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Characterizations of Regular Gamma Semi-Groups Using Quazi-Ideals

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Abstract. In this paper some important concept are given for the gamma semi-groups and also some characterizations of regular gamma semi-groups. Characterizations of the regular elements are given for quasi-ideals. We take a Theorem as an important result about. From this theorem we have a corollary giving a very important property of quasi-ideals in a Γ-semigroup M. We have proved an important corollary in a Γ-semigroup that every quasi-ideal is a bi-ideal, but its reverse proposition is not true in general. The following corollary contains the conditions that the vice-versa proposition be true.

Keywords: Γ-semigroup regular, quasi-ideal, bi-ideal, principal ideal (quasi-ideal)

1. Preliminary Knowledge

Definition 1.1. *Let $M = \{a, b, c, \dots\}$ and $\Gamma = \{x, y, z, \dots ,\}$ be two non-empty sets. M is called a Γ-semigroup if :*

1. $axb \in M$.

2. $(axb) = ax(byc), \forall a, b, c, \in M$ *and* $x, y \in \Gamma$.

Definition 1.2. *An element $a \in M$ is called regular in a Γ-semigroup M if $a \in a\Gamma M\Gamma a$, where $a\Gamma M\Gamma a = \{(axb)ya : b \in M \text{ and } x, y \in \Gamma\}$.*

Definition 1.3. *A Γ-semigroup M is called regular if every element is regular.*

Definition 1.4. *Quasi-ideal of Γ-semigroup M is called every non-empty subset Q of M such that $Q\Gamma M \cap M\Gamma Q \subseteq Q$.*

Definition 1.5. *Bi-ideal of a Γ-semigroup M is called every Γ-subsemigroup B of M, such that $B\Gamma M\Gamma B \subseteq B$.*

Proposition 1.6. *Let M be a Γ-semigroup and $a \in M$.*

Then the following identities are true

1. $(a)_l = M\Gamma a \cup a$
2. $(a)_r = a\Gamma M \cup a$
3. $(a) = M\Gamma a \cup a\Gamma M \cup (M\Gamma a)\Gamma M \cup a$.

Proposition 1.7. *Let A be a two-sided ideal of a Γ-semigroup M and Q a quasi- ideal of A. Then Q is the bi-ideal of M.*

Corollary 1.1. *Every quasi-ideal of a Γ-semigroup M is the bi-ideal of M.*

1.1. **Characterizations of regular Γ-semigroups in the language of quasi-ideals.**

Theorem 1.8. *The four following conditions in a Γ-semigroup M are equivalent*

1. *M is regular*
2. *For every right ideal R and every left ideal L of M we have*

$$R \cap L = R\Gamma L$$

3. *For every right ideal R and every left ideal L of M we have*
   (a) $R\Gamma R = R$

   (b) $L\Gamma L = L$

   (c) *$R \cap L = R\Gamma L$ is the quasi-ideal of Γ-semigroup M*

4. *Every quasi-ideal Q of Γ-semigroup M has the form $Q = Q\Gamma M\Gamma Q$.*

Proof: (1) $\Leftrightarrow$ (2). This equivalence is proved.
(2) $\Rightarrow$ (3) a) $R\Gamma R \subseteq R\Gamma M \subseteq R$ b)
To prove the reverse we take an element $a \in R$, this means $a \in M$ and, because M is a regular Γ-semigroup then, $a \in a\Gamma M\Gamma a \subseteq R\Gamma M\Gamma a \subseteq R\Gamma a \subseteq R\Gamma R$, thus $a \in R$ implies $a \in R\Gamma R$. This proves the set equality $R = R\Gamma R$. In analogues way is proved that $L\Gamma L = L$.
c) To prove this proposition suffices to say that the intersection of any two ideals right and left is a quasi-ideal of Γ-semigroup M, a theorem proved before .
4) $\Rightarrow$ (1). Let Q be any quasi-ideal of Γ-semigroup M having the form $Q = Q\Gamma M\Gamma Q$ (*) We prove that M is a regular Γ-semigroup. Q is a quasi-ideal, therefore it is an intersection of a principal right ideal $(a)_r$ and a principal left ideal $(a)_l$ of M, generated by the element a. Suppose that $a \in Q$. Then $Q = (a)_r \cap (a)_l = (a)_q$, where $(a)_q$ is a principal quasi-ideal of Γ-semigroup M. So, we have:

$$(a)_r \cap (a)_l = ((a)_r \cap (a)_l)\Gamma M\Gamma((a)_r \cap (a)_l) \subseteq (a)_l\Gamma M\Gamma(a)_r$$

Thus way, from $a \in (a)_r \cap (a)_l$ follows that

$$a \in (a)_r \Gamma M \Gamma (a)_l = (a \cup a\Gamma M)(\Gamma M \Gamma)(M\Gamma a \cup a) = a\Gamma M\Gamma(a \cup M\Gamma a) \cup$$
$$\cup a\Gamma M\Gamma M\Gamma(a \cup M\Gamma a) = a\Gamma M\Gamma a \cup a\Gamma M\Gamma(M\Gamma a) \cup (a\Gamma M)\Gamma M\Gamma a \cup$$
$$\cup(a\Gamma M)\Gamma M\Gamma(M\Gamma a) \subseteq a\Gamma M\Gamma a \cup a\Gamma M\Gamma a \cup a\Gamma M\Gamma a \cup a\Gamma M\Gamma a = a\Gamma M\Gamma a$$

So, a is a regular element of M.
The implication (1) $\Rightarrow$ (4) is evident because for $\forall a \in Q$ we have that $a \in M$ and $a \in a\Gamma M\Gamma a$, so (4) is satisfied.
Theorem 1.1. c an have another form, for one element only.

Before we do this must be given the concept of a principal idempotent ideal (left, right), applying an analogy with the idempotent element in a group.

Definition 1.9. *A principal ideal(left, right) is called idempoten in a* Γ*-semigroup if for* $\forall x \in \Gamma$ *hold true the equalities:*

$$(a)_r = (a)_r x(a)_r (a)_l = (a)_l y(a)_l (1)$$

Definition 1.10. *A subset* A *of* Γ*-semigroup* M *is called idempotent if the identity* $A = A\Gamma A$ *holds true.*

1.2. **Characterizations of a regular element using quasi-ideals.**

Theorem 1.11. *In a* Γ*-semigroup* M *the four following conditions are equivalent:*

1. *a is regular element of* Γ*-semigroup* M
2. $(a)_r \cap (a)_l = (a)_q = (a)_r\Gamma(a)_l$.
3. $(a)_r = (a)_r\Gamma(a)_r, \forall x \in \Gamma$.
   $(a)_l = (a)_l\Gamma(a)_l, \forall y \in \Gamma$.
   $(a)_r \cap (a)_l$ *is quasi-ideal of* M.
4. $(a)_q = (a)_q\Gamma(a)_q$.

Proof: (1) $\Rightarrow$ (2) Let be $a \in M$, where a is a regular element. From [4] follows that the generated principal right ideal has the form $(a)_r = axM$. The same way $(a)_l = Mya$. It follows that,

$$(a)_r\Gamma(a)_l = (axM)\Gamma(Mya) \subseteq (M\Gamma M)\Gamma(Mya) \subseteq (M\Gamma Mya \subseteq Mya = (a)_l.$$

In the same way is proved that,

$$(a)_r\Gamma(a)_l \subseteq (a)_l$$

which means $(a)_r\Gamma(a)_l \subseteq (a)_l \cap (a)_r$.

To bring the equivalence to the end suffices to accept that $(a)_q \subseteq (a)_r\Gamma(a)_l$. Let be $a \in (a)_q$ and because it is regular from the definition of a Γ -semigroup we have

$$a = (axb)ya \subseteq (a)_r\Gamma(a)_l$$

where $x, y \in \Gamma$, $b \in M$. This is true because $axb \in axM = (a)_r$ and $a \in (a)_l$.
The reverse (2) $\Rightarrow$ (1) is evident.
Prove that (1) $\Rightarrow$ (3). Let have e, f - two idempotents in Γ-semigroup M and $(a)_r = exM, (a)_l = Myf$, where $x, y \in \Gamma$, also $e = exe, f = fyf$ (Lema 1.2,Lemma 1.3, ON-Γ-SEMIGROUP - III, N.K.Saha). We have that

$$e = exe(exM)x(exM) \subseteq (a)_r\Gamma(a)_r \subseteq (exM)\Gamma(M\Gamma M) \subseteq exM = (a)_r$$

Also $(a)_r = exM = (exe)xM = ex(exM) \subseteq (a)_r\Gamma(a)_r$. Thus way we have:

$$(a)_r\Gamma(a)_r = (a)_r$$

This means that $(a)_r$ is an idempotent in Γ-semigroup M. In analogues way can be proved that

$$(a)_l\Gamma(a)_l = (a)_l$$

so $(a)_l$ is idempotent in M. The point three is evident as an intersection of two ideals respectively right and left.
(3) $\Leftrightarrow$ (2) From lemma for a regular Γ-semigroup we have

$$(a)_r = axM \text{ and } (a)_l = Mya.$$

Suppose that $a \in (a)_r = (a)_r\Gamma(a)_r = (axM)\Gamma(Mya) \subseteq a\Gamma M\Gamma a$; it follows that a is a regular element of M.
Reversely, if we suppose that a is a regular element, i.e. $a \in M$ then

$$a \in (a)_r\Gamma(a)_r = (ax_1M)x(ax_1M) = ax_1(Mxa)x_1M \subseteq ax_1M = (a)_r$$

On the other side let have $a \in (a)_r$, then we will have

$$a \in (a)_r = ex_1M \subseteq ex(ex1M) \subseteq (ex_1M)\Gamma(ex_1M) = (a)_r\Gamma(a)_r$$

In the same way can be proved that $(a)_l = (a)_l\Gamma(a)_l$, and the third point is evident.
(2) $\Rightarrow$ (4) We have that $(a)_q = (a)_r \cap (a)_l = (a)_r\Gamma(a)_l$, where $(a)_q$ is a principal quasi-ideal satisfying the condition $(a)_q\Gamma M \cup M\Gamma(a)_q \subseteq (a)_q$.
$(a)_q\Gamma M\Gamma(a)_q \subseteq (a)_q\Gamma M\Gamma M \subseteq (a)_q\Gamma M(a)_q\Gamma M\Gamma(a)_q \subseteq M\Gamma M\Gamma(a)_q \subseteq \Gamma M(a)_q$.
Therefore

$$(a)_q\Gamma M\Gamma(a)_q \subseteq (a)_q\Gamma M \cap \Gamma M(a)_q \subseteq (a)_q$$

Also, $(a)_q = (a)_r \cap (a)_l = (a)_r\Gamma(a)_l = (axM)\Gamma(Mya) \subseteq (a)_q\Gamma M\Gamma(a)_q$, because $a \in (a)_q$.
(4) $\Rightarrow$ (1). We proved that

$$(a)_r \cap (a)_l = (a)_q = (a)_r\Gamma(a)_l$$

so sufficies to prove that a is regular. But $(a)_q = (a)_q\Gamma M\Gamma(a)_q$, hence, if $a \in (a)_q$ then

$$a \in axMya \subseteq a\Gamma M\Gamma a$$

So a is regular element of M.

Remark. A Γ-semigroup M is regular if the conditions $(1), (2), (3), (4)$ are met for all the elements a of M.

The following theorem describes an important property of the quasi-ideals in a Γ-semigroup M.

Theorem 1.12. *Every two-sided ideal of a regular Γ-semigroup M is a regular Γ-subsemigroup of M.*

Proof: Let have $a \in I$, any element. From $a \in M$, we have that $a = (axb)ya$ where $b \in M$ and $x, y \in \Gamma$, hence

$$a = (axb)ya = (((axb)ya)xb)ya \tag{1}$$

Considering the element $axa \in I$, then $a = (ax((bya)xb)ya$. If we prove that $b_1 = (bya)xb \in I$ then $a = (axb)ya \in a\Gamma I\Gamma a$ and so, a is regular. For $b_1 = (bya)xb$ have $bya \in M\Gamma I \subseteq I$ hence $(bya)xb \in I\Gamma M \subseteq I$. Because for every $a \in I$ is regular then the ideal I is regular. The two-sided ideal I is Γ-subsemigroup of M because $I\Gamma I \subseteq I\Gamma M \subseteq I$. From this theorem we have a corollary giving a very important property of quasi-ideals in a Γ-semigroup M.

We have proved an important corollary in a Γ-semigroup that every quasi-ideal is a bi-ideal, but its riverse statement is not true in general. The following corollary contains the conditions that the vice-versa statement be true.

Corollary 1.2. *Let M be a regular Γ-semigroup. The four following propositions are true:*

1. *Every quasi-ideal Q of Γ-semigroup M can be written in the form*
$$Q = R \cup L = R\Gamma L$$
where $R(L)$ are right (left) ideals of a Γ-Semigroup M generated by Q.
2. *If Q is a quasi-ideal of Γ-semigroup M then*
$$Q = Q\Gamma Q$$
3. *Every bi-ideal of Γ-semigroup M is a quasi-ideal of M.*
4. *Every bi-ideal of a two-sided ideal of Γ-semigroup M is a quasi-ideal of Γ-semigroup M.*

Proof (1) *This is evident by Theorem 1, because Q is a generating set for R and L.*

(2) *Suffices to prove that $Q = Q\Gamma Q$, because being regular have that $Q = Q\Gamma Q\Gamma Q$. Having that Q is a Γ-subsemigroup follows that $Q \subseteq Q\Gamma Q$. On the other side we have that $Q = Q\Gamma Q\Gamma Q \subseteq (Q\Gamma Q)\,\Gamma Q \subseteq Q\Gamma Q$.*

(3) *Let B be a bi-ideal of Γ-semigroup M, it follows that $B\Gamma M\Gamma B \subseteq B$. To prove that it is a quasi-ideal of M suffices to prove that $B\Gamma M \cup M\Gamma B \subseteq B$. We have that $(B\Gamma M)\Gamma M = B\Gamma(M\Gamma M) \subseteq B\Gamma M$ and $M\Gamma(M\Gamma M)\Gamma M = (M\Gamma M)\Gamma B \subseteq M\Gamma B$, which tells that $B\Gamma M$ and $M\Gamma B$ are respectively right*

and left ideals of M. From Theorem 1 we have: $B\Gamma M \cup M\Gamma B \subseteq (B\Gamma M)\Gamma(M\Gamma B) \subseteq B\Gamma M\Gamma B \subseteq B$ (what we wanted).

(4) *Let B be a bi-ideal of the two-sided ideal I of Γ-semigroup M. From the Theorem 3.2.2. we have that the bi-ideal B of I is bi-ideal of Γ-semigroup M, so from point* (3) *we have the quasi-ideal of Γ-semigroup M.*

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