)^{m-1}q^{\binom{m-1}{2}}z^{m-1} \times \{\frac{\frac{D^{(m-1)}_{z,q}{}_r\Phi_s(a_1,...,a_r,b_1,...,b_s;q,z)}{(a_1;q)_{m-1}} - \frac{D^{(m-1)}_{z,q}{}_r\Phi_s(qa_1,...,a_r,b_1,...,b_s;q,z)}{(qa_1;q)_{m-1}}}{a_1(1-q)}\}$$

$$= \frac{(-1)^{m-1}q^{\binom{m-1}{2}}}{a_1(1-q)}\{\sum_{n=0}^{\infty}[n]_q...[n-m+2]_q(\frac{\prod_{k=0}^{n-1}(1-a_1q^k)}{\prod_{k=0}^{m-2}(1-a_1q^k)} - \frac{\prod_{k=0}^{n-1}(1-a_1q^{k+1})}{\prod_{k=0}^{m-2}(1-a_1q^{k+1})})\}$$

$$\times \frac{(a_2;q)_n...(a_r;q)_n}{(b_1;q)_n...(b_s;q)_n}[(-1)^n q^{\binom{n}{2}}]^{1+s-r}\frac{z^n}{(q;q)_n}$$

$$= \frac{(-1)^{m-1}q^{\binom{m-1}{2}}}{a_1(1-q)(a_1;q)_m} \times \{\sum_{n=0}^{\infty}[n]_q...[n-m+2]_q\prod_{k=0}^{n-1}(1-a_1q^k)\{-a_1q^{m-1}(1-qn-m+1\}$$

$$\times \frac{(a_2;q)_n...(a_r;q)_n}{(b_1;q)_n...(b_s;q)_n}[(-1)^n q^{\binom{n}{2}}]^{1+s-r}\frac{z^n}{(q;q)_n}$$

$$= \frac{(-1)^{m-1}q^{\binom{m-1}{2}+m-1}}{a_1(1-q)(a_1;q)_m}\{\sum_{n=0}^{\infty}[n]_q...[n-m+1]_q\prod_{k=0}^{n-1}(1-a_1q^k)\frac{(a_2;q)_n...(a_r;q)_n}{(b_1;q)_n...(b_s;q)_n}[(-1)^n q^{\binom{n}{2}}]^{1+s-r}\frac{z^n}{(q;q)_n}$$

So,

$$D^{(m)}_{a_1,q\,r}\Phi_s(a_1,...,a_r,b_1,...,b_s;q,z) = \frac{(-1)^m q^{\binom{m}{2}}z^m}{(a_1;q)_m}D^{(m)}_{z,q}{}_r\Phi_s(a_1,...,a_r,b_1,...,b_s;q,z) \tag{2.4}$$

By mathematical induction we get (2.1) for $i = 1$. Similarly, by the same technique we can obtain the first required for all the parameters a_i, $\quad i = 1,...,r$. Also, for the parameter b_1 we have

$$D_{b_1,q\,r}\Phi_s(a_1,...,a_r,b_1,...,b_s;q,z) = \sum_{n=0}^{\infty}\frac{\frac{1}{(b_1;q)_n} - \frac{1}{(qb_1;q)_n}}{b_1 - b_1q} \times$$

$$\frac{(a_1;q)_n...(a_r;q)_n}{(b_2;q)_n...(b_s;q)_n}[(-1)^n q^{\binom{n}{2}}]^{1+s-r}\frac{z^n}{(q;q)_n}$$

$$=\sum_{n=0}^{\infty}\frac{[n]_q}{(1-b_1q^n)}\frac{(a_1;q)_n...(a_r;q)_n}{(b_1;q)_n...(b_s;q)_n}[(-1)^n q^{\binom{n}{2}}]^{1+s-r}\frac{z^n}{(q;q)_n}$$

$$=\frac{1}{1-b_1}\sum_{n=0}^{\infty}[n]_q\frac{(a_1;q)_n...(a_r;q)_n}{(b_1q;q)_n...(b_s;q)_n}[(-1)^n q^{\binom{n}{2}}]^{1+s-r}\frac{z^n}{(q;q)_n}$$

So

$$D_{b_1,q\,r}\Phi_s(a_1,...,a_r,b_1,...,b_s;q,z)=\frac{z}{1-b_1}D_{z,q\,r}\Phi_s(a_1,...,a_r,b_1q,b_2,...,b_s;q,z) \tag{2.5}$$

Suppose that

$$D^{(m-1)}_{b_1,q}\,{}_r\Phi_s(a_1,...,a_r,b_1,...,b_s;q,z)=\frac{z}{(b_1;q)_{m-1}}D^{(m-1)}_{z,q}\{z^{m-2}\,{}_r\Phi_s(a_1,...,a_r,q^{m-1}b_1,...,b_s;q,z)\}$$

This implies that

$$D^{(m)}_{b_1,q}\,{}_r\Phi_s(a_1,...,a_r,b_1,...,b_s;q,z)=D_{b_1,q}\{D^{(m-1)}_{b_1,q}\,{}_r\Phi_s(a_1,...,a_r,b_1,...,b_s;q,z)\}$$

$$=\frac{z}{b_1(1-q)}\{\frac{D^{(m-1)}_{z,q}\{z^{m-2}\,{}_r\Phi_s(a_1,...,a_r,q^{m-1}b_1,...,b_s;q,z)\}}{(b_1;q)_{m-1}}$$

$$-\frac{D^{(m-1)}_{z,q}\{z^{m-2}\,{}_r\Phi_s(a_1,...,a_r,q^{m}b_1,...,b_s;q,z)\}}{(qb_1;q)_{m-1}}\}$$

$$=\frac{z}{(b_1;q)_m}D^{(m-1)}_{z,q}\{\frac{\frac{1-b_1q^{m-1}}{\prod_{k=0}^{n-1}(1-b_1q^{k+m-1})}-\frac{1-b_1}{\prod_{k=0}^{n-1}1-b_1q^{k+m}}}{b_1(1-q)}\}\times$$

$$\frac{(a_1;q)_n...(a_r;q)_n}{(b_2;q)_n...(b_s;q)_n}[(-1)^n q^{\binom{n}{2}}]^{1+s-r}\frac{z^n}{(q;q)_n}$$

hence

$$D^{(m)}_{b_1,q}\,{}_r\Phi_s(a_1,...,a_r,b_1,...,b_s;q,z)=\frac{z}{(b_1;q)_n}D^{(m)}_{z,q}\{z^{m-1}\,{}_r\Phi_s(a_1,...,a_r,q^{n}b_1,...,b_s;q,z)\}$$

By mathematical induction we have

$$D^{(n)}_{b_1,q}\,{}_r\Phi_s(a_1,...,a_r,b_1,...,b_s;q,z)=\frac{z}{(b_1;q)_n}D^{(n)}_{z,q}\{z^{n-1}\,{}_r\Phi_s(a_1,...,a_r,q^{n}b_1,...,b_s;q,z)\}$$

using the same technique for all parameters b_i, $\quad i=1,2,..,s$ we get (2.2) which completes the proof of the Theorem. As a consequence of the above Theorem we have the following Corollaries:

Corollary 2.2.
The Heine's hypergeometric series

$$ {}_2\Phi_1(a,b;c;q,z)=\sum_{n=0}^{\infty}\frac{(a;q)_n(b;q)_n}{(c;q)_n(q;q)_n}z^n \tag{2.6} $$

satisfies the following system of q-parametric difference equations

$$ \{D^{(n)}_{a,q}-\frac{(-1)^nq^{\binom{n}{2}}z^n}{(a;q)_n}D^{(n)}_{z,q}\}{}_2\Phi_1(a,b;c;q,z)=0,\quad n\geqslant 2 \tag{2.7} $$

$$ \{D^{(n)}_{b,q}-\frac{(-1)^nq^{\binom{n}{2}}z^n}{(b;q)_n}D^{(n)}_{z,q}\}{}_2\Phi_1(a,b;c;q,z)=0,\quad n\geqslant 2 \tag{2.8} $$

$$ \{D^{(n)}_{c,q}-\frac{z^n}{(c;q)_n}D^{(n)}_{z,q}z^{n-1}\}{}_2\Phi_1(a,b;c;q,z)=0,\quad n\geqslant 2 \tag{2.9} $$

Corollary 2.3.
The basic hypergeometric function ${}_r\Phi_s(a_1,...,a_r,b_1,...,b_s;q,z)$ satisfies the difference equations

$$ \{(a_i;q)_nD^{(n)}_{a_i;q}-(a_j;q)_nD^{(n)}_{a_j;q}\}{}_r\Phi_s(a_1,...,a_r,b_1,...,b_s;q,z)=0,\quad i,j=1,2,...,r \tag{2.10} $$

and

$$ \{(b_i;q)_nD^{(n)}_{b_i;q}-(b_j;q)_nD^{(n)}_{b_j;q}\}{}_r\Phi_s(a_1,...,a_r,b_1,...,b_s;q,z)=0,\quad i,j=1,2,...,s \tag{2.11} $$

3 Basically Completely Monotonic Functions (CM_q)

Jackson [13] introduced the q-integral defined by

$$ \int_0^x f(t)d_qt:=\sum_{n=0}^{\infty}f(xq^n)(xq^n-xq^{n+1}) \tag{3.1} $$

and defined an integral on $(0,\infty)$by

$$ \int_0^\infty f(t)d_qt:=(1-q)\sum_{n=-\infty}^{\infty}q^nf(q^n) \tag{3.2} $$

Notice that

$$ lim_{N\to\infty}\int_0^{q^{-N}}f(x)d_qx=\int_0^\infty f(x)dx $$

The idea here is that on $(1,\infty)$ the division points are at $q^{-1},q^{-2},q^{-3},...$when$0<q<1$.

Definition 3.1. A function $f :]0, \infty[\to \mathbb{R}$ will be called basically completely monotonic if it satisfies the following axioms :

(1) f has q-derivative of all orders.

(2) $(-1)^n D_q^{(n)} f(x) \geqslant 0$ for all $x > 0$, $n = 0, 1, 2, \cdots$
We denote the class of all basically completely monotonic functions on $]0, \infty[$ by CM_q

Theorem 3.2
The sum, the product, and the pointwise limit of basically completely monotonic functions are also basically completely monotonic and every function $f \in CM_q$ has an integral representation of the form

$$f(s) = \int_0^\infty E_q^{-sx} \mu(x) d_q x := L_q(\mu) \tag{3.3}$$

for some measure μ on the real line.

Proof
Let $\{f_m\} \subseteq CM_q$, $f_m \to f$ as $m \to \infty$. Since the q-derivative and the limit commute, then we have

$$\begin{aligned} 0 &\leqslant \lim_{m\to\infty} (-1)^n D_q^{(n)} f_m(x) \\ &= (-1)^n D_q^{(n)} \lim_{m\to\infty} f_m(x) \\ &= (-1)^n D_q^{(n)} f(x), \end{aligned}$$

so, $f \in CM_q$. Let $f, g \in CM_q$, the linearity of D_q implies $f + g \in CM_q$. Also, since

$$D_q(f(x)g(x)) = f(qx) D_q g(x) + (D_q f(x)) g(x),$$

and $f(x), g(x)$ are nonnegative for all $x > 0$, so at $n = 1$ we have

$$(-1) D_q(f(x)g(x)) \geqslant 0 \quad \text{for all} \quad x > 0.$$

Suppose

$$(-1)^{(n-1)} D_q^{(n-1)}(f(x)g(x)) \geqslant 0 \quad \text{for all} \quad x > 0,$$

this implies

$$(-1)^{(n)} D_q^{(n)}(f(x)g(x)) = (-1)^{(n-1)} D_q^{(n-1)}[(-1) D_q(f(x)g(x))] \geqslant 0 \quad \text{for all} \quad x > 0,$$

By using mathematical induction we get $fg \in CM_q$. Since $E_q^{x+y} \neq E_q^x E_q^y$ in general [15], so we must use the definition of q-addition (compare, [2] and [17])

$$(x \oplus_q y)^n = \sum_{k=0}^{n} \binom{n}{k}_q x^k y^{n-k}, n = 0, 1, 2, \cdots, y \neq x$$

and so,

$$E_q^{x \oplus_q y} = E_q^x E_q^y.$$

Defining a semicharacter $\rho_a :]0,\infty[\to \mathbb{R}$ by $\rho_a(s) = E_q^{-as}$, it is clear that $0 \leqslant \rho_a(s) \leqslant 1$ i.e., ρ_a is decreasing then the integral

$$\int_0^\infty E_q^{-sx} w(x) d_q x$$

exists for some weight function $w(x)$ defined on $\mathbb{R}$. Applying corollary 4.5. Page 114 [4], we get

$$f(s) = \int_0^\infty E_q^{-sx} w(x) d_q x$$

which completes the proof of the Theorem.

Example

The function E_q^{-x} belongs to the class CM_q, where

$$E_q^x = \sum_{j=0}^{\infty} q^{j(j-1)/2} \frac{x^j}{[j]!} \tag{3.4}$$

From now on we denote by S_q the set of all functions $f :]0,\infty[\to \mathbb{R}$ which can be expressed in the form

$$f(y) = a + \int_0^\infty \frac{1}{y \oplus_q x} \mu(x) d_q x$$

for a unique constant $a \geqslant 0$.

Theorem 3.3

The set S_q is a convex cone contained in CM_q.

Proof

Since,

$$\begin{aligned} \int_0^\infty \frac{1}{y \oplus_q x} w(x) d_q x &= \int_0^\infty \int_1^\infty E_q^{-(y\oplus_q x)\xi} d_q\xi w(x) d_q x \\ &= \int_1^\infty \int_0^\infty E_q^{-y\xi} E_q^{-x\xi} w(x) d_q x d_q \xi \\ &= \int_1^\infty E_q^{-y\xi} L_q w(\xi) d_q \xi \end{aligned}$$

For $n \geqslant 1$ and $y > 0$ we find,

$$\frac{(-1)^n}{[n]_q!} D_q^{(n)} f(y) = \int_0^\infty \frac{1}{(y \oplus_q x)^{(n+1)}} w(x) d_q x$$

by applying Cauchy-Schwartz inequality we get that the sequence

$$\Phi_n(y) = \frac{(-1)^n}{[n]_q!} D_q^{(n)} f(y) \tag{3.5}$$

satisfying the relation

$$(-1)^{(k)} D_q^{(k)} [\log \Phi(y)] \geqslant 0, \quad \text{for} \quad k = 1, 2, \cdots$$

and noting that Φ satisfies (3.5) if and only if $-D_q(\log \Phi)$ belongs to CM_q, we get the desired.

Definition 3.4. A linear homomorphic function f on $\mathbb{R}^d$ will be called totally basically completely monotonic if it satisfies:

$(1)\partial_{x_i}^{(m,q)} f(x) = \partial_{x_i}^{(q)} \partial_{x_i}^{(m-1,q)} f(x)$ exist for all $m \in \mathbb{N}_0$

$(2)(-1)^m \partial_{x_i}^{(m,q)} f(x) \geqslant 0$ for all $m \in \mathbb{N}_0$

where, the q-difference operators $\partial_{x_i}^{(q)}, \quad i = 1, 2, ..., d$ are given by

$$\partial_{x_i}^{(q)} f(x) = \partial_{x_i}^{(q)} f(x_1, \cdots, x_d) \equiv \frac{f(x_1, \cdots, x_{i-1}, qx_i, x_{i+1}, \cdots, x_d) - f(x)}{(q-1)x}$$

and

$$\partial_{x_i}^{(1,q)} = \partial_{x_i}^{(q)} \quad and \quad \partial_{x_i}^{(0,q)} = 1.$$

Remark. It is clear that if $d = 1$, then

$$D_q f(x) = \partial_x^{(q)} f(x)$$

Lemma 3.5

Let f, g be two linear homogenous functions on $\mathbb{R}^d$ then we have

$$\partial_{x_i}^{(m,q)} (f(x)g(x)) = f(\widetilde{x_i}) \partial_{x_i}^{(m,q)} g(x) + g(x) \partial_{x_i}^{(m,q)} f(x)$$

where $x = (x_1, \cdots, x_d) \in \mathbb{R}^d$ and $m = (m_1, \cdots, m_d) \in \mathbb{N}_0^d$ and

$$\widetilde{x_i} = (x_1, \cdots, x_{i-1}, q^{n_i} x_i^{n_i}, x_{i+1}, \cdots, x_d)$$

We leave the proof for the readers

Theorem 3.6

The sum, the product, and the point wise limit of the elements of the set TM_q of all totally basically completely monotonic functions are also totally basically completely monotonic functions and every function $f \in TM_q$ has an integral representation of the form

$$f(s) = \underbrace{\int_0^\infty ... \int_0^\infty}_{m-times} E_q^{-s.x} \mu(x) d_q x \tag{3.6}$$

for some Borel measure μon $\mathbb{R}^m$where $s = (s_1, \cdots, s_m) \in \mathbb{R}^m$, $x = (x_1, \cdots, x_m)$ and $s.x = s_1 x_1 \oplus_q \cdots \oplus_q s_m x_m$

Proof

It is clear that the operators $\partial_{x_i}^{(q)}$ for all $i \in \mathbb{N}_0$ is linear, so $f + g \in TM_q$ for all $f, g \in TM_q$ Since the operator $\partial_{x_i}^{(q)}$ and the limit commute then we directly have that TM_q is a closed set over the Euclidean space $\mathbb{R}^m$. Let $f, g \in TM_q$, by virtue of the last Lemma and from the non negativity of $f(x), g(x)$; $x \in \mathbb{R}^m_+$ we get:

$$(-1)\partial^{(q)}_{x_i}(f(x)g(x)) \geqslant 0 \quad \forall x \in \mathbb{R}^m_+$$

Letting

$$(-1)^{n-1}\partial^{(n-1,q)}_{x_i}(f(x)g(x)) \geqslant 0 \quad \forall x \in \mathbb{R}^m_+$$

then

$$(-1)^{n}\partial^{(n,q)}_{x_i}(f(x)g(x)) = (-1)^{n-1}\partial^{(n-1,q)}_{x_i}[(-1)\partial^{(q)}_{x_i}f(x)g(x)] \geqslant 0$$

applying mathematical induction we get $fg \in CM_q$. Combining Theorem 3.2 with proposition 4.7 page 115 [4] we get

$$f(s) = \underbrace{\int_0^\infty ... \int_0^\infty}_{m-times} E_q^{-s_1.x_1} E_q^{-s_2.x_2} ... E_q^{-s_m.x_m} \mu_1(x_1) d_q x_1 \mu_2(x_2) d_q x_2 ... \mu_m(x_m) d_q x_m$$

for some measures $\mu_1, \mu_2, ..., \mu_m$ on $\mathbb{R}$. Putting $\mu(x) = \mu_1(x_1)\mu_2(x_2)...\mu_m(x_m)$ implies (3.6) which completes the proof of the theorem.

4 u-Exponential Series

Recalling from [10], that, for an arbitrary analytic function $u(\alpha)$ an arbitrary solution $\Gamma_u(\alpha)$ of the functional equation

$$\Gamma_u(\alpha + 1) = u(\alpha)\Gamma_u(\alpha) \tag{4.1}$$

is called a u-gamma function. Obviously, the function $\Gamma_u(\alpha)$, if it exists at all, is defined up to a factor $c(\alpha)$that is a 1-periodic function. Furthermore, any analytic function $F(\alpha)$ is a u-gamma function, where $u(\alpha) = \frac{F(\alpha+1)}{F(\alpha)}$. The classical examples are $u(\alpha) = \alpha$ and $u(\alpha) = \frac{q^\alpha - 1}{q-1}$. In the first case, the solution of (4.1) is given by the Euler gamma function $\Gamma(\alpha)$, and the second case, by the corresponding q-analogue $\Gamma_q(\alpha)$. For $|q| < 1$ the function Γ_q can be represent by the infinite product[3]

$$\Gamma_q(\alpha) := (1-q)^{1-\alpha} \prod_{n=0}^{\infty} \frac{1-q^{n+1}}{1-q^{n+\alpha}}. \tag{4.2}$$

Suppose that $\Gamma_u(\alpha)$ be a u-gamma function, that is, a solution of the functional equation (4.1). We assume that

$$\Gamma_u(\alpha) \neq 0 \quad for \quad \alpha \in \mathbb{Z};$$

$$there \quad exist \quad \alpha \in \mathbb{Z}: \quad \Gamma_u(\alpha) \neq \infty \tag{4.3}$$

Definition 4.1. The u-exponential series $exp_u x$ is the following formal series:

$$exp_u x = \sum_{n=-\infty}^{+\infty} \frac{x^n}{\Gamma_u(n+1).} \tag{4.4}$$

Condition (4.3) implies that all the coefficients of this series are finite and at least one of them is non-zero. Since Γ_u is defined up to a 1-periodic factor, (4.4) determines the series $exp_u x$ up to a costant factor. For example, if $u(\alpha) = \alpha$, then $exp_u x = ce^x$, where c is an arbitrary constant.

Now suppose that $u(\alpha)$ is a rational function of q^α, where $0 < q < 1$. We represent $u(\alpha)$ in the form

$$u(\alpha) = c\frac{(1-\mu_1 q^\alpha)...(1-\mu_s q^\alpha)}{(1-\lambda_1 q^\alpha)...(1-\lambda_r q^\alpha)}$$

For simplicity, let $c = 1$. Then

$$\Gamma_u(\alpha) = \frac{\Gamma_v(q^\alpha \mu_1)...\Gamma_v(q^\alpha \mu_s)}{\Gamma_v(q^\alpha \lambda_1)...\Gamma_v(q^\alpha \lambda_r)},$$

where$v(\alpha) = 1 - \alpha$ and hence

$$\Gamma_v(\alpha) = \prod_{k=0}^{\infty} (1 - \alpha q^k)^{-1} \tag{4.5}$$

Thus

$$exp_u x = \sum_{n\in\mathbb{Z}} \frac{\Gamma_v(q^{n+1}\lambda_1)...\Gamma_v(q^{n+1}\lambda_r)}{\Gamma_v(q^{n+1}\mu_1)...\Gamma_v(q^{n+1}\mu_s)} x^n, \tag{4.6}$$

where $\Gamma_v(\alpha)$ is given by (4.5). Here, as before, the λ_i and μ_j are parameters.

All the terms in (4.6) are finite provided that $q^{k+\lambda_i} \neq 1$ for $k \in \mathbb{Z}$ and $i = 1, ..., r$. If $q^{k+\lambda_i} = 1$ for at least one pair $k \in \mathbb{Z}$, $i = 1, ..., s$, then the series (4.6) terminates as $n \longrightarrow -\infty$. Since in this case its coefficients are bounded, it follows that, in contrast with the classical Pochhammer series, the series (4.6) converges in a neighbourhood of $x = 0$ for an arbitrary r and s. In particular, for $\mu_s = 1$ we obtain the following series, convergent in a neighbourhood of $x = 0$:

$$exp_u x = \sum_{n=0}^{\infty} \frac{\Gamma_v(q^{n+1}\lambda_1)...\Gamma_v(q^{n+1}\lambda_r)}{\Gamma_v(q^{n+1}\mu_1)...\Gamma_v(q^{n+1}\mu_{s-1})} \frac{x^n}{\Gamma_v(n+1)}.$$

Considering the two series $G_{\lambda_i,u}x := \frac{1}{\Gamma_\nu(\lambda_i)} exp_u$ and $H_{\mu_i,u}x := \Gamma_\nu(\mu_i) exp_u$, and by virtue of our calculation in the second section we can easily prove the following Theorem:

Theorem 4.2.

$$D^{(n)}_{\lambda_i,q}G_{\lambda_i,u}x \quad = D^{(n)}_{\lambda_i,q}\{\tfrac{1}{\Gamma_\nu(\lambda_i)}exp_u x\} \quad = \frac{(-1)^n q^{\binom{n}{2}} x^n}{(\lambda_i,q)_n} D^{(n)}_{x,q}\{G_{\lambda_i,u}x\} \tag{4.7}$$

$$D^{(n)}_{\mu_i,q}H_{\mu_i,u}x \quad = D^{(n)}_{\mu_i,q}\{\Gamma_\nu(\mu_i)exp_u x\} \quad = \frac{x}{(\mu_i,q)_n} D^{(n)}_{x,q}\{x^n H_{\mu_i,u}x\} \tag{4.8}$$

5 q-Appell's Hypergeometric series

In the 1880's the French mathematician Appell[1], introduced four families of hypergeometric functions of two variables x and y commonly denoted by F_1, F_2, F_3 and F_4[7,16]. Recalling from [14] that the q-analogues of the four Appell's hypergeometric functions are given by:

$$\Phi^{(1)}(\alpha,\beta_1,\beta_2;\gamma;x,y;q) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\alpha;q)_{m+n}(\beta_1;q)_m(\beta_2;q)_n}{m!n!(\gamma;q)_{m+n}} x^m y^n \tag{5.1}$$

$$\Phi^{(2)}(\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2;x,y;q) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\alpha;q)_{m+n}(\beta_1;q)_m(\beta_2;q)_n}{m!n!(\gamma_1;q)_m(\gamma_2;q)_n} x^m y^n \tag{5.2}$$

$$\Phi^{(3)}(\alpha_1,\alpha_2,\beta_1,\beta_2;\gamma;x,y;q) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\alpha_1;q)_m(\alpha_2;q)_n(\beta_1;q)_m(\beta_2;q)_n}{m!n!(\gamma;q)_{m+n}} x^m y^n \tag{5.3}$$

$$\Phi^{(4)}(\alpha,\beta;\gamma_1,\gamma_2;x,y;q) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\alpha;q)_{m+n}(\beta;q)_{m+n}}{m!n!(\gamma_1;q)_m(\gamma_2;q)_n} x^m y^n \tag{5.4}$$

For the aim of simplicity we denote $\Phi^{(1)}(A\alpha, B\beta_1, C\beta_2; D\gamma; Ex, Fy; q)$ by $\Phi^{(1)}(A,B,C;D;E,F)$. By virtue of our calculation in section 2, and by helping of the identity $[n+m]_q = \frac{1}{2}\{(1+q^n)[m]_q + (1+q^m)[n]_q\}$ we directly get the following main results for $n \geqslant 2$:

Theorem 5.1.

$$D^{(n)}_{\alpha,q}\Phi^{(1)}(1,1,1;1;1,1) = \frac{(-1)^n q^{\binom{n}{2}}}{2(\alpha,q)_n}\{x^n D^{(n)}_x[\Phi^{(1)}(1,1,1;1;1,1) + \Phi^{(1)}(1,1,1;1;1,q^n)]$$

$$+y^n D^{(n)}_y[\Phi^{(1)}(1,1,1;1;1,1) + \Phi^{(1)}(1,1,1;1;q^n,1)]$$

$$+[n]_{q^{-1}} x^{n-1} y D^{(n-1)}_x D_y[\Phi^{(1)}(1,1,1;1;q,1) + \Phi^{(1)}(1,1,1;1;1,q^{n-1})]$$

$$+[n]_{q^{-1}}xy^{n-1}D_xD_y^{(n-1)}[\Phi^{(1)}(1,1,1;1;1,q)+\Phi^{(1)}(1,1,1;1;q^{n-1},1)]+...\} \tag{5.5}$$

$$D_{\beta_1,q}^{(n)}\Phi^{(1)}(1,1,1;1;1,1)=\frac{(-1)^nq^{\binom{n}{2}}x^n}{(\beta_1,q)_n}D_{x,q}^{(n)}\Phi^{(1)}(1,1,1;1;1,1) \tag{5.6}$$

$$D_{\beta_2,q}^{(n)}\Phi^{(1)}(1,1,1;1;1,1)=\frac{(-1)^nq^{\binom{n}{2}}y^n}{(\beta_2,q)_n}D_{y,q}^{(n)}\Phi^{(1)}(1,1,1;1;1,1) \tag{5.7}$$

$$D_{\gamma,q}^{(n)}\Phi^{(1)}(1,1,1;1;1,1)=\frac{1}{2(\gamma,q)_n}\{xD_x^{(n)}x^{n-1}[\Phi^{(1)}(1,1,1;q^n;1,1)+\Phi^{(1)}(1,1,1;q^n;1,q^n)]$$

$$+yD_y^{(n)}y^{n-1}[\Phi^{(1)}(1,1,1;q^n;1,1)+\Phi^{(1)}(1,1,1;1;q^n,1)]$$

$$+[n]_qxyD_x^{(n-1)}D_yx^{n-2}[\Phi^{(1)}(1,1,1;q^n;q,1)+\Phi^{(1)}(1,1,1;q^n;1,q^{n-1})]$$

$$+[n]_qxyD_xD_y^{(n-1)}y^{n-2}[\Phi^{(1)}(1,1,1;q^n;1,q)+\Phi^{(1)}(1,1,1;q^n;q^{n-1},1)]+...\} \tag{5.8}$$

Theorem 5.2.

$$D_{\alpha,q}^{(n)}\Phi^{(2)}(1,1,1;1,1;1,1)=\frac{(-1)^nq^{\binom{n}{2}}}{(\alpha,q)_n}\{x^nD_x^{(n)}[\Phi^{(2)}(1,1,1;1,1;1,1)+\Phi^{(2)}(1,1,1;1,1;1,q^n)]$$

$$+y^nD_y^{(n)}[\Phi^{(2)}(1,1,1;1,1;1,1)+\Phi^{(2)}(1,1,1;1,1;q^n,1)]$$

$$+[n]_{q^{-1}}x^{n-1}yD_x^{(n-1)}D_y[\Phi^{(2)}(1,1,1;1,1;q,1)+\Phi^{(2)}(1,1,1;1,1;1,q^{n-1})]$$

$$+[n]_{q^{-1}}xy^{n-1}D_xD_y^{(n-1)}[\Phi^{(2)}(1,1,1;1,1;1,q)+\Phi^{(2)}(1,1,1;1,1;q^{n-1},1)]+...\} \tag{5.9}$$

$$D_{\beta_1,q}^{(n)}\Phi^{(2)}(1,1,1;1,1;1,1)=\frac{(-1)^nq^{\binom{n}{2}}x^n}{(\beta_1,q)_n}D_{x,q}^{(n)}\Phi^{(2)}(1,1,1;1,1;1,1) \tag{5.10}$$

$$D_{\beta_2,q}^{(n)}\Phi^{(2)}(1,1,1;1,1;1,1)=\frac{(-1)^nq^{\binom{n}{2}}y^n}{(\beta_2,q)_n}D_{y,q}^{(n)}\Phi^{(2)}(1,1,1;1,1;1,1) \tag{5.11}$$

$$D_{\gamma_1,q}^{(n)}\Phi^{(2)}(1,1,1;1,1;1,1)=\frac{x}{(\gamma;q)_n}D_{x,q}^{(n)}\{x^{n-1}\Phi^{(2)}(1,1,1;q^n,1;1,1)\} \tag{5.12}$$

$$D_{\gamma_2,q}^{(n)}\Phi^{(2)}(1,1,1;1,1;1,1)=\frac{y}{(\gamma;q)_n}D_{y,q}^{(n)}\{y^{n-1}\Phi^{(2)}(1,1,1;1,q^n;1,1)\} \tag{5.13}$$

Theorem 5.3.

$$D^{(n)}_{\alpha_1,q}\Phi^{(3)}(1,1,1,1;1;1,1) = \frac{(-1)^n q^{\binom{n}{2}} x^n}{(\alpha_1,q)_n} D^{(n)}_{x,q}\Phi^{(3)}(1,1,1,1;1;1,1) \tag{5.14}$$

$$D^{(n)}_{\alpha_2,q}\Phi^{(3)}(1,1,1,1;1;1,1) = \frac{(-1)^n q^{\binom{n}{2}} y^n}{(\alpha_2,q)_n} D^{(n)}_{y,q}\Phi^{(3)}(1,1,1,1;1;1,1) \tag{5.15}$$

$$D^{(n)}_{\beta_1,q}\Phi^{(3)}(1,1,1,1;1;1,1) = \frac{(-1)^n q^{\binom{n}{2}} x^n}{(\beta_1,q)_n} D^{(n)}_{x,q}\Phi^{(3)}(1,1,1,1;1;1,1) \tag{5.16}$$

$$D^{(n)}_{\beta_2,q}\Phi^{(3)}(1,1,1,1;1;1,1) = \frac{(-1)^n q^{\binom{n}{2}} y^n}{(\beta_2,q)_n} D^{(n)}_{y,q}\Phi^{(3)}(1,1,1,1;1;1,1) \tag{5.17}$$

$$\begin{aligned} D^{(n)}_{\gamma,q}\Phi^{(3)}(1,1,1,1;1;1,1) = \frac{1}{2(\gamma,q)_n}\{ & xD^{(n)}_x x^{n-1}[\Phi^{(3)}(1,1,1,1;q^n;1,1) + \Phi^{(3)}(1,1,1,1;q^n;1,q^n)] \\ & + yD^{(n)}_y y^{n-1}[\Phi^{(3)}(1,1,1,1;q^n;1,1) + \Phi^{(3)}(1,1,1,1;1;q^n,1)] \\ & + [n]_q xyD^{(n-1)}_x D_y x^{n-2}[\Phi^{(3)}(1,1,1,1;q^n;q,1) + \Phi^{(3)}(1,1,1,1;q^n;1,q^{n-1})] \\ & + [n]_q xyD_x D^{(n-1)}_y y^{n-2}[\Phi^{(3)}(1,1,1,1;q^n;1,q) + \Phi^{(3)}(1,1,1,1;q^n;q^{n-1},1)] + ...\} \end{aligned} \tag{5.18}$$

Theorem 5.4.

$$\begin{aligned} D^{(n)}_{\alpha,q}\Phi^{(4)}(1,1;1,1;1,1) = \frac{(-1)^n q^{\binom{n}{2}}}{2(\alpha,q)_n}\{ & x^n D^{(n)}_x[\Phi^{(4)}(1,1;1,1;1,1) + \Phi^{(4)}(1,1;,1,1;1,q^n)] \\ & + y^n D^{(n)}_y[\Phi^{(4)}(1,1;1,1;1,1) + \Phi^{(4)}(1,1;1,1;q^n,1)] \\ & + [n]_{q^{-1}} x^{n-1} y D^{(n-1)}_x D_y[\Phi^{(4)}(1,1;1,1;q,1) + \Phi^{(4)}(1,1;1,1;1,q^{n-1})] \\ & + [n]_{q^{-1}} x y^{n-1} D_x D^{(n-1)}_y[\Phi^{(4)}(1,1;1,1;1,q) + \Phi^{(4)}(1,1;1,1;q^{n-1},1)] + ...\} \end{aligned} \tag{5.19}$$

$$\begin{aligned} D^{(n)}_{\beta,q}\Phi^{(4)}(1,1;1,1;1,1) = \frac{(-1)^n q^{\binom{n}{2}}}{2(\beta,q)_n}\{ & x^n D^{(n)}_x[\Phi^{(4)}(1,1;1,1;1,1) + \Phi^{(4)}(1,1;,1,1;1,q^n)] \\ & + y^n D^{(n)}_y[\Phi^{(4)}(1,1;1,1;1,1) + \Phi^{(4)}(1,1;1,1;q^n,1)] \end{aligned}$$

$$+[n]_{q^{-1}}x^{n-1}yD_x^{(n-1)}D_y[\Phi^{(4)}(1,1;1,1;q,1)+\Phi^{(4)}(1,1;1,1;1,q^{n-1})]$$

$$+[n]_{q^{-1}}xy^{n-1}D_xD_y^{(n-1)}[\Phi^{(4)}(1,1;1,1;1,q)+\Phi^{(4)}(1,1;1,1;q^{n-1},1)]+...\} \quad (5.20)$$

$$D_{\gamma_1,q}^{(n)}\Phi^{(4)}(1,1;1,1;1,1)=\frac{x}{(\gamma;q)_n}D_{x,q}^{(n)}\{x^{n-1}\Phi^{(4)}(1,1;q^n,1;1,1)\} \quad (5.21)$$

$$D_{\gamma_2,q}^{(n)}\Phi^{(4)}(1,1;1,1;1,1)=\frac{y}{(\gamma;q)_n}D_{y,q}^{(n)}\{y^{n-1}\Phi^{(4)}(1,1;1,q^n;1,1)\} \quad (5.22)$$

6 Concluding Remarks

The first major treatise on the basic hypergeometric functions of two variables asserts that, the q-Appell hypergeometric functions $\Phi^{(1)}$,$\Phi^{(2)}$,$\Phi^{(3)}$ and $\Phi^{(4)}$ reduce to the ordinary Heine's hypergeometric function in the cases$x=0$ and $y=0$. So all our results obtained in section 5 can be considered as an extension of Corollary 2.2. Also, we remark that, the basic hypergeometric function${}_r\Phi_s(a_1,...,a_r;b_1,...,b_s;q;z)$ is basically completely monotonic with respect to the parameter a_i, $\quad i=1,2,..,r$ if the parametr $a_i$is less than or equal to unity and the function ${}_r\Phi_s$ has positive q-derivative of all orders. Finally, the basic hypergeometric function${}_r\Phi_s(a_1,...,a_r;b_1,...,b_s;q;z)$ is totally basically completely monotonic if all parametrs are less than or equal to unity and the function ${}_r\Phi_s$has positive q-derivative of all orders.

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