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# A Note on Hardy-Hilbert Type Integral Inequality

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**Abstract**

In this paper it is shown that a new Hardy-Hilbert type integral inequality can be established by introducing a proper integral kernel function, and that the constant factor is proved to be the best possible. In particular, for case $p = 2$, a new Hilbert type integral inequality is given .And as the mathematics aesthetics, several important constants $\pi$ and Euler number $E_n$ appear simultaneously in the coefficient. As applications, some equivalent forms are studied.

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**Keywords:** Hardy-Hilbert integral inequality, integral kernel function, Riemann Zeta function, Euler number, weight function

## 1 Introduction and Lemmas

Let $f(x) \in L^p(0, +\infty)$ and $g(x) \in L^q(0, +\infty)$ It is well known that the inequality of the form

$$\int_0^\infty \int_0^\infty \frac{f(x)\,g(y)}{x+y}dxdy \le \frac{\pi}{\sin\frac{\pi}{p}}\left\{\int_0^\infty f^p(x)\,dx\right\}^{\frac{1}{p}}\left\{\int_0^\infty g^q(x)\,dx\right\}^{\frac{1}{q}} \tag{1.1}$$

is called Hardy-Hilbert's integral inequality, where the coefficient $\frac{\pi}{\sin\frac{\pi}{p}}$ is the best possible.

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In the papers [1-2], the following inequality of the form

$$\int_0^\infty\int_0^\infty \frac{(\ln x-\ln y)\, f(x)\, g(y)}{x\ -y}dxdy \le \left(\frac{\pi}{\sin\frac{\pi}{p}}\right)^2 \left\{\int_0^\infty f^p(x)\,dx\right\}^{\frac{1}{p}} \left\{\int_0^\infty g^q(x)\,dx\right\}^{\frac{1}{q}} \tag{1.2}$$

was established, and the coefficient $\left(\frac{\pi}{\sin\frac{\pi}{p}}\right)^2$ is also the best possible.

Owing to the importance of the Hardy-Hilbert inequality and the Hardy-Hilbert type inequality in analysis and applications, some mathematicians have been studying them. Recently, various improvements and extensions of (1.1) and (1.2) appear in a great deal of papers (see [3]-[7]etc.). Specially, Gao and Hsu enumerated more than 40 the research articles in the paper [3]. The aim of the present paper is to build some new Hardy-Hilbert type integral inequalities by introducing a proper integral kernel function and by using the technique of analysis, and to discuss the constant factor of which is related to the Riemann Zeta function, and then to study some equivalent forms of them.

In the sake of convenience, we introduce some notations and define some functions.

Let $0<\alpha<1$ and $n$ be a positive integer. Define a function $\zeta^*$ by

$$\zeta^*(n,\ \alpha)=\sum_{k=0}^{\infty}\tfrac{(-1)^k}{(\alpha+k)^n}. \tag{1.3}$$

And further define the function $\zeta_p$ by

$$\zeta_p=n!\left(\zeta^*(n+1,\ \tfrac{1}{p})+\zeta^*(n+1,\ 1-\tfrac{1}{p})\right),(n\in N_0) \tag{1.4}$$

where $\frac{1}{p}+\frac{1}{q}=1$ and $p>1$. It is obvious that $\zeta_p=\zeta_q$.

In order to prove our main results, we need the following lemmas.

**Lemma 1.1.** Let $0<\alpha<1$ and $n$ be a nonnegative integer. Then

$$\int_0^1 t^{\alpha-1}\left(\ln\tfrac{1}{t}\right)^n\tfrac{1}{1+t}dt=n!\zeta^*(n+1,\ \alpha). \tag{1.5}$$

where $\zeta^*$ is defined by (1.3).

This result has been given in the paper [8]. Hence its proof is omitted here.

**Lemma 1.2.** With the assumptions as Lemma 1.1, then

$$\int_0^\infty u^{\alpha-1}\left|\ln\frac{1}{u}\right|^n \frac{1}{1+u}du = n!\left\{\zeta^*(n+1,\ \alpha)+\zeta^*(n+1,\ 1-\alpha)\right\}, \tag{1.6}$$

where $\zeta^*$ is defined by (1.3).

**Proof.** It is easy to deduce that

$$\int_0^\infty u^{\alpha-1}\left|\ln\frac{1}{u}\right|^n \frac{1}{1+u}du = \int_0^1 u^{\alpha-1}\left|\ln\frac{1}{u}\right|^n \frac{1}{1+u}du + \int_1^\infty u^{\alpha-1}\left|\ln\frac{1}{u}\right|^n \frac{1}{1+u}du$$

$$= \int_0^1 u^{\alpha-1}\left|\ln\frac{1}{u}\right|^n \frac{1}{1+u}du + \int_0^1 v^{-\alpha}\left|\ln v\right|^n \frac{1}{1+v}dv$$

$$= \int_0^1 u^{\alpha-1}\left(\ln\frac{1}{u}\right)^n \frac{1}{1+u}du + \int_0^1 v^{(1-\alpha)-1}\left(\ln\frac{1}{v}\right)^n \frac{1}{1+v}dv.$$

By Lemma 1.1, the equality (1.6) follows at once.

# 2 Main Results

In this section, we will prove our assertions by using the above Lemmas.

**Theorem 2.1.** Let $f,\ g \geq 0,\ \frac{1}{p}+\frac{1}{q}=1$ and $p>1$, and $n$ be a nonnegative integer, If $0<\int_0^\infty f^p(x)dx<+\infty$ and $0<\int_0^\infty g^q(x)dx<+\infty$, then

$$\int_0^\infty\int_0^\infty \frac{\left|\ln x-\ln y\right|^n f(x)g(y)}{x+y}dxdy \leq \zeta_p\left(\int_0^\infty f^p(x)dx\right)^{\frac{1}{p}}\left(\int_0^\infty g^q(x)dx\right)^{\frac{1}{q}}, \tag{2.1}$$

where $\zeta_p$ is defined by (1.4), and the coefficient $\zeta_p$ is the best possible.

**Proof.** We may apply the Hölder inequality to estimate the left-hand side of (2.1) as follows:

$$\int_0^\infty\int_0^\infty \frac{\left|\ln x-\ln y\right|^n f(x)g(y)}{x+y}dxdy = \int_0^\infty\int_0^\infty \left(\frac{\left|\ln x-\ln y\right|^n}{x+y}\right)^{\frac{1}{p}}\left(\frac{x}{y}\right)^{\frac{1}{pq}} f(x)\left(\frac{\left|\ln x-\ln y\right|^n}{x+y}\right)^{\frac{1}{q}}\left(\frac{y}{x}\right)^{\frac{1}{pq}} g(y)dxdy$$

$$\leq \left\{ \int_0^\infty \int_0^\infty \frac{|\ln x - \ln y|^{\,n}}{x+y} \left(\frac{x}{y}\right)^{\frac{1}{q}} f^p(x)dxdy \right\}^{\frac{1}{p}} \left\{ \int_0^\infty \int_0^\infty \frac{|\ln x - \ln y|^{\,n}}{x+y} \left(\frac{y}{x}\right)^{\frac{1}{p}} g^q(y)dxdy \right\}^{\frac{1}{q}}$$

$$= \left( \int_0^\infty \omega_q(x) f^p(x)dx \right)^{\frac{1}{p}} \left( \int_0^\infty \omega_p(x) g^q(x)dx \right)^{\frac{1}{q}} \tag{2.2}$$

where $\omega_r(x) = \int_0^\infty \frac{|\ln x - \ln y|^{\,n}}{x+y} \left(\frac{x}{y}\right)^{\frac{1}{r}} dy\ , \qquad r = p,\ q,$

By using Lemma 1.2, it is easy to deduce that

$$\omega_r(x) = \int_0^\infty \frac{\left|\ln \frac{x}{y}\right|^n}{x\left(1+\frac{y}{x}\right)} \left(\frac{x}{y}\right)^{\frac{1}{r}} dy = \int_0^\infty u^{-\frac{1}{r}} \left|\ln \frac{1}{u}\right|^n \frac{1}{1+u} du = \zeta_r.$$

Notice that $\frac{1}{p} + \frac{1}{q} = 1$ and $p > 1$, thereby we have

$$\omega_r = \omega_p = \omega_q = \zeta_p. \tag{2.3}$$

where $\zeta_p$ is defined by (1.4).

It follows from (2.2) and (2.3) that the inequality (2.1) is valid.

It remains to need only to show that $\zeta_p$ in (2.1) is the best possible. $\forall \varepsilon > 0$. Define two functions by

$\tilde{f}(x) = x^{-\frac{1+\varepsilon}{p}}$ and $\tilde{g}(y) = y^{-\frac{1+\varepsilon}{q}}$. It is easy to deduce that $\int_\varepsilon^{+\infty} \tilde{f}^p(x)\,dx = \frac{1}{\varepsilon^{1+\varepsilon}}$ and

$\int_\varepsilon^\infty \tilde{g}^q(y)\,dy = \frac{1}{\varepsilon^{1+\varepsilon}}$. If $\zeta_p$ is not the best possible, then there exists $C > 0$, such that

$$\int_\varepsilon^\infty \int_\varepsilon^\infty \frac{|\ln x - \ln y|^{\,n} \tilde{f}(x)\tilde{g}(y)}{x+y} dxdy \leq C \left( \int_\varepsilon^\infty \tilde{f}^p(x)\,dx \right)^{\frac{1}{p}} \left( \int_\varepsilon^\infty \tilde{g}^q(y)\,dy \right)^{\frac{1}{q}} = \frac{C}{\varepsilon^{1+\varepsilon}} < \frac{\zeta_p}{\varepsilon^{1+\varepsilon}} \tag{2.4}$$

On the other hand, we have

$$\int_{\varepsilon}^{\infty}\int_{\varepsilon}^{\infty}\frac{|\ln x-\ln y|^{\,n}\tilde{f}(x)\tilde{g}(y)}{x+y}dxdy=\int_{\varepsilon}^{\infty}\int_{\varepsilon}^{\infty}\frac{\left\{x^{-(1+\varepsilon)/p}\right\}\left\{|\ln x-\ln y|^{\,n}y^{-(1+\varepsilon)/q}\right\}}{x+y}dxdy$$

$$=\int_{\varepsilon}^{\infty}\left\{\int_{\varepsilon}^{\infty}\frac{|\ln x-\ln y|^{\,n}y^{-(1+\varepsilon)/q}}{x+y}dy\right\}\left\{x^{-(1+\varepsilon)/p}\right\}dx$$

$$=\int_{\varepsilon}^{\infty}\left\{\int_{\varepsilon/x}^{\infty}\frac{\left|\ln\frac{1}{t}\right|^{n}t^{-\frac{1+\varepsilon}{q}}}{1+t}dt\right\}\left\{x^{-1-\varepsilon}\right\}dx$$

$$=\frac{1}{\varepsilon^{1+\varepsilon}}\int_{\varepsilon/x}^{\infty}\frac{\left|\ln\frac{1}{t}\right|^{n}t^{-\frac{1+\varepsilon}{q}}}{1+t}dt. \tag{2.5}$$

When $\varepsilon$ is small enough, by using Lemma 1.2 we can obtain from (2.5)

$$\int_{\varepsilon}^{\infty}\int_{\varepsilon}^{\infty}\frac{|\ln x-\ln y|^{\,n}\tilde{f}(x)\tilde{g}(y)}{x+y}dxdy>\frac{1}{\varepsilon^{1+\varepsilon}}\left\{\zeta_p\ +\ \circ(1)\right\}\quad(\ \varepsilon\to 0\ ) \tag{2.6}$$

Clearly, when $\varepsilon$ is small enough, the inequality (2.4) is in contradiction with the inequality (2.6). Therefore, $\zeta_p$ in (2.1) is the best possible. Thus the proof of Theorem is completed.

**Remark.** We point out that $\zeta_p$ can be expressed by Riemann's Zeta function. In fact, let $z>1$, then the Riemann Zeta function is that $\zeta(z,\alpha)=\sum\limits_{k=0}^{\infty}\frac{1}{(\alpha+k)^z}$. Hence the equality (1.3) can be written in the following form:

$$\zeta^*(z,\ \alpha)=\frac{1}{2^z}\left(\sum_{m=0}^{\infty}\frac{1}{(\frac{\alpha}{2}+m)^z}-\sum_{m=0}^{\infty}\frac{1}{(\frac{\alpha+1}{2}+m)^z}\right)=\frac{1}{2^z}\left\{\zeta\left(z,\ \tfrac{\alpha}{2}\right)-\zeta\left(z,\ \tfrac{\alpha+1}{2}\right)\right\}\ ,$$

When $n\geq 1$, we can write (1.4) in form:

$$\zeta_p = \frac{n!}{2^{n+1}} \left\{ \zeta(n+1, \tfrac{1}{2p}) - \zeta(n+1, \tfrac{1}{2p} + \tfrac{1}{2}) + \zeta(n+1, \tfrac{1}{2q}) - \zeta(n+1, \tfrac{1}{2q} + \tfrac{1}{2}) \right\}.$$

When $p = 2$, it is known from (1.4) and (1.3) that

$$\zeta_2 = n!2^{n+2} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{n+1}}, (n \in N_0) \tag{2.7}$$

Hence we have the following Hilbert type inequality.

**Theorem 2.2.** If $0 < \int_0^\infty f^2(x)\,dx < +\infty$ and $0 < \int_0^\infty g^2(x)\,dx < +\infty$, then

$$\int_0^\infty \int_0^\infty \frac{|\ln x - \ln y|^n f(x)g(y)}{x+y} dxdy \leq \zeta_2 \left\{ \int_0^\infty f^2(x)\,dx \right\}^{\frac{1}{2}} \left\{ \int_0^\infty g^2(x)\,dx \right\}^{\frac{1}{2}}. \tag{2.8}$$

where $\zeta_2$ is defined by (2.7), and $\zeta_2$ is the best possible.

In particular, when $n$ is an even, we obtain the following important result:

**Corollary 2.3.** With the assumptions as Theorem 2.2, then

$$\int_0^\infty \int_0^\infty \frac{|\ln x - \ln y|^{2m} f(x)g(y)}{x+y} dxdy \leq \left(\pi^{2m+1} E_m\right) \left\{ \int_0^\infty f^2(x)\,dx \right\}^{\frac{1}{2}} \left\{ \int_0^\infty g^2(x)\,dx \right\}^{\frac{1}{2}}, \tag{2.9}$$

where $E_m$ is Euler number viz. $E_1 = 1$, $E_2 = 5$, $E_3 = 61$, $E_4 = 1385$, $E_5 = 50521$, $\cdots$. Define $E_0 = 1$, and the constant factor $\pi^{2m+1}E_m$ is the best possible, where $m \in N_0$.

**Proof.** We need only to show that $\pi^{2m+1}E_m$ is the constant factor of (2.9). When $n = 2m$, we can write $\zeta_2$ defined by (2.7) in the following form:

$$\zeta_2 = (2m)!2^{2m+2} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{2m+1}}. \tag{2.10}$$

It is known from the paper [9] that

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{2m+1}} = \frac{\pi^{2m+1}}{2^{2m+2}(2m)!} E_m. \tag{2.11}$$

where $E_m$ is Euler number viz. $E_1 = 1$, $E_2 = 5$, $E_3 = 61$, $E_4 = 1385$, $E_5 = 50521$, $\cdots$.

Notice that $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots\cdots = \frac{\pi}{4}$. Hence we can define $E_0 = 1$, It follows from (2.10) and (2.11) that $\zeta_2 = \pi^{2m+1} E_m$, where $m \in N_0$.

Specially, when $m = 1$, $\zeta_2$ is reduced to $\pi^3$. Hence we have the following result.

**Corollary 2.4.** With the assumptions as Theorem 2.2, then

$$\int_0^\infty \int_0^\infty \frac{|\ln x - \ln y|^2 f(x) g(y)}{x+y} dxdy \le \pi^3 \left\{ \int_0^\infty f^2(x)\, dx \right\}^{\frac{1}{2}} \left\{ \int_0^\infty g^2(x)\, dx \right\}^{\frac{1}{2}}. \tag{2.12}$$

where the coefficient $\pi^3$ is the best possible.

# 3 Some Equivalent Forms

As applications, we will build the following inequalities.

**Theorem 3.1.** Let $n$ be a nonnegative integer, $\frac{1}{p} + \frac{1}{q} = 1$ and $p > 1$. If $f$ is a nonnegative real function such that $0 < \int_0^\infty f^p(x) dx < +\infty$, then

$$\int_0^\infty \left\{ \int_0^\infty \frac{|\ln x - \ln y|^n}{x+y} f(x)\, dx \right\}^p dy \le \zeta_p^p \int_0^\infty f^p(x)\, dx, \tag{3.1}$$

where $\zeta_p$ is defined by (1.4) and the constant factor $\zeta_p^p$ is the best possible. Inequality (3.1) is equivalent to (2.1).

**Proof.** First, we show that the inequality (3.1) is equivalent to (2.1). Setting a real function $g(y)$as

$$g(y) = \left\{ \int_0^\infty \frac{|\ln x - \ln y|^n}{x+y} f(x)\, dx \right\}^{p-1}, \quad y \in (0, +\infty)$$

By using (2.1), we have

$$\int_0^\infty \left\{ \int_0^\infty \frac{|\ln x - \ln y|^n}{x+y} f(x)\, dx \right\}^p dy = \int_0^\infty \int_0^\infty \frac{|\ln x - \ln y|^n}{x+y} f(x)\, g(y) dxdy$$

$$\leq \zeta_p \left\{ \int_0^\infty f^p(x)dx \right\}^{\frac{1}{p}} \left\{ \int_0^\infty g^q(y)dy \right\}^{\frac{1}{q}}$$

$$= \zeta_p \left\{ \int_0^\infty f^p(x)dx \right\}^{\frac{1}{p}} \left\{ \int_0^\infty \left( \int_0^\infty \frac{|\ln x - \ln y|^n}{x+y} f(x)dx \right)^{q(p-1)} dy \right\}^{\frac{1}{q}}$$

$$= \zeta_p \left\{ \int_0^\infty f^p(x)dx \right\}^{\frac{1}{p}} \left\{ \int_0^\infty \left( \int_0^\infty \frac{|\ln x - \ln y|^n}{x+y} f(x)dx \right)^{p} dy \right\}^{\frac{1}{q}} \tag{3.2}$$

It follows from (3.2) that the inequality (3.1) is valid after some simplifications.

On the other hand, assume that the inequality (3.1) keeps valid, then by applying in turn Hölder's inequality and (3.1), we have

$$\int_0^\infty \int_0^\infty \frac{|\ln x - \ln y|^n}{x+y} f(x)\, g(y)\, dxdy = \int_0^\infty \left\{ \int_0^\infty \frac{|\ln x - \ln y|^n}{x+y} f(x)\, dx \right\} g(y)\, dy$$

$$\leq \left\{ \int_0^\infty \left( \int_0^\infty \frac{|\ln x - \ln y|^n}{x+y} f(x)\, dx \right)^p dy \right\}^{\frac{1}{p}} \left\{ \int_0^\infty g^q(y)\, dy \right\}^{\frac{1}{q}}$$

$$\leq \left\{ \zeta_p^p \int_0^\infty f^p(x)\, dx \right\}^{\frac{1}{p}} \left\{ \int_0^\infty g^q(y)\, dy \right\}^{\frac{1}{q}}$$

$$= \zeta_p \left\{ \int_0^\infty f^p(x)\, dx \right\}^{\frac{1}{p}} \left\{ \int_0^\infty g^q(y)\, dy \right\}^{\frac{1}{q}} \tag{3.3}$$

If the constant factor $\zeta_p^p$ in (3.1) is not the best possible, then it is known from (3.3) that the constant factor $\zeta_p$ in (2.1) is also not the best possible. This is a contradiction. Theorem is proved.

**Theorem 3.2.** If $0 < \int_0^\infty f^2(x)dx < +\infty$, then

$$\int_0^\infty \left\{ \int_0^\infty \frac{|\ln x - \ln y|^n}{x+y} f(x)\, dx \right\}^2 dy \le \zeta_2^2 \int_0^\infty f^2(x)\, dx, \tag{3.4}$$

where $\zeta_2$ is defined by (2.7), and the constant factor $\zeta_2^2$ is the best possible. Inequality (3.4) is equivalent to (2.8).

In particular, when $n$ is an even, we obtain the following important result:

**Corollary 3.3.** With the assumptions as Theorem 3.2, then

$$\int_0^\infty \left\{ \int_0^\infty \frac{|\ln x - \ln y|^{2m}}{x+y} f(x)\, dx \right\}^2 dy \le \left(\pi^{2m+1} E_m\right)^2 \int_0^\infty f^2(x)\, dx, \tag{3.5}$$

where $E_m$ is Euler number and the constant factor $\pi^{2m+1}E_m$ is the best possible, where $m \in N_0$. Inequality (3.5) is equivalent to (2.9).

**Corollary 3.4.** With the assumptions as Theorem 3.2, then

$$\int_0^\infty \left\{ \int_0^\infty \frac{|\ln x - \ln y|^2}{x+y} f(x)\, dx \right\}^2 dy \le \pi^6 \int_0^\infty f^2(x)\, dx, \tag{3.6}$$

where the constant factor $\pi^6$ is the best possible. Inequality (3.6) is equivalent to (2.12).

The proofs of Theorem 3.2 and Corollaries (3.3) and (3.4) are similar to one of Theorem 3.1, they are omitted here.

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