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# On Fourier Parabolic and Wave Equations

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**Abstract**

This paper is devoted to a diffusion (heat) and wave equations for the function of two independent variables. We establish a criterion for existence and uniqueness of the solution of Fourier parabolic equation using Taylor series. A formal solution for a wave equation is investigated.

**Mathematics Subject Classifications**: 35A10, 58A99

**Keywords**: Euler function, formal solution, Fourier parabolic (diffusion or heat) equation, holomorphic function, Taylor expansion, wave equation.

# 1 Introduction

The purpose of this article is to study some partial differential equations [2] including diffusion and wave equations. We give the existence and uniqueness of the solution of diffusion equation also called Fourier parabolic or heat equation. A formal solution for wave equation is shown to be unique and the convergence depends on the initial conditions. The series theory is used for testing the convergence of the solution to our problems, and iterative method gives the relation between the coefficients of the Taylor series [1, 4, 5].

The following problem $(P)$ was studied by [2] and [6],

$$(P)\begin{cases} D^{\beta}u = \sum\limits_{|\alpha|\leq m} a_{\alpha}D^{\alpha}u + f \\ D_j^k(u-\varphi)_{|\Omega'_j} = 0, (j,k) \in \mathcal{I}_{\beta}, \end{cases}$$

where $(a_\alpha)_{|\alpha|\le m}$, $f$ and $\varphi$ are arbitrary holomorphic functions. The initial values are supported by the hypersurface $z_j = 0$. We study the following Fourier parabolic equation using results from [2] and [6]:

$$(P')\begin{cases} D_0 u = D_1^2 u \\ u\,(0, z_1) = u_0(z_1). \end{cases}$$

and find a formal solution of the wave equation

$$(P'')\begin{cases} D_0^2 u - D_1^2 u = 0 \\ u\,(0, z_1) = u_0\,(z_1) = \varphi\,(z_1) \\ D_0 u(0, z_1) = \psi\,(z_1)\,. \end{cases}$$

We begin by presenting some basic notations. Let $\mathcal{C}^{n+1}$ be the $(n+1)$-dimensional complex space with variables $z = (z_0, z_1, ..., z_n)$ and $\Omega$ be an open subset of $\mathbb{C}^{n+1}$ containing the origin. We use the standard multi-index notation. More precisely, let $\mathbb{Z}$ be the set of integers, $> 0$ or $\le 0$, and $\mathbb{Z}_+$ be the set of integers $\ge 0$. Then $\mathbb{Z}_+^{n+1}$ is the set of all $\alpha = (\alpha_0, \alpha_1, ..., \alpha_n)$ with $\alpha_j \in \mathbb{Z}_+$ for each $j = 0, 1, ..., n$. The length of $\alpha \in \mathbb{Z}_+^{n+1}$ is $|\alpha| = \alpha_0 + \alpha_1 + ... + \alpha_n$. We write

$$D^\alpha = \left(\frac{\partial}{\partial z_0}\right)^{\alpha_0}\left(\frac{\partial}{\partial z_1}\right)^{\alpha_1}...\left(\frac{\partial}{\partial z_n}\right)^{\alpha_n},$$

and use the notation $D_j = \frac{\partial}{\partial z_j}$. Also, let $D^\alpha = D_0^{\alpha_0} D_1^{\alpha_1}...D_n^{\alpha_n}$. Let $u$ be a continuous function in $\Omega$. By $\mathcal{C}^k(\Omega)$, $k \in \mathbb{Z}_+$, $0 \le k \le \infty$, we denote the set of all functions $u$ defined in $\Omega$, whose derivatives $D^\alpha u\,(z)$ exist and continuous for $|\alpha| \le k$. If $u \in \mathcal{C}^\infty(\Omega)$, we may consider the Taylor expansion at the origin

$$u\,(z) = \sum_{\alpha \in \mathbb{Z}_+^{n+1}} \frac{D^\alpha u(0)}{\alpha!} z^\alpha. \tag{1}$$

By $\mathcal{H}(\Omega)$ we denote the set of all holomorphic functions in $\Omega$, that is, functions $u(z) \in \mathcal{C}^\infty(\Omega)$ given by their Taylor expansion in some neighborhood of the origin in $\Omega$. A linear partial differential operator $P(z; D)$ is defined by

$$P\,(z; D) = \sum_{|\alpha| \le m} a_\alpha\,(z)\, D^\alpha \tag{2}$$

where the coefficients $a_\alpha(z)$ are in $\mathcal{H}(\Omega)$. If for some $\alpha$ of length $m$ the coefficient $a_\alpha\,(z)$ does not vanish identically in $\Omega$, $m$ is called the order of $P(z; D)$. We set $\mathcal{I}_\beta = \{(j, k) : j = 0, 1, ..., n, \text{and } k = 0, 1, ..., \beta_j - 1\}$ and adopt the following plan:

**1.** Existence and uniqueness of the solution of a diffusion equation: the parabolic case.

**2.** Uniqueness of the formal solution of wave equation: the hyperbolic case.

## 2 Existence and uniqueness of the solution of a diffusion equation: parabolic case

We consider problem $(P)$ and ask the following question: What happens if we take less initial conditions $|\beta|$ than the order of the operator? Let us study the case of Fourier parabolic equation (heat equation):

$$(P')\begin{cases} D_0 u = D_1^2 u \\ u(0,z_1) = u_0(z_1) = v(z_1), \end{cases}$$

with $z = (z_0, z_1)\,, \beta = (1,0), |\beta| = 1$ and $m = 2$.

We start by investigating the uniqueness of the solution of the problem $P'$, and suppose that it has a holomorphic solution in neighborhood of the origin, then we look for this solution under the form

$$u(z_0,z_1) = \sum_{j=0}^{\infty} u_j(z_1)\frac{z_0^j}{j!}, \tag{3}$$

where $u_j \in \mathcal{H}(\Omega_0')$. We have

$$D_1^2 u\,(z_0,z_1) = \sum_{j=0}^{\infty} u_j^{(2)}\,(z_1)\,\frac{z_0^j}{j!}, \tag{4}$$

and

$$D_0 u\,(z_0,z_1) = \sum_{j=0}^{\infty} u_{j+1}\,(z_1)\,\frac{z_0^j}{j!}. \tag{5}$$

Hence $D_0u(z_0,z_1) = D_1^2 u(z_0,z_1)$ can be written as

$$\sum_{j=0}^{\infty} u_{j+1}(z_1)\frac{z_0^j}{j!} = \sum_{j=0}^{\infty} u_j^{(2)}(z_1)\frac{z_0^j}{j!}. \tag{6}$$

By identification, (see equation (6)), we have

$$u_{j+1}\,(z_1) = u_j^{(2)}\,(z_1)\,, \forall j \in \mathbb{Z}_+. \tag{7}$$

For $j = 0$, (7) can be written as

$$\begin{aligned} u_1\,(z_1) &= u_0^{(2)}\,(z_1) \\ &= v^{(2)}\,(z_1)\,. \end{aligned} \tag{8}$$

For $j = 1$, (7) can be written as

$$\begin{aligned} u_2(z_1) &= u_1^{(2)}(z_1) \\ &= v^{(2)}(z_1), \end{aligned} \tag{9}$$

and (8) leads to

$$u_2(z_1) = v^{(4)}(z_1). \tag{10}$$

For $j = k$, we have

$$u_k(z_1) = u_1^{(2)}(z_1) = v^{(2)}(z_1), \tag{11}$$

and (8) gives

$$\begin{aligned} u_k(z_1) &= u_0^{(2k)}(z_1) \\ &= v^{(2k)}(z_1). \end{aligned} \tag{12}$$

We can prove easily by induction on $k$ that

$$u_k(z_1) = v^{(2k)}(z_1), \forall k \in \mathbb{Z}_+, \tag{13}$$

in other words $u_k$ are uniquely determined by the formula (13). Consequently if $u$ exists, it is necessarily defined by the following formula:

$$u(z_0, z_1) = \sum_{j=0}^{\infty} v^{(2j)}(z_1) \frac{z_0^j}{j!}. \tag{14}$$

Next we study the existence of the solution. We claim that we do not have the existence of the solution for each data $v$. As examples, consider

(A) $v(z_1) = \frac{1}{1-z_1}$; and

(B) $v(z_1) = e^{z_1}$.

Let us start by studying the example (A)

$$v(z_1) = \frac{1}{1 - z_1}. \tag{15}$$

and compute the derivatives $v^{(2j)}(z_1)$:

$$\begin{aligned} v^{(1)}(z_1) &= \frac{1}{(1 - z_1)^2} \\ &= \frac{1!}{(1 - z_1)^{1+1}}, \end{aligned} \tag{16}$$

$$
\begin{aligned}
v^{(2)}(z_1) &= \frac{2}{(1-z_1)^3} \\
&= \frac{2!}{(1-z_1)^{2+1}}.
\end{aligned}
\tag{17}
$$

We obtain the following formula by induction on $k$:

$$
v^{(k)}(z_1) = \frac{k!}{(1-z_1)^{k+1}}. \tag{18}
$$

For $k = 2j$, (18) becomes:

$$
v^{(2j)}(z_1) = \frac{(2j)!}{(1-z_1)^{2j+1}}, \tag{19}
$$

hence (14) gives

$$
\begin{aligned}
u(z_0, z_1) &= \sum_{j=0}^{\infty} \frac{(2j)!}{j!} \frac{z_0^j}{(1-z_1)^{2j+1}} \\
&= \frac{1}{1-z_1} \sum_{j=0}^{\infty} \frac{(2j)!}{j!} \left( \frac{z_0}{(1-z_1)^2} \right)^j.
\end{aligned}
\tag{20}
$$

Now consider the following Taylor expansion of the solution $u(z_0, z_1)$

$$
u(z_0, z_1) = \sum_{j=0}^{\infty} \frac{(2j)!}{j!} Z^j, \tag{21}
$$

then by D'Alembert rule this series has a radius of convergence which is zero, hence $u(z_0, z_1)$ does not converge $\left( \frac{z_0}{(1-z_1)^2} \longmapsto Z \right)$, that is, the problem $P'$ does not have a holomorphic solution in a neighborhood of the origin.

(B) Now, we study the second example

$$
v(z_1) = e^{z_1}. \tag{22}
$$

Note that

$$
v^{(2j)}(z_1) = e^{z_1}, \tag{23}
$$

therefore $u\left(z_0, z_1\right)$ can be written as

$$\begin{aligned} u\left(z_0, z_1\right) &= \sum_{j=0}^{\infty} e^{z_1} \frac{z_0^j}{j!} \\ &= e^{z_1} \sum_{j=0}^{\infty} \frac{z_0^j}{j!} \\ &= e^{z_1} e^{z_0} \\ &= e^{z_0+z_1}, \end{aligned} \tag{24}$$

which is a holomorphic function on $\mathcal{C}^2$, in other words, the problem $P'$ admits a unique solution. In conclusion, we have the uniqueness but we do not have existence for every data.

The following theorem gives a class for which we have a solution.

**Theorem 2.1** *If the initial condition $v$ satisfies the following condition: there is $K$ compact neighborhood of the origin of $\mathcal{C}$, there are two positive constants $M$ and $\rho$ such that*

$$\left|v^{(j)}\left(z_1\right)\right| \leq M \rho^j \Gamma\left(\frac{j}{2}+1\right), \forall j \in \mathbb{Z}_+, \forall z_1 \in K. \tag{25}$$

*Then $u$ defined by*

$$u\left(z_0, z_1\right)=\sum_{j=0}^{\infty} u_j\left(z_1\right) \frac{z_0^j}{j!} \tag{26}$$

*is the unique solution of the problem $P'$ on certain connected open neighborhood of the origin with $u\left(0, z_1\right)=u_0\left(z_1\right)=v\left(z_1\right)$, where $\Gamma$ denotes the Euler function.*

*Proof:* We have by (18):

$$u_j\left(z_1\right)=v^{(2j)}\left(z_1\right), \forall j \in \mathbb{Z}_+. \tag{27}$$

For $z_1 \in K$, we have

$$\left|v^{(2j)}\left(z_1\right)\right| \leq M \rho^{2j} \Gamma\left(j+1\right). \tag{28}$$

Using the property of Euler function $\Gamma$, one can write

$$\left|v^{(2j)}\left(z_1\right)\right| \leq M \rho^{2j} j! \tag{29}$$

for every $z_1 \in K$. Hence

$$\left|\frac{z_0^j}{j!} v^{(2j)}\left(z_1\right)\right| \leq M\left(\rho^2\left|z_0\right|\right)^j. \tag{30}$$

Consequently, this series converges for $\left|z_0\right|<\frac{1}{\rho^2}$ which completes the proof of the Theorem.

# 3 Uniqueness of the formal solution of wave equation: hyperbolic case

In this section, we study the uniqueness of the formal solution of wave equation:

$$(P^{***}) \begin{cases} D_0^2 u - D_1^2 u = 0 \\ u(0, z_1) = u_0(z_1) = \varphi(z_1) \\ D_0 u(0, z_1) = \psi(z_1), \end{cases}$$

where $z = (z_0, z_1)$, $\beta = (2, 0)$, $|\beta| = 2$, $m = 2$.

Let us prove the uniqueness of the formal solution of the problem $(P^{***})$. To do so, suppose that this solution exists, we look for it as the Taylor expansion

$$u(z_0, z_1) = \sum_{j=0}^{\infty} u_j(z_1) \frac{z_0^j}{j!}, \tag{31}$$

where $u_j \in \mathcal{H}(\Omega_0')$. The proof is similar to the method given in the previous section. We have

$$D_1^2 u(z_0, z_1) = \sum_{j=0}^{\infty} u_j^{(2)}(z_1) \frac{z_0^j}{j!}, \tag{32}$$

and

$$D_0^2 u(z_0, z_1) = \sum_{j=0}^{\infty} u_{j+2}(z_1) \frac{z_0^j}{j!}. \tag{33}$$

Hence $D_0 u(0, z_1) = u_1(z_1) = \psi(z_1)$ from the third equation of $P^{***}$. The first equation of $P^{***}$ becomes

$$\sum_{j=0}^{\infty} u_{j+2}(z_1) \frac{z_0^j}{j!} = \sum_{j=0}^{\infty} u_j^{(2)}(z_1) \frac{z_0^j}{j!}. \tag{34}$$

By identification of the coefficients, (34) is equivalent to

$$u_{j+2}(z_1) = u_j^{(2)}(z_1), \forall j \in \mathbb{Z}_+. \tag{35}$$

For $j = 0$, (35) can be written as

$$u_2(z_1) = u_0^{(2)}(z_1) = \varphi^{(2)}(z_1). \tag{36}$$

For $j = 1$, (35) can be written as

$$\begin{aligned} u_3(z_1) &= u_1^{(2)}(z_1) \\ &= \psi^{(2)}(z_1). \end{aligned} \tag{37}$$

For $j = 2$, (35) can be written as

$$\begin{aligned} u_4(z_1) &= u_2^{(2)}(z_1) \\ &= \varphi^{(4)}(z_1). \end{aligned} \tag{38}$$

One can check easily that

$$u_{2k}(z_1) = \varphi^{(2k)}(z_1), \qquad \text{and} \qquad u_{2k+1}^{(2)}(z_1) = \psi^{(2k)}(z_1), \tag{39}$$

$\forall k \in \mathbb{Z}_+$. Hence $u_k$ are uniquely determined by the formula (39) and we have

$$u(z_0, z_1) = \sum_{j=0}^{\infty} \left[ \varphi^{(2j)}(z_1) \frac{z_0^{2j}}{(2j)!} + \psi^{(2j+1)}(z_1) \frac{z_0^{2j+}}{(2j+1)!} \right], \tag{40}$$

and the proof is complete.

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