

Special Finsler Spaces Admitting Metric Like Tensor Field

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Abstract

In this work we modify the special Finsler spaces like C-reducible, semi-C-reducible, quasi-C-reducible are admitting the tensor field $X_{hk} = h_{hk} + X_{00}l_hl_k$, which satisfies the condition $C_{ij}^h X_{hk} = C_{ijk}$. Similarly, we have also worked out for S3-like, C^h -recurrent, P-reducible and T-conditions of Finsler spaces.

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1 Introduction

The terminology and notations are referred to [1], [4] and [6]. Let $F^n = (M^n, L)$ be a Finsler space on a differentiable manifold M endowed with a fundamental function $L(x, y)$. We use the following notations: [4][6]

$$\begin{aligned} a) \quad g_{ij} &= \frac{1}{2} \dot{\partial}_i \dot{\partial}_j L^2, \quad \dot{\partial}_i = \frac{\partial}{\partial y^i}, \\ b) \quad C_{ijk} &= \frac{1}{2} \dot{\partial}_k g_{ij}, \end{aligned}$$

$$\begin{aligned}
c) \quad & h_{ij} = g_{ij} - l_i l_j, \\
d) \quad & C^h h_{hk} = C_k, C^h l_h = 0, \\
e) \quad & h_k^m h_{mj} = h_{jk}, h_j^m l_m = 0, \\
f) \quad & C_{hr}^m g_{mj} = C_{hjr}, \\
g) \quad & l_m l^m = 0, p^m l_m = 0, \\
h) \quad & h_j^m X_{mk} = X_{jk} - X_{k0} l_j, \\
i) \quad & X_{i0} l_j = X_{j0} l_i, X_{i0} = X_{00} l_i.
\end{aligned} \tag{1}$$

There are three kinds of torsion tensors in cartan's theory of Finsler spaces. Two of them are $h(h\nu)$ -torsion tensor C_{ijk} and $(\nu)h\nu$ -torsion tensor P_{ijk} , which are symmetric in all their indices. The contravariant components of $(\nu)h\nu$ -torsion tensor is given by $C_{ij}^h = g^{hk} C_{ijk}$, which may be treated as Christoffel symbols of second kind of each tangent Riemannian space of Finsler space F^n . Here, g^{hk} is the inverse of metric tensor g_{hk} of F^n . If l_i is the normalized element of support h_{ij} is the angular metric tensor given by $h_{ij} = g_{ij} - l_i l_j$, then

$$C_{ij}^h g_{hk} = C_{ijk} = C_{ij}^h h_{hk}. \tag{2}$$

If b_i are components of a concurrent vector field, then $b_{i/j} = -g_{ij}$ and $b_{i|j} = 0$, where $/j$ and $|j$ denote the h and ν -covariant derivatives with respect to cartan's connection CT . From this it follows that b_i are functions of position only, and $C_{ij}^h b_h = 0$. Thus if we consider a tensor field is given by $B_{ij} = g_{ij} + \alpha l_i l_j + \beta b_i b_j$, where α and β are scalar functions, then

$$C_{ij}^h B_{hk} = C_{ijk}. \tag{3}$$

The purpose of the present paper is to study the existence of any symmetric covariant tensor X_{hk} which satisfies

$$X_{hk} = h_{hk} + X_{00} l_h l_k. \tag{4}$$

Throughout the paper we are concerned with non-Riemannian Finsler space having positive definite metric tensor g_{ij} . From (4) we have,

$$C^h X_{hk} = C_k \quad \text{and} \quad C_{ij}^h X_{h0} = 0, \tag{5}$$

where $C^h = C_{ij}^h g^{ij}$ and 0 denotes the contraction with l^i .

2 The Existence Of Covariant Tensor X_{hk} In C-Reducible Finsler Space :

In a C-reducible Finsler space the $(h)h\nu$ -torsion tensor C_{ij}^h is given by, [2][5]

$$C_{ij}^h = (C^h h_{ij} + C_i h_j^h + C_j h_i^h)/(n+1). \tag{6}$$

Now contracting above equation by X_{hk} , then from equations (4) and (1)(d), we have

$$\begin{aligned}
C_{ij}^h X_{hk} &= X_{hk}(C^h h_{ij} + C_i h_j^h + C_j h_i^h)/(n+1), \\
C_{ij}^h X_{hk} &= 1/(n+1)[C^h h_{ij}(h_{hk} + X_{00} l_h l_k) + C_i h_j^h (h_{hk} + X_{00} l_h l_k) + C_j h_i^h (h_{hk} + X_{00} l_h l_k)] \\
C_{ij}^h X_{hk} &= 1/(n+1)[C_k h_{ij} + C_i h_{jk} + C_i h_j^h X_{00} l_h l_k + C_j h_{ik} + C_j h_i^h X_{00} l_h l_k] \\
C_{ij}^h X_{hk} &= 1/(n+1)[C_k h_{ij} + C_i h_{jk} + C_i X_{00} l_h l_k (\delta_j^h - l^h l_j) + C_j h_{ik} + C_j X_{00} l_h l_k (\delta_j^h - l^h l_j)] \\
C_{ij}^h X_{hk} &= 1/(n+1)[C_k h_{ij} + C_i h_{jk} + C_i X_{00} l_k (l_j - l_j) + C_j h_{ik} + C_j X_{00} l_k (l_i - l_i)] \\
C_{ij}^h X_{hk} &= 1/(n+1)[C_k h_{ij} + C_i h_{jk} + C_j h_{ik}] \\
C_{ij}^h X_{hk} &= C_{ijk}. \tag{7}
\end{aligned}$$

Theorem 2.1 *In a C-reducible Finsler space the covariant tensor field X_{hk} satisfies (4) is of the form (7).*

Consider a Semi-C-reducible Finsler space C_{ij}^h is given by [3],

$$C_{ij}^h = (C^h h_{ij} + C_i h_j^h + C_j h_i^h)p/(n+1) + (C^h C_i C_j)q/C^2. \tag{8}$$

Now contracting above equation by X_{hk} and using equations (4) and (1)(d), we have

$$\begin{aligned}
C_{ij}^h X_{hk} &= X_{hk}(C^h h_{ij} + C_i h_j^h + C_j h_i^h)p/(n+1) + X_{hk}(C^h C_i C_j)q/C^2, \\
C_{ij}^h X_{hk} &= (C^h h_{ij}(h_{hk} + X_{00} l_h l_k) + C_i h_j^h (h_{hk} + X_{00} l_h l_k) + C_j h_i^h (h_{hk} + X_{00} l_h l_k)) \\
&\quad p/(n+1) + (C^h C_i C_j)(h_{hk} + X_{00} l_h l_k)q/C^2, \\
C_{ij}^h X_{hk} &= (C_k h_{ij} + C_i h_{jk} + C_j h_{ik})p/(n+1) + (C_k C_i C_j)q/C^2, \\
C_{ij}^h X_{hk} &= C_{ijk}. \tag{9}
\end{aligned}$$

Theorem 2.2 *In a Semi-C-reducible Finsler space the tensor field X_{hk} satisfies (4) is of the form (9).*

Consider a Quasi-C-reducible Finsler space C_{ij}^h is given by [3],

$$C_{ij}^h = (C^h A_{ij} + C_i A_j^h + C_j A_i^h). \tag{10}$$

Now contracting above equation by X_{hk} and using equations (4) and (1)(d), we have

$$\begin{aligned}
C_{ij}^h X_{hk} &= (C^h A_{ij} + C_i A_j^h + C_j A_i^h)X_{hk}, \\
C_{ij}^h X_{hk} &= (C^h A_{ij}(h_{hk} + X_{00} l_h l_k) + C_i A_j^h (h_{hk} + X_{00} l_h l_k) \\
&\quad + C_j A_i^h (h_{hk} + X_{00} l_h l_k)), \\
C_{ij}^h X_{hk} &= (C_k A_{ij} + C_i A_{jk} + C_j A_{ik}), \\
C_{ij}^h X_{hk} &= C_{ijk}. \tag{11}
\end{aligned}$$

Theorem 2.3 *In a Quasi-C-reducible Finsler space the tensor field X_{hk} satisfies (4) is of the form (11) if $A^h l_h = 0$.*

3 The Existence Of Covariant Tensor X_{hk} In S3-Like Finsler Space:

In a S3-like Finsler space, whose ν -curvature tensor of cartons connection CT is given by [5],

$$L^2 S_{ihk}^m = S(h_{ih}h_k^m - h_{ik}h_h^m). \quad (12)$$

Contacting above equation by X_{mj} and using equations (4) and (1)(d), we have

$$\begin{aligned} L^2 S_{ihk}^m X_{mj} &= S(h_{ih}h_k^m X_{mj} - h_{ik}h_h^m X_{mj}), \\ L^2 S_{ihk}^m X_{mj} &= S[h_{ih}h_k^m (h_{mj} - X_{00}l_m l_j) - h_{ik}h_h^m (h_{mj} - X_{00}l_m l_j)], \\ L^2 S_{ihk}^m X_{mj} &= S[h_{ih}h_{jk} - X_{00}h_{ih}h_k^m l_m l_j - h_{ik}h_{hj} + X_{00}h_{ik}h_h^m l_m l_j], \\ L^2 S_{ihk}^m X_{mj} &= S[h_{ih}h_{jk} - h_{ik}h_{hj}], \\ L^2 S_{ihk}^m X_{mj} &= S_{hijk}. \end{aligned} \quad (13)$$

Theorem 3.1 *In a S3-like Finsler space, the covariant tensor field X_{mj} satisfies (4) is of the form (13).*

Next we consider S4-like Finsler space, whose ν -curvature tensor of cartons connection CT is given by [7],

$$L^2 S_{ihk}^m = h_h^m M_{ik} + h_{ik}M_h^m - h_{hk}M_i^m - h_i^m M_{hk}. \quad (14)$$

Contacting above equation by X_{mj} and using equations (4) and (1)(d), we have

$$\begin{aligned} L^2 S_{ihk}^m X_{mj} &= h_h^m M_{ik} X_{mj} + h_{ik}M_h^m X_{mj} - h_{hk}M_i^m X_{mj} - h_i^m M_{hk} X_{mj}, \\ L^2 S_{ihk}^m X_{mj} &= h_{hj}M_{ik} + h_{ik}M_{hj} - h_{hk}M_{ij} - h_{ij}M_{hk}, \\ L^2 S_{ihk}^m X_{mj} &= L^2 S_{hijk}. \end{aligned} \quad (15)$$

Theorem 3.2 *In a S4-like Finsler space, the covariant tensor field X_{ij} satisfies (4) is of the form (15).*

Next we consider a ν -curvature tensor of Cartons connection CT is given by [2],

$$S_{ihk}^m = C_{hr}^m C_{ik}^r - C_{kr}^m C_{ih}^r. \quad (16)$$

Contacting above equation by X_{mj} and using equations (4) and (1)(d), we have

$$\begin{aligned} S_{ihk}^m X_{mj} &= C_{hr}^m C_{ik}^r X_{mj} - C_{kr}^m C_{ih}^r X_{mj}, \\ S_{ihk}^m X_{mj} &= C_{hr}^m C_{ik}^r (h_{mj} + X_{00}l_m l_j) - C_{kr}^m C_{ih}^r (h_{mj} + X_{00}l_m l_j), \\ S_{ihk}^m X_{mj} &= C_{hjr} C_{ik}^r - C_{krj} C_{ih}^r, \\ S_{ihk}^m X_{mj} &= S_{hijk}. \end{aligned} \quad (17)$$

Theorem 3.3 *In a ν -curvature tensor, the covariant tensor field X_{mj} satisfies (4) is of the form (17).*

Now we concerned with a space of scalar curvature in Berwald's sence. It is characterized by the equation is [2]

$$R_{jk}^i = h_k^i k_j - h_j^i k_k. \quad (18)$$

where h_{ik} is the angular metric tensor and the scalar curvature K is a function scalar field.

Contacting above equation by X_{il} and using equations (4) and (1)(d), we get

$$\begin{aligned} R_{jk}^i X_{il} &= h_k^i K_j X_{il} - h_j^i K_k X_{il}, \\ R_{jk}^i X_{il} &= h_k^i K_j (h_{il} + X_{00} l_i l_l) - h_j^i K_k X_{il} (h_{il} + X_{00} l_i l_l), \\ R_{jk}^i X_{il} &= K_j h_{ki} - K_k h_{jl}, \\ R_{jk}^i X_{il} &= R_{ijk}. \end{aligned} \quad (19)$$

Theorem 3.4 *In a space of scalar curvature tensor, the covariant tensor field X_{il} satisfies (4) is of the form (19).*

4 The Existence Of Covariant Tensor X_{hk} In P-Reducible Finsler Space :

The P-reducible Finsler space is given as [5],

$$P_{jk}^m = (h_j^m P_k + h_{jk} P^m + h_k^m P_j)/(n+1), \quad (20)$$

Contacting above equation by X_{mi} and using equations (4) and (1)(d), we have

$$\begin{aligned} P_{jk}^m X_{mi} &= (h_j^m P_k X_{mi} + h_{jk} P^m X_{mi} + h_k^m P_j X_{mi})/(n+1), \\ P_{jk}^m X_{mi} &= (h_j^m P_k (h_{mi} + X_{00} l_m l_i) + h_{jk} P^m (h_{mi} + X_{00} l_m l_i) + \\ &\quad h_k^m P_j (h_{mi} + X_{00} l_m l_i))/(n+1), \\ P_{jk}^m X_{mi} &= (h_{ij} P_k + h_{jk} P_i + h_{ki} P_j)/(n+1), \\ P_{jk}^m X_{mi} &= P_{ijk}. \end{aligned} \quad (21)$$

Theorem 4.1 *In a P-reducible Finsler space, the covariant tensor field X_{mi} satisfies (4) is of the form (21).*

5 The Existence Of Covariant Tensor X_{hk} In C^h -Recurrent Finsler Space:

Now we consider a C^h -recurrent Finsler space is given as [2],

$$C_{jk/h}^m = \alpha_h C_{jk}^m. \quad (22)$$

Contacting above equation by X_{mi} and using equations (4) and (1)(d), we can written as

$$\begin{aligned}
C_{jk/h}^m X_{mi} &= \alpha_h C_{jk}^m X_{mi}, \\
C_{jk/h}^m X_{mi} &= \alpha_h C_{jk}^m (h_{mi} + X_{00} l_m l_i), \\
C_{jk/h}^m X_{mi} &= \alpha_h C_{ijk}, \\
C_{jk/h}^m X_{mi} &= C_{ijk/h}.
\end{aligned} \tag{23}$$

Theorem 5.1 *In a C^h -recurrent Finsler space, the covariant tensor field X_{mi} satisfies (4) is of the form (23).*

6 The Existence Of Covariant Tensor X_{hk} In T-Condition :

Finsler space satisfying T-condition can be defined as,[6]

$$T_{ijk}^m = LC_{ij/k}^m + l^m c_{ijk} + l_i c_{jk}^m + l_j c_{ik}^m + l_k c_{ij}^m = 0, \tag{24}$$

Contacting above equation by X_{hm} and using equations (4) and (1)(d), we have

$$\begin{aligned}
T_{ijk}^m X_{hm} &= LC_{ij/k}^m X_{hm} + l^m C_{ijk} X_{hm} + l_i C_{jk}^m X_{hm} + l_j C_{ik}^m X_{hm} + l_k C_{ij}^m X_{hm} = 0, \\
T_{ijk}^m X_{hm} &= LC_{ij/k}^m X_{hm} + l^m C_{ijk} (h_{hm} + X_{00} l_h l_m) + l_i C_{jk}^m (h_{hm} + X_{00} l_h l_m) + \\
&\quad l_j C_{ik}^m (h_{hm} + X_{00} l_h l_m) + l_k C_{ij}^m (h_{hm} + X_{00} l_h l_m) = 0, \\
T_{ijk}^m X_{hm} &= LC_{hij/k} + l_h C_{ijk} + X_{00} l_h l_m l^m C_{ijk} + l_i C^{hjk} + l_j C^{hik} + l_k C^{hij} = 0, \\
T_{ijk}^m X_{hm} &= LC_{hij/k} + l_h C_{ijk} + l_i C^{hjk} + l_j C^{hik} + l_k C^{hij} + X_{00} l_h l_m l^m C_{ijk} = 0, \\
T_{ijk}^m X_{hm} &= T_{hijk} + X_{00} l_h c_{ijk} = 0, \\
T_{ijk}^m X_{hm} &= T_{hijk} = 0.
\end{aligned} \tag{25}$$

Theorem 6.1 *If the Finsler space satisfying T-condition, then the covariant tensor field X_{hm} satisfies (4) is of the form (25) provided $X_{00} l_h c_{ijk} = 0$.*

Finsler space satisfying generalized T-condition can be defined as [5],

$$T_j^h = LC_{/j}^h + l^h C_j + l_j C^h = 0. \tag{26}$$

Contacting above equation by X_{ih} and using equations (4) and (1)(d), we obtain

$$\begin{aligned}
T_j^h X_{ih} &= LC_{/j}^h X_{ih} + l^h C_j X_{ih} + l_j C^h X_{ih} = 0, \\
T_j^h X_{ih} &= LC_{/j}^h (h_{ih} + X_{00} l_i l_h) + l^h C_j (h_{ih} + X_{00} l_i l_h) + l_j C^h (h_{ih} + X_{00} l_i l_h) = 0, \\
T_j^h X_{ih} &= LC_{i/j} + l_i C_j + l_j C_i = 0, \\
T_j^h X_{ih} &= T_{ij}.
\end{aligned} \tag{27}$$

Theorem 6.2 *If the Finsler space satisfying generalized T-Condition, then the covariant tensor field X_{ih} satisfies (4) is of the form (27).*

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