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Regularization for a Common Solution of a System of Nonlinear Ill-Posed Equations

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Abstract

The purpose of this paper is to give a theoretical analysis for the variational variant of the Tikhonov regularization method for solving a system of nonlinear ill-posed equations in real Hilbert spaces.

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1. INTRODUCTION

Let X, Y_j be the Hilbert spaces where the scalar product and the norm of X are denoted by the symbols $\langle ., . \rangle_X$ and $\|.\|_X$, respectively. Let $A_j, j = 1, ..., N$, be the nonlinear operators from a closed convex subset $\mathcal{D} \subseteq X$ into Y_j having the following properties:

(i) A_j is continuous;

(ii) A_j is weakly closed, i.e. for any sequence $\{x_n\}$ converging weakly in X to x where $A_j(x_n)$ converging weakly in Y_j to y we have $A_j(x) = y$.

Consider the following problem: find an element $x_0 \in \mathcal{D}$ such that

$$A_j(x_0) = f_j, \quad \forall j = 1, ..., N, \tag{1.1}$$

where f_j is given in Y_j a priori. Set

$$S_j = \{\overline{x} \in \mathcal{D} : A_j(\overline{x}) = f_j\}, j = 1, ..., N, \quad S = \cap_{j=1}^N S_j.$$

Here, we suppose that $S \neq \emptyset$. From the properties of A_j it is easy to see that S_j is closed in X. Therefore, S is also closed.

We are specially interested in the situation where the data f_j is not exactly known, i.e., we have only an approximation $f_j^{\delta_j}$ of the data f_j satisfying

$$\|f_j - f_j^{\delta_j}\|_{Y_j} \leq \delta_j, \quad \delta_j \to 0. \tag{1.2}$$

This problem is studied in [1] - [3] and [5]. The system of equations (1.1) can be written [5] in the form

$$\mathcal{A}(x) = f, \tag{1.3}$$

where $\mathcal{A} : X \to Y := Y_1 \times ... \times Y_N$ by $\mathcal{A}(x) := (A_1(x), ..., A_N(x))$, and $f := (f_1, ..., f_N)$. Note that (1.3) can be seen as a special case of (1.1) with $N = 1$. Howerver, one potential advantage of (1.1) over (1.3) can be that it might better reflect the structure of the underlying information $(f_1, ..., f_N)$ leading to the couplet system, than a plain concantenation into one single data element f could. In particular, the second advantage is that in estimating convergence rates of regularization solution which is showed later we need only the smooth property for one among A_j, while for (1.3) we need [5] the property for all $A_j, j = 1, ..., N$.

With the above conditions on A_j each j-equation (1.1) is ill-posed. By this we mean that the solution set S_j does not depend continuously on the data f_j. Therefore, to find a solution of each j-equation in (1.1) we have to use stable methods. One of those methods is the variational variant of Tikhonov's regularization that consists of minimizing the functional

$$\|A_j(x) - f_j^{\delta_j}\|_{Y_j}^2 + \alpha\|x - x_*\|_X^2 \tag{1.4}$$

where x_* is some element in $X \setminus S_j$, $\alpha > 0$ is the small parameter of regularization. It proved in [4] that each j-minimization problem in (1.4) has unique solution $x_j^{\alpha\delta_j}$, and if $\delta_j^2/\alpha, \alpha \to 0$ then $\{x_j^{\alpha\delta_j}\}$ contains a converging subsequence, and the limit point $\tilde{x}_j$ of any converging subsequence has the following property

$$\|\tilde{x}_j - x_*\|_X = \min_{x \in S_j} \|x - x_*\|_X, \quad j = 1, ..., N.$$

Our problem: find $x_\alpha^{\delta_j}$ such that $x_\alpha^{\delta_j} \to x_0$ as $\delta_j, \alpha \to 0$, a relation $\alpha = \alpha(\delta_j)$ such that $x_{\alpha(\delta_j)}^{\delta_j} \to x_0$ as $\delta_j \to 0$, and finally estimate the value $\|x_{\alpha(\delta_j)}^{\delta_j} - x_0\|$ where x_0 is a x_*-minimal norm element in S (x_*-MNS).

To do this, consider the problem

$$\sum_{j=1}^N \|A_j(x) - f_j^{\delta_j}\|_{Y_j}^2 + \alpha\|x - x_*\|_X^2 \to \min_{\mathcal{D}}. \tag{1.5}$$

Notice that when $Y_j = X^*$, the conjugate of a Banach spaces X, and A_j is the derivative of some weakly lower semicontinuous and proper convex functional, the stated problem was considered in [1] on the base of the theory of monotone operators.

Above and below, the symbols $\rightharpoonup$ and $\to$ denote the weak convergence and convergence in the norm, respectively, and $a \sim b$ is meant $a = O(b)$ and $b = O(a)$.

2. MAIN RESULTS

Under the assumptions on A_j it can be easy to show that problem (1.5) admits a solution. Since A_j are nonlinear, the solution will not be unique, in general. We shall first address two questions. Is the problem (1.5) stable in the sense of continuous dependence of the solution on the data $f_j^{\delta_j}$? Secondly, do the solutions of (1.5) converge toward a solution of (1.1) as $\alpha, \delta_j \to 0$. In [3], stability has been proved for the case $N = 1$. For the convenience of the reader, we provide the whole argument in the case of arbitrary $N \geq 1$.

Theorem 2.1. *Let $\alpha > 0$, $f_j^{\delta_{jk}} \to f_j^{\delta_j}$ with $\delta_j \geq 0$, and x_k be a minimizer of (1.5) with $f_j^{\delta_j}$ replaced by $f_j^{\delta_{jk}}$. Then there exists a convergent subsequence of $\{x_k\}$ and the limit of every convergent subsequence is a minimizer of (1.5).*

Proof. For any $x \in \mathcal{D}$ we have

$$\sum_{j=1}^{N} \|A_j(x_k) - f_j^{\delta_{jk}}\|_{Y_j}^2 + \alpha\|x_k - x_*\|_X^2 \leq \sum_{j=1}^{N} \|A_j(x) - f_j^{\delta_{jk}}\|_{Y_j}^2 + \alpha\|x - x_*\|_X^2. \tag{2.1}$$

Hence, $\|x_k\|_X$ and $\|A_j(x_k)\|_{Y_j}$ are bounded for each j. Consequence, there exist a subsequence $\{x_m\}$ of $\{x_k\}$ and $\tilde{x}$ such that

$$x_m \rightharpoonup \tilde{x}, A_j(x_m) \rightharpoonup A_j(\tilde{x}), j = 1, ..., N.$$

By the weak lower semicontinuity of the norm and the weakly closed property of A_j we have

$$\|\tilde{x} - x_*\|_X \leq \liminf \|x_m - x_*\|_X$$

and

$$\|A_j(\tilde{x}) - f_j^{\delta_j}\|_{Y_j} \leq \liminf \|A_j(x_m) - f_j^{\delta_{jm}}\|_{Y_j}.$$

Therefore,

$$\sum_{j=1}^{N} \|A_j(\tilde{x}) - f_j^{\delta_j}\|_{Y_j}^2 \leq \sum_{j=1}^{N} \liminf \|A_j(x_m) - f_j^{\delta_{jm}}\|_{Y_j}^2. \tag{2.2}$$

Moreover, (2.1) implies

$$\begin{aligned}\sum_{j=1}^{N}\|A_j(\tilde{x})-f_j^{\delta_j}\|_{Y_j}^2+\alpha\|\tilde{x}-x_*\|_X^2 \le& \sum_{j=1}^{N}\liminf\|A_j(x_m)-f_j^{\delta_{jm}}\|_{Y_j}^2\\ &+\alpha\liminf\|x_m-x_*\|_X^2\\ &\le \sum_{j=1}^{N}\limsup\|A_j(x_m)-f_j^{\delta_{jm}}\|_{Y_j}^2+\alpha\limsup\|x_m-x_*\|_X^2\\ &\le \sum_{j=1}^{N}\lim\|A_j(x)-f_j^{\delta_{jm}}\|_{Y_j}^2+\alpha\lim\|x-x_*\|_X^2\\ &= \sum_{j=1}^{N}\|A_j(x)-f_j^{\delta_j}\|_{Y_j}^2+\alpha\|x-x_*\|_X^2\end{aligned}$$

for all $x\in\mathcal{D}$. This implies that $\tilde{x}$ is a minimizer of (1.5) and that

$$\begin{aligned}\lim_{m\to\infty}&\left(\sum_{j=1}^{N}\|A_j(x_m)-f_j^{\delta_{jm}}\|_{Y_j}^2+\alpha\|x_m-x_*\|_X^2\right)\\ &=\sum_{j=1}^{N}\|A_j(\tilde{x})-f_j^{\delta_j}\|_{Y_j}^2+\alpha\|\tilde{x}-x_*\|_X^2.\end{aligned}\tag{2.3}$$

Now, assume that $x_m\not\to\tilde{x}$. Then $c:=\limsup\|x_m-x_*\|_X>\|\tilde{x}-x_*\|_X$ and there exists a subsequence $\{x_n\}$ of $\{x_m\}$ such that $x_n\rightharpoonup\tilde{x}, A_j(x_n)\rightharpoonup A_j(\tilde{x})$ and $\|x_n-x_*\|_X\to c$. As a consequence of (2.3), we obtain

$$\lim_{n\to\infty}\sum_{j=1}^{N}\|A_j(x_n)-f_j^{\delta_{jn}}\|_{Y_j}^2=\sum_{j=1}^{N}\|A_j(\tilde{x})-f_j^{\delta_j}\|_{Y_j}^2+\alpha(\|\tilde{x}-x_*\|_X^2-c^2).$$

Therefore,

$$\lim_{n\to\infty}\sum_{j=1}^{N}\|A_j(x_n)-f_j^{\delta_{jn}}\|_{Y_j}^2<\sum_{j=1}^{N}\|A(\tilde{x})-f_j^{\delta_j}\|_{Y_j}^2$$

in contradiction to (2.2). This argument shows that $x_m\to\tilde{x}$.

Further, without loss of generality, assume that $\delta_j=\delta,\delta\to 0$.

Theorem 2.2. *Let $\alpha(\delta)$ be such that $\alpha(\delta)\to 0, \delta^2/\alpha(\delta)\to 0$ as $\delta\to 0$. Then every sequence $\{x_{\alpha_k}^{\delta_k}\}$, where $\delta_k\to 0, \alpha_k=\alpha(\delta_k)$ and $x_{\alpha_k}^{\delta_k}$ is a solution of* (1.5), *has a convergent subsequence. The limit of every convergent subsequence is an x_*-MNS. If, in addition, the x_*-MNS x_0 is unique, then*

$$\lim_{\delta\to 0}x_{\alpha(\delta)}^{\delta}=x_0.$$

Proof. From (1.5) we have

$$\begin{aligned}\sum_{j=1}^{N}\|A_j(x^{\delta}_{\alpha(\delta)})-f^{\delta}_j\|^2_{Y_j}+\alpha(\delta)\|x^{\delta}_{\alpha(\delta)}-x_*\|^2_X \\ \le \sum_{j=1}^{N}\|A_j(x)-f^{\delta}_j\|^2_{Y_j}+\alpha(\delta)\|x-x_*\|^2_X \\ \le \sum_{j=1}^{N}\|f_j-f^{\delta}_j\|^2_{Y_j}+\alpha(\delta)\|x-x_*\|^2_X \\ \le N\delta^2+\alpha(\delta)\|x-x_*\|^2_X, \quad x\in S.\end{aligned} \tag{2.4}$$

Hence,

$$\|x^{\delta}_{\alpha(\delta)}-x_*\|^2_X \le N\frac{\delta^2}{\alpha(\delta)}+\|x-x_*\|^2_X \quad \forall x\in S. \tag{2.5}$$

Consequently, $\{x^{\delta}_{\alpha(\delta)}\}$ is bounded, i.e. there exists a positive constant d_1 such that $\|x^{\delta}_{\alpha(\delta)}\|\le d_1$. Therefore, every sequence $\{x^{\delta_k}_{\alpha_k}\}$, where $\delta_k\to 0, \alpha_k=\alpha(\delta_k)$ and $x^{\delta_k}_{\alpha_k}$ is a solution of (1.5), has a weak convergent subsequence. Let $\{x^{\delta_m}_{\alpha_m}\}\subset\{x^{\delta_k}_{\alpha_k}\}$ be such that $x^{\delta_m}_{\alpha_m}\rightharpoonup\overline{x}$ as $m\to\infty$. From (2.4) we see that

$$\|A_j(x^{\delta_m}_{\alpha_m})-f^{\delta_m}_j\|^2_{Y_j}\le N\delta_m^2+\alpha_m\|x-x_*\|^2_X, 1\le j\le N. \tag{2.6}$$

First, note that $A_j(x^{\delta_m}_{\alpha_m})\rightharpoonup A_j(\tilde{x})$ as $m\to\infty$. Tending $m\to\infty$ in (2.6), we obtain $\|A_j(\overline{x})-f_j\|_{Y_j}=0$, i.e. $\overline{x}\in S_j, 1\le j\le N$. From (2.5) it implies that $\overline{x}$ is a x_*-MNS of (1.1), and $\|x^{\delta_m}_{\alpha_m}-x_*\|_X\to\|\overline{x}-x_*\|_X$. Since X is a Hilbert space, then $x^{\delta_m}_{\alpha_m}\to\overline{x}$ as $m\to\infty$. Theorem is proved.

Now, assume that $x^{\delta}_{\alpha(\delta)}\to x_0$, as $\delta\to 0$. The convergence rates of $\{x^{\delta}_{\alpha(\delta)}\}$ is defined by the following theorem.

Theorem 2.3. *Let the following conditions hold:*

(i) A_1 *is Fréchet differentiable*

(ii) there exists $L>0$ *such that* $\|A_1'(x_0)-A_1'(z)\|_{Y_j}\le L\|x_0-z\|_X$ *for* z *in some neighbouhood* $\mathcal{U}$ *of* x_0

(iii) there exists $\omega\in Y_1$ *such that* $x_0-x_*=A_1'(x_0)^*\omega$

(iv) $L\|\omega\|_{Y_1}<1$.

Then for the choice $\alpha\sim\delta^p, 0<p<2$, *we obtain*

$$\|x^{\delta}_{\alpha(\delta)}-x_0\|_X=O(\delta^{1-\frac{p}{2}}).$$

Proof. Using (2.4) with $x=x_0$ we obtain

$$\begin{aligned}\sum_{j=1}^{N}\|A_j(x^{\delta}_{\alpha(\delta)})-f^{\delta}_j\|^2_{Y_j}+\alpha(\delta)\|x^{\delta}_{\alpha(\delta)}-x_0\|^2_X \\ \le N\delta^2+\alpha(\delta)(\|x_0-x_*\|^2_X-\|x^{\delta}_{\alpha(\delta)}-x_*\|^2_X+\|x^{\delta}_{\alpha(\delta)}-x_0\|^2_X).\end{aligned}$$

Hence,

$$\|A_1(x^\delta_{\alpha(\delta)}) - f^\delta_1\|^2_{Y_j} + \alpha(\delta)\|x^\delta_{\alpha(\delta)} - x_0\|^2_X \le N\delta^2 + 2\alpha(\delta)\langle\omega, A'_1(x_0)(x_0 - x^\delta_{\alpha(\delta)})\rangle. \tag{2.7}$$

Note that conditions (i) and (ii) imply

$$A_1(x^\delta_{\alpha(\delta)}) = A_1(x_0) + A'_1(x_0)(x^\delta_{\alpha(\delta)} - x_0) + r^\delta_\alpha \tag{2.8}$$

with

$$\|r^\delta_\alpha\| \le \frac{1}{2}L\|x^\delta_{\alpha(\delta)} - x_0\|^2. \tag{2.9}$$

Combining (2.7)-(2.9) leads to

$$\begin{aligned}\|A_1(x^\delta_{\alpha(\delta)}) - f^\delta_1\|^2_{Y_j} + \alpha(\delta)\|x^\delta_{\alpha(\delta)} - x_0\|^2_X &\le N\delta^2 \\ &+ 2\alpha(\delta)\langle\omega, (f_1 - f^\delta_1) + (f^\delta_1 - A_1(x^\delta_{\alpha(\delta)}) + r^\delta_\alpha\rangle \\ &\le N\delta^2 + 2\|\omega\|_{Y_1}\alpha(\delta)\delta + 2\|\omega\|_{Y_1}\alpha(\delta)\|A_1(x^\delta_{\alpha(\delta)}) - f^\delta_1\|_{Y_j} \\ &+ \alpha(\delta)L\|\omega\|_{Y_1}\|x^\delta_{\alpha(\delta)}) - x_0\|^2_X.\end{aligned}$$

and hence

$$\begin{aligned}&\|A_1(x^\delta_{\alpha(\delta)}) - f^\delta_1\|^2_{Y_j} + \alpha(\delta)(1 - L\|\omega\|_{Y_1})\|x^\delta_{\alpha(\delta)} - x_0\|^2_X \le \\ &N\delta^2 + 2\|\omega\|_{Y_1}\alpha(\delta)\delta + 2\|\omega\|_{Y_1}\alpha(\delta)\|A_1(x^\delta_{\alpha(\delta)}) - f^\delta_1\|_{Y_j}.\end{aligned} \tag{2.10}$$

Together with (iv) and the implication

$$(a, b, c \ge 0, a^2 \le ab + c^2) \Rightarrow a \le b + c$$

(2.10) implies

$$\|A_1(x^\delta_{\alpha(\delta)}) - f^\delta_1\|_{Y_j} \le \Big[N\delta^2 + 2\|\omega\|_{Y_1}\alpha(\delta)\delta\Big]^{1/2} + 2\|\omega\|_{Y_1}\alpha(\delta).$$

Together with (2.9), this implies

$$\|x^\delta_{\alpha(\delta)} - x_0\|_X \le \frac{(N\delta^2 + 2\|\omega\|_{Y_1}\alpha(\delta)\delta)^{1/2}}{\sqrt{\alpha(\delta)}(1 - L\|\omega\|_{Y_1})^{1/2}}. \tag{2.11}$$

The assertion now follows from (2.11), if $\alpha(\delta) \sim \delta^p, 0 < p < 2$.

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