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Some Fixed Point and Coincident Point Theorem in Generalized $\mathcal{M}$ - Fuzzy Metric Space

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Abstract

The aim of this paper is to establish fixed-point theorems and coincident point theorem for generalized contraction mappings in generalized $\mathcal{M}$ - fuzzy metric spaces.

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1. INTRODUCTION

In 1965, Zadeh introduced the famous theory of fuzzy sets and used it as a tool for dealing with uncertainty arising out of lack of information about certain complex system. Fixed point theorems in fuzzy mathematics are emerging with vigorous hope and vital trust. It appears that Kramosil and Michalek's study of fuzzy metric spaces paves a way for very soothing machinery to develop fixed point theorems for contractive type maps.

Recently Sedghi and Shobe [8] introduced D* - metric space as a probable modification of the definition of D - metric introduced by Dhage, and prove some basic properties in D* metric spaces. Using D*- metric concepts, They [8] define $\mathcal{M}$ - fuzzy metric space and proved a common fixed point theorem in it.

In this paper we prove fixed point theorem for contractive mappings and common fixed point theorem for three maps in the setting of generalized $\mathcal{M}$ -fuzzy metric space.

2. PRELIMINARIES

2.1 Definition

A fuzzy set $\mathcal{M}$ in an arbitrary set X is a function with domain X and values in [0, 1].

2.2 Definition

A mapping *: [0, 1] x [0, 1] $\rightarrow$ [0, 1] is called a triangular norm (shortly t-norm) if it satisfies the following conditions:

(i) $*(a, 1) = a$ for every $a \in [0, 1]$

(ii) $*(a, b) = *(b, a)$ for every $a, b \in [0, 1]$

(iii) $*(a, c) \geq *(b, d)$ for $a \geq b$; $c \geq d$

(iv) $*(a, *(b, c,)) = *(*(a, b), c)$ for all $a, b, c \in [0,1]$

2.3 Example

$a * b = ab$ for $a, b \in [0, 1]$.

2.4 Example

$a * b = \min\{a, b\}$ for $a, b \in [0, 1]$.

2.5 Definition

The triple (X, $\mathcal{M}$, $*$) is a $\mathcal{M}$ -fuzzy metric space if X is an arbitrary set, $*$ is a continuous t - norm and $\mathcal{M}$ is a fuzzy set in X^3 x $(0,\infty)$ satisfying the following conditions for each x, y, z, a $\in$ X and t, s > 0

(1) $\mathcal{M}(x, y, z, t) > 0$, for all $x, y, z \in X$

(2) $\mathcal{M}(x, y, z, t) = 1$ iff $x = y = z$, for all $t > 0$,

(3) $\mathcal{M}(x, y, z, t) = \mathcal{M}(p\{x, y, z\}, t)$, where p is a permutation function,

(4) $\mathcal{M}(x, y, a, t) * \mathcal{M}(a, z, z, s) \leq \mathcal{M}(x, y, z, t + s)$,

(5) $\mathcal{M}(x, y, z, .) : [0, \infty) \to [0, 1]$ is continuous.

2.6 Lemma

Let (X, $\mathcal{M}$, $*$) be a $\mathcal{M}$ – fuzzy metric space. Then for every t > 0 and for every x, y $\in$ X we have $\mathcal{M}(x, x, y, t) = \mathcal{M}(x, y, y, t)$.

Proof:

For each $\in > 0$ by triangular inequality

We have

(i) $\mathcal{M}(x, x, y, \in + t) \geq \mathcal{M}(x, x, x, \in) * \mathcal{M}(x, y, y, t)$
$= \mathcal{M}(x, y, y, t)$

(ii) $\mathcal{M}(y, y, x, \in + t) \geq \mathcal{M}(y, y, y, \in) * \mathcal{M}(y, x, x, t)$
$= \mathcal{M}(y, x, x, t)$.

By taking limits of (i) and (ii) when $\in \to 0$,
we obtain $\mathcal{M}(x, x, y, t) = \mathcal{M}(x, y, y, t)$

2.7 Definition

Let X be a nonempty set. A generalized metric (or D* -metric) on X is a function: $D^*: X^3 \to [0,\infty)$ that satisfies the following conditions for each x, y, z $\in$ X.

(1) $D^*(x, y, z) \geq 0$,

(2) $D^*(x, y, z) = 0$ if and only if $x = y = z$,

(3) $D^*(x, y, z) = D^*(p\{x, y, z\})$, (symmetry) where p is a permutation function,

(4) $D^*(x, y, z) \leq D^*(x, y, a) + D^*(a, z, z)$.

The pair (X, D*) is called a generalized metric (or D*-metric) space.

2.8 Definition

Let (X, $\mathcal{M}$, $*$) be a $\mathcal{M}$ - fuzzy metric space, For t > 0, the open ball $B_{\mathcal{M}}(x, r, t)$ with center x $\in$ X and radius o < r < 1 is defined by

$B_{\mathcal{M}}(x, r, t) = \{ y \in X : \mathcal{M}(x, y, y, t) > 1 - r\}$. A subset A of X is called open set if for each $x \in A$ there exist $t > 0$ and $0 < r < 1$ such that $B_{\mathcal{M}}(x, r, t) \subseteq A$.

2.9 Definition

A sequence $\{x_n\}$ in X converges to x if and only if $\mathcal{M}(x, x, x_n, t) \to 1$ as $n \to \infty$, for each $t > 0$. It is called a Cauchy sequence if for each $0 < \in < 1$ and $t > 0$, there exist $n_0 \in N$ such that $\mathcal{M}(x_n, x_n, x_m, t) > 1 - \in$ for each $n, m \geq n_0$. Then $\mathcal{M}$ - fuzzy metric (X, $\mathcal{M}$, *) is said to be complete if every Cauchy sequence is convergent.

2.10 Lemma [8]

Let (X, $\mathcal{M}$, *) be a $\mathcal{M}$ – fuzzy metric space. Then $\mathcal{M}(x, y, z, t)$ is nondecreasing with respect to t, for all x, y, z in X.

2.11 Example

Let X be a nonempty with D*- metric on X. Denote a * b = a.b for all a, b $\in$ [0, 1]. For each $t \in (0, \infty)$, Define

$$\mathcal{M}(x, y, z, t) = \frac{t}{t + D^*(x, y, z)}$$

2.12 Theorem

Let (X, $\mathcal{M}$, *) be a generalized $\mathcal{M}$ -fuzzy metric space and $T : X \to X$ be a mapping such that for all $x, y, z \in X$ and $t > 0$,With $x \neq y$ or $y \neq z$ or $z \neq x$
$\mathcal{M}(Tx, Ty, Tz, t) > \min\{ \mathcal{M}(x, y, z, t), \mathcal{M}(x, Tx, Ty, t), \mathcal{M}(z, Ty, Tz, t)\}$ for any point $x_0 \in X$ such that sequence $\{ T^n (x_0)\}$ has a subsequence converges to u. Then u is unique fixed point of T.

Proof:

Let $x_0 \in X$ be any arbitrary fixed element in X.
Then there exists $x_1 \in X$ such that $x_1 = Tx_0$
Similarly there exists $x_2 \in X$ such that $x_2 = Tx_1 = T^2x_0$
Continuing in this way get a sequence $x_n = T^nx_0$ for all $n \geq 1$ in X.
Suppose $x_n = x_{n+1}$ for some n.
Then $x_n = Tx_n$, Thus $x_n = u$ is a fixed point of T
Let us assume that $x_n \neq x_{n+1}$ for all n.
For $n \geq 1$, We have
$\mathcal{M}(x_n, x_n, x_{n+1}, t) = \mathcal{M}(Tx_{n-1}, Tx_{n-1}, Tx_n, t)$

$> \min \{ \mathcal{M}(x_{n-1}, x_{n-1}, x_n, t), \mathcal{M}(x_{n-1}, Tx_{n-1}, Tx_{n-1}, t),$
$\mathcal{M}(x_n, Tx_{n-1}, Tx_n, t)\}.$
$= \min \{ \mathcal{M}(x_{n-1}, x_{n-1}, x_n, t), \mathcal{M}(x_{n-1}, x_n, x_n, t), \mathcal{M}(x_n, x_n, x_{n+1}, t)\}.$

Hence $\mathcal{M}(x_n, x_n, x_{n+1}, t) > \mathcal{M}(x_{n-1}, x_{n-1}, x_n, t)$ for all $n \geq 1$.
Thus $\{ \mathcal{M}(x_n, x_n, x_{n+1}, t) \}$ is monotonically increasing sequence of positive real numbers bounded above by 1, It is convergent to a positive real number, say L.
Therefore $\lim_{n\to\infty} \mathcal{M}(x_n, x_n, x_{n+1}, t) = L$
Also the sequence $\mathcal{M}(x_n, x_n, x_{n+1}, t)\}$ has a sub sequence $\{ \mathcal{M}(x_{n_k}, x_{n_k}, x_{n_k+1}, t) \}$ converges to L
$\lim_{k\to\infty} \mathcal{M}(x_{n_k}, x_{n_k}, x_{n_k+1}, t) = L$
To prove that L = 1
Suppose L < 1
Since $x_n = T^n x_o$ has a subsequence x_{n_k} converges to u
We have $\lim_{k\to\infty} \mathcal{M}(x_{n_k}, x_{n_k}, u, t) = 1$ $\rightarrow$ (2.12.1)

$$1 > L = \lim_{k\to\infty} \mathcal{M}(x_{n_k}, x_{n_k}, x_{n_k+1}, t)$$
$$\geq \lim_{k\to\infty} \{ \mathcal{M}(x_{n_k}, x_{n_k}, u, t/2) * \mathcal{M}(u, x_{n_k+1}, x_{n_k+1}, t/2)$$
$$= 1 * 1 \quad \text{using } (2.12.1)$$

Now $1 > L = \ldots$ (as above)

Which is contradiction.
Therefore $\lim_{k\to\infty} \mathcal{M}(x_{n_k}, x_{n_k}, x_{n_k+1}, t) = 1$
Now we prove u is fixed point of T.
Suppose $u \neq T(u)$ we have

$$\mathcal{M}(u, u, Tu, t) = \lim_{k\to\infty} \mathcal{M}(x_{n_k+1}, x_{n_k+2}, Tu, t)$$
$$= \lim_{k\to\infty} \mathcal{M}(Tx_{n_k}, Tx_{n_k+1}, Tu, t)$$
$$> \lim_{k\to\infty} \min \{ \mathcal{M}(x_{n_k}, x_{n_k+1}, u, t), \mathcal{M}(x_{n_k}, Tx_{n_k}, Tx_{n_k+1}, t),$$
$$\mathcal{M}(u, Tx_{n_k+1}, Tu, t)\}$$
$$> \lim_{k\to\infty} \min \{ \mathcal{M}(x_{n_k}, x_{n_k+1}, u, t), \mathcal{M}(x_{n_k}, x_{n_k+1}, x_{n_k+2}, t),$$
$$\mathcal{M}(u, x_{n_k+2}, Tu, t)\}$$
$$= \min \{ \mathcal{M}(u, u, u, t), \mathcal{M}(u, u, u, t), \mathcal{M}(u, u, Tu, t) \}$$
$$= \mathcal{M}(u, u, Tu, t)$$

Which is contradiction.
Thus $u = Tu$.
Uniqueness: Suppose there exists $v \in X$ such that $Tv = v$ and $v \neq u$.
Now consider $\mathcal{M}(u, u, v, t) = \mathcal{M}(Tu, Tu, Tv, t)$

$$> \min \{ \mathcal{M}(u, u, v, t), \mathcal{M}(u, Tu, Tu, t), \mathcal{M}(v, Tu, Tv, t)\}$$
$$= \min \{ \mathcal{M}(u, u, v, t), \mathcal{M}(u, u, u, t), \mathcal{M}(v, u, v, t)\}$$
$$> \mathcal{M}(u, u, v, t)$$

which is contradiction. Therefore u is a unique fixed point of T.

2.13 Theorem

Let (X, $\mathcal{M}$, *) be a generalized fuzzy metric space and T : X→X be a mapping such that for all x ∈ X and t > 0. $\mathcal{M}(Tx, T^2x, T^3x, t) > \mathcal{M}(x, Tx, T^2x, t)$ for any $x_o \in X$ the sequence $x_n = (T^n x_o)$ has a subsequence (x_{n_k}) converges to u. Then u is unique fixed point of T.

Proof:

Let $x_o \in X$ be any arbitrary fixed element in X
Define the sequence $x_{n+1} = Tx_n = T^{n+1}x_o$ for all n ≥ 1.
If for some $x_{n+1} = x_n$.
Thus $Tx_n = x_n$.
Hence $u = x_n$ fixed point of T.
We assume that $x_{n+1} \neq x_n$ for all n.
For n ≥ 1 We have
$\mathcal{M}(x_n, x_n, x_{n+1}, t) = \mathcal{M}(Tx_{n-1}, Tx_{n-1}, Tx_n, t)$
$> \mathcal{M}(x_{n-1}, x_{n-1}, x_n, t)$
Thus { $\mathcal{M}(x_n, x_n, x_{n+1}, t)$} is monotonically increasing sequence of positive real numbers bounded above by 1, it is convergent a positive real number, say L ≤ 1.
Hence { $\mathcal{M}(x_n, x_n, x_{n+1}, t)$} has a subsequence { $\mathcal{M}(x_{n_k}, x_{n_k}, x_{n_k+1}, t)$ } which is convergent to l.
Also the sequence $\mathcal{M}(x_n, x_n, x_{n+1}, t)$} has a sub sequence { $\mathcal{M}(x_{n_k}, x_{n_k}, x_{n_k+1}, t)$ } that is converges to L
$\lim_{k\to\infty} \mathcal{M}(x_{n_k}, x_{n_k}, x_{n_k+1}, t) = L$
Now, we prove that L = 1
Suppose L < 1.
Since {x_n } has a subsequence { x_{n_k} }converges to u
We have $\lim_{k\to\infty} \mathcal{M}(x_{n_k}, x_{n_k}, u, t) = 1$
We have $1 > L = \lim_{k\to\infty} \mathcal{M}(x_{n_k}, x_{n_k}, x_{n_k+1}, t)$
$\geq \lim_{k\to\infty} \{ \mathcal{M}(x_{n_k}, x_{n_k}, u, t/2) * \mathcal{M}(u, x_{n_k+1}, x_{n_k+1}, t/2)\}$
$= 1 * 1$
Which is contradiction
Therefore $\lim_{k\to\infty} \mathcal{M}(x_{n_k}, x_{n_k}, x_{n_k+1}, t) = 1$
Now we have prove u is fixed point of T.
That is u = Tu.
Consider for all t > 0
$\mathcal{M}(u, u, Tu, t) = \lim_{k\to\infty} \mathcal{M}(x_{n_k+1}, x_{n_k+2}, Tu, t)$

$$= \lim_{k\to\infty} \mathcal{M}(Tx_{n_k}, Tx_{n_k+1}, Tu, t)$$
$$> \lim_{k\to\infty} \mathcal{M}(x_{n_k}, x_{n_k+1}, u, t)$$
$$= 1$$

Thus $\mathcal{M}(u, u, Tu, t) = 1$.
Therefore $u = Tu$
Now we prove the uniqueness
Suppose $u \neq v$ such that $Tv = v$
$\mathcal{M}(u, u, v, t) = \mathcal{M}(Tu, Tu, Tv, t)$
$> \mathcal{M}(u, u, v, t)$
This implies $u = v$

2.14 Remark

Putting $y = Tx$ and $z = Ty$ in the above theorem , we get following theorem as corollary.

2.15 Theorem
Let $(X, \mathcal{M}, *)$ be a generalized $\mathcal{M}$ -fuzzy metric space and $T : X\to X$ be a mapping such that for all $x , y , z \in X$ and $t > 0$,With $x \neq y$ or $y \neq z$ or $z \neq x$
$\mathcal{M}(Tx, T^2x, T^3x, t) > \min \{ \mathcal{M}(x, Tx, T^2x, t), \mathcal{M}(T^2x, T^2x, T^3x, t)\}$ for any point $x_0 \in X$ such that sequence $\{ T^n (x_0)\}$ has a subsequence converges to u. Then u is unique fixed point of T.

2.16 Theorem

Let $(X, \mathcal{M}, *)$ be a generalized complete $\mathcal{M}$ - fuzzy metric space and $T_1, T_2, T_3 : X \to X$ be any three mappings s such that $\mathcal{M}(T_1x, T_2T_1x, T_3T_2T_1x, t) \geq \mathcal{M}(x, T_1x, T_2T_1x, t)$ for all $x \in X$ with $x \neq T_1x \neq T_2T_1x$ and for all $t > 0$. If $x_0 \in X$ such that the sequence $x_n = T_3T_2T_1x_{n-3}$ for $n \geq 3$ has a subsequence $\{ x_{n_k} \}$ converges to u. Then u is a unique common fixed point T_1, T_2 and T_3.

Proof:

Let $x_o \in X$ be an arbitrary fixed element in X.
Define the sequence $\{x_n\}$ in X follows.

$x_1 = T_1x_o$
$x_2 = T_2x_1$
$x_3 = T_3x_2$
.
.
.

$x_n = T_3T_2T_1\, x_{n-3.}$

Now for $n \geq 1$

$\mathcal{M}(x_{n+3}, x_{n+2}, x_{n+2}, t) = \mathcal{M}(T_3T_2T_1x_n, T_2T_1x_n, T_2T_1x_n, t)$

$> \mathcal{M}(T_2T_1x_n, T_1\, x_n, T_1x_n, t)$

$= \mathcal{M}(x_{n+2}, x_{n+1}, x_{n+1}, t)$

Thus $\mathcal{M}(x_{n+2}, x_{n+1}, x_{n+1}, t)$ is monotonically increasing a sequence of positive real number bounded above by 1 it is convergent to a real number say L.

Therefore $\lim\limits_{n\to\infty} \mathcal{M}(x_{n+2}, x_{n+1}, x_{n+1}, t) = L$

Also the sequence $\{ \mathcal{M}(x_{n+2}, x_{n+1}, x_{n+1}, t)\}$ has a sub sequence

$\{ \mathcal{M}(x_{n_k+2}, x_{n_k+1}, x_{n_k+1}, t) \}$ converges to L

$\lim\limits_{k\to\infty} \mathcal{M}(x_{n_k+2}, x_{n_k+1}, x_{n_k+1}, t) = L$

To prove that L = 1

Suppose L < 1

Since $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ converges to u

We have $\lim\limits_{k\to\infty} \mathcal{M}(x_{n_k}, x_{n_k}, u, t) = 1 \qquad \to (2.16.1)$

Now $1 > L = \lim\limits_{k\to\infty} \mathcal{M}(x_{n_k+1}, x_{n_k+1}, x_{n_k+2}, t)$

$\geq \lim\limits_{k\to\infty} \{ \mathcal{M}(x_{n_k+1}, x_{n_k+1}, u, t/2) * \mathcal{M}(u, x_{n_k+2}, x_{n_k+2}, t/2)$

$= 1 * 1$ using (2.14.1)

Which is contradiction

Therefore L = 1.

Therefore $\lim\limits_{k\to\infty} \mathcal{M}(x_{n_k+1}, x_{n_k+1}, x_{n_k+2}, t) = 1$

Now we prove that u = Tu

Suppose $u \neq T_1u$.

Now consider $\mathcal{M}(u, u, T_1u, t) = \lim\limits_{n\to\infty} \mathcal{M}(x_{n_k+3}, x_{n_k+2}, T_1u, t)$

$= \lim\limits_{n\to\infty} \mathcal{M}(T_3T_2T_1\, x_{n_k}, T_2T_1\, x_{n_k}, T_1u, t)$

$\geq \lim\limits_{n\to\infty} \mathcal{M}(T_2T_1\, x_{n_k}, T_1\, x_{n_k}, u, t)$

$\geq \lim\limits_{n\to\infty} \mathcal{M}(x_{n_k+2}, x_{n_k+1}, u, t)$

$= 1$

There fore $\mathcal{M}(u, u, T_1u, t) = 1$ for each $t > 0$

Hence $u = T_1u$

Similarly we prove that $u = T_2\, T_1\, u = T_2u$ and $T_3\, T_2\, u = T_3u = u$

There u is common fixed point T_1, T_2 and T_3.

Uniqueness: Suppose $u \neq v$ such that $v = T_1v = T_2v = T_3v$.

$\mathcal{M}(u, v, v, t) = \mathcal{M}(T_3T_2T_1u, T_2T_1v, T_1v, t)$

$\geq \mathcal{M}(T_2T_1u, T_1\, v, v, t)$

$\geq \mathcal{M}(u, v, v, t)$

Which is contradiction

Hence u = v.

Thus u is a unique common fixed point of T_1, T_2 and T_3.
This completes the proof.

REFERENCES

[1] Dhage.B.C, Indian.J.pure.aple.math.(1989) 30(4),395.

[2] Grabiec.M, Fuzzy set system (1983) 27,385.

[3] Grabiec.M, Fixed points in fuzzy metric spaces, fuzzy sets and systems 27 (1988) 385 – 389

[4] Jungck.G, Compatible mappings and common fixed points, Internet J.math.& math.sci.9(1986) 771-779.

[5] Kramosil.O and Michalek.J, Fuzzy metric and statistical metric spaces, kybernetica, 11 (1975), 326 – 334.

[6] Naidu S.V.R,Rao.K.P.R and Srinivasa Rao.N. On the topology of D- metric spaces and the generation of D-metric spaces from metric spaces, Internet. J.Math.Math.Sci. (2004),No.51,2719-2740.

[7] Naidu S.V.R,Rao.K.P.R and Srinivasa Rao.N. On convergent sequences and fixed point theorems in d- metric spaces,Internet.J.Math.Math.Sci. 2005:12 (2005), 1969-1988.

[8] S.Sedghi and N.Shobe, Fixed point theorem in $\mathcal{M}$ – fuzzy metric spaces with property(E) , Advances in Fuzzy Mathematics, Vol.1, No.1 (2006), 55-65.

[9] Zadeh.L.A., Fuzzy sets, information and control, 8(1965) 338 –353.

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