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Convergence Theorems of CQ Iteration Processes for a Finite Family of Averaged Mappings in Hilbert Spaces[1]

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Abstract. In this paper, we study convergence theorems of CQ iteration processes for averaged mappings in Hilbert spaces. Let K be a bounded closed convex subset of a real Hilbert space H and Let $\{T_i\}_{i=1}^{N} : K \to K$ be a finite family averaged mappings such that $F = \cap_{i=1}^{N} F(T_i) \neq \emptyset$. Assume that $\{\alpha_n\}$ is real number sequences in (0,1) such that $\lim_{n\to\infty} \alpha_n = 0$. Define a sequence $\{x_n\}_{n=1}^{\infty}$ in K by the algorithm:

$$\begin{cases} x_1 \in K \ chosen \ \ arbitrarily, \\ y_n = (1-\alpha_n)T_n x_n, \\ C_n = \{z \in K : \|y_n - z\|^2 \leq \|x_n - z\|^2 + \alpha_n \|z\|^2\}, \\ Q_n = \{z \in K : \langle x_n - z, x_1 - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \bigcap Q_n} x_1. \end{cases}$$

$T_n = T_{nmodN}$, Then $\{x_n\}$ converges in norm to $P_F x_1$.

Keywords: averaged mappings; CQ iteration processes;Hilbert spaces

1. Introduction

Let H be a real Hilbert space, K be a nonempty closed convex subset of H. Let T is a selfmapping of K. $F(T)$ denotes the set of fixed points of T. The mapping T is said to be
(1)nonexpansive, if $\|Tx - Ty\| \leq \|x - y\|$, for all $x, y \in K$;
(2)averaged mapping, if there exists a nonexpansive mapping S: $K \to K$ and a number $k \in (0, 1)$ such that

$$T = (1-k)I + kS.$$

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Denote by $F(S)$ the set of fixed points of S; that is $F(S) = \{x \in K : Sx = x\}$. If $F(S) \neq \emptyset$, then we obtain immediately that $F(S) = F(T)$, Let $\{T_i\}_{i=1}^N : K \to K$ be a finite family averaged mappings, then from $T_i = (1-k)I + kS_i$, if $\cap_{i=1}^N F(S_i) \neq \emptyset$, then we also have $\cap_{i=1}^N F(S_i) = \cap_{i=1}^N F(T_i)$.

It is clear that an averaged mapping is nonexpansive, but not vice versa. An example of an averaged mapping is the metric projection from a Hilbert space onto a closed convex subset. It is known that averaged mappings are always asymptotically regular and their Picard iterates converge weakly.

In the recently, Yonghong Yao, Haiyun Zhou and Rudong Chen [2] have proved strong convergence theorems of a new iterative algorithm for averaged mappings in Hilbert spaces , and proved the following theorem.

Theorem 1.1. *Let H be a real Hilbert space . Let $T : H \to H$ be an averaged mapping with $F(T) \neq \emptyset$. Let $\{\alpha_n\}$ be a real numbers in (0,1). For given $x_0 \in H$, let the sequence $\{x_n\}$ be generated iteratively by*

$$x_{n+1} = (1-\alpha_n)Tx_n, n \geq 0.$$

Assume that the following control conditions hold:
(i) $\lim_{n\to\infty} \alpha_n = 0$;
(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$.
Then $\{x_n\}$ converges strongly to a fixed point of T.

In this paper, thanks to the iterative algorithm by Yonghong Yao, Haiyun Zhou and Rudong Chen [2], we establish Convergence theorems of CQ iteration processes for a finite family averaged mappings in Hilbert spaces.

2. Preliminaries

We will use the notations:
1. $\rightharpoonup$ for weak convergence and for $\to$ strong convergence.
2. $\omega_w(x_n) := \{x : \exists x_{n_j} \rightharpoonup x\}$ denotes weak ω-limit set of x_n.

Lemma 2.1. *([1])Let H be a real Hilbert space. Then there hold the following well-known results:*
(1)$\|tx + (1-t)y\|^2 \leq t\|x\|^2 + (1-t)\|y\|^2, \forall x, y \in H, \forall t \in [0,1]$.
(2)$\|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y\rangle, \forall x, y \in H$.

Lemma 2.2. *([5])(Demi-closed principle). Let K be a nonempty closed convex subset of a real Hilbert space H . Let S: $K \to K$ be a nonexpansive mapping. Then S is demi-closed on K, i.e., if $x_n \rightharpoonup x \in K$ and $x_n - Sx_n \to 0$, then $x = Sx$.*

Lemma 2.3. *([1])Let H be a real Hilbert space. Given a closed convex subset $K \subset H$ and points $x, y, z \in H$. Given also a real number $a \in R$. The set*

$$D := \{v \in K : \|y - v\|^2 \leq \|x - v\|^2 + \langle z, v\rangle + a\}$$

is convex (and closed).

Recall that a closed convex subset K of a real Hilbert space H, the nearest point projection P_K from H onto K assigns to each $x \in H$ its nearest point denoted by $P_K x$ in K from x to K; that is the unique point in K with the property

$$\|x - P_K x\| \leq \|x - y\|, \forall y \in K.$$

Lemma 2.4. *([1])Let K be a closed convex subset of a real Hilbert space H and let P_K be the projection from H onto K. Given $x \in K$ and $z \in K$. Then $z = P_K x$ if and only if there holds the relation:*

$$\langle x - z, y - z\rangle \leq 0, \forall y \in K.$$

Lemma 2.5. *([1])Let K be a closed convex subset of a Hilbert space H and let $\{x_n\}$ be a sequence in H and $u \in H$. Let $q = P_K u$. If $\{x_n\}$ is such that $\omega_w(x_n) \subset K$ and satisfies the condition*

$$\|x_n - u\| \leq \|u - q\|, \forall n.$$

Then $x_n \to q$.

3. Main Results

In this section, we will prove our main result.

Theorem 3.1. *Let K be a bounded closed convex subset of a real Hilbert space H and Let $\{T_i\}_{i=1}^N : K \to K$ be a finite family averaged mappings such that $F = \cap_{i=1}^N F(T_i) \neq \emptyset$. Assume that $\{\alpha_n\}$ is real number sequences in (0,1) such that $\lim_{n\to\infty} \alpha_n = 0$. Define a sequence $\{x_n\}_{n=1}^\infty$ in K by the algorithm:*

$$\begin{cases} x_1 \in K \text{ chosen arbitrarily}, \\ y_n = (1-\alpha_n)T_n x_n, \\ C_n = \{z \in K : \|y_n - z\|^2 \leq \|x_n - z\|^2 + \alpha_n\|z\|^2\}, \\ Q_n = \{z \in K : \langle x_n - z, x_1 - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \bigcap Q_n} x_1. \end{cases} \tag{3.1}$$

$T_n = T_{n mod N}$, Then $\{x_n\}$ converges in norm to $P_F x_1$.

Proof. It is obvious that Q_n is closed and convex , C_n is closed. Since K is bounded , let $M = diam K$, that M is a positive constant. From (3.1) we have

$$\|y_n - z\|^2 \leq \|x_n - z\|^2 + \alpha_n\|z\|^2 \leq \|x_n - z\|^2 + M^2.$$

Then we observe that C_n is convex by Lemma 2.3. So $C_n \bigcap Q_n$ is closed and convex, and $\{x_n\}$ is well defined. Next, we show that $F \subset C_n$ for all n.

Indeed, we have , for all $p \in F$, and $T_n x_n = (1-k)x_n + kS_n x_n$.

$$\begin{array}{rl} & \|y_n - p\|^2 \\ = & \|(1-\alpha_n)T_n x_n - p\|^2 \\ = & \|(1-\alpha_n)((1-k)x_n + kS_n x_n) - p\|^2 \\ = & \|(1-\alpha_n)((1-k)x_n + kS_n x_n - p) - \alpha_n p\|^2 \\ \leq & (1-\alpha_n)\|(1-k)x_n + kS_n x_n - p\|^2 + \alpha_n\|p\|^2 \\ = & (1-\alpha_n)\|(1-k)(x_n - p) + k(S_n x_n - p)\|^2 + \alpha_n\|p\|^2 \\ \leq & (1-\alpha_n)((1-k)\|x_n - p\|^2 + k\|S_n x_n - p\|^2) + \alpha_n\|p\|^2 \\ \leq & \|x_n - p\|^2 + \alpha_n\|p\|^2 \end{array} \tag{3.2}$$

So $p \in C_n$ for all n. Next, we show that

$$F \subset Q_n, \forall n \geq 0. \tag{3.3}$$

We prove this by induction. For n=1, we have $F \subset K = Q_1$. Assume that $F \subset Q_n$. Since $x_{n+1} = P_{C_n \bigcap Q_n} x_1$, by Lemma 2.4, we have

$$\langle x_{n+1} - z, x_1 - x_{n+1}\rangle \geq 0, \forall z \in C_n \cap Q_n.$$

As $F \subset Q_n \cap C_n$ by the induction assumption, the last inequality holds, in particular, for all $z \in F$. This together with the definition of Q_{n+1} implies that $F \subset Q_{n+1}$. Hence the $F \subset Q_n$ holds for all $n \geq 0$. And also $F \subset Q_n \cap C_n$ holds for all $n \geq 0$.

By the definition of Q_n, we have $x_n = P_{Q_n} x_1$, and since $F \subset Q_n$, we have

$$\|x_n - x_1\| \leq \|p - x_1\|, \forall p \in F.$$

in particular,

$$\|x_n - x_1\| \leq \|q - x_1\|, where q = P_F x_1. \tag{3.4}$$

The fact that $x_{n+1} \in Q_n$ implies that $\langle x_{n+1} - x_n, x_n - x_1\rangle \geq 0$. This together with Lemma 2.1, implies

$$\begin{array}{rl} & \|x_{n+1} - x_n\|^2 \\ = & \|(x_{n+1} - x_1) - (x_n - x_1)\|^2 \\ = & \|x_{n+1} - x_1\|^2 - \|x_n - x_1\|^2 - 2\langle x_{n+1} - x_n, x_n - x_1\rangle \\ \leq & \|x_{n+1} - x_1\|^2 - \|x_n - x_1\|^2 \end{array} \tag{3.5}$$

This implies that the sequence $\{\|x_n - x_1\|\}$ is increasing. Since K is bounded, then $\{x_n\}$ is bounded, we get that $\lim_{n\to\infty}\|x_n - x_1\|$ exists. It turn out from (3.5) that

$$\|x_{n+1} - x_n\| \to 0, n \to \infty. \tag{3.6}$$

By the fact $x_{n+1} \in C_n$ implies that

$$\|y_n - x_{n+1}\|^2 \leq \|x_n - x_{n+1}\|^2 + \alpha_n\|x_{n+1}\|^2$$

However, since $\lim_{n\to\infty}\alpha_n = 0$, and $\{x_n\}$ is bounded, we have

$$\|y_n - x_{n+1}\| \to 0, n \to \infty. \tag{3.7}$$

And also have

$$\|y_n - x_n\| \leq \|y_n - x_{n+1}\| + \|x_{n+1} - x_n\| \to 0, n \to \infty. \tag{3.8}$$

Noticing that $T_nx_n = y_n + \alpha_n T_n x_n$, and since K is bounded, we have

$$\begin{aligned} & \|T_nx_n - x_n\| \\ = & \|y_n - x_n + \alpha_n T_n x_n\| \\ \leq & \|y_n - x_n\| + \alpha_n \|T_n x_n\| \to 0, n \to \infty. \end{aligned} \tag{3.9}$$

Following $T_nx_n = (1-k)x_n + kS_nx_n$, we have

$$\|S_nx_n - x_n\| = \frac{1}{k}\|T_nx_n - x_n\| \to 0, n \to \infty. \tag{3.10}$$

We know that $\|x_{n+1} - x_n\| \to 0$, so that for all $j = 1, 2, \cdots, N$,

$$\|x_n - x_{n+j}\| \to 0, n \to \infty. \tag{3.11}$$

So for any $i = 1, 2, \cdots, N$, we also have

$$\begin{aligned} & \|x_n - S_{n+i}x_n\| \\ \leq & \|x_n - x_{n+i}\| + \|x_{n+i} - S_{n+i}x_{n+i}\| + \|S_{n+i}x_{n+i} - S_{n+i}x_n\| \\ \leq & \|x_n - x_{n+i}\| + \|x_{n+i} - S_{n+i}x_{n+i}\| + \|x_{n+i} - x_n\| \\ \leq & 2\|x_n - x_{n+i}\| + \|x_{n+i} - S_{n+i}x_{n+i}\| \end{aligned} \tag{3.12}$$

Thus, it follows from (3.11) and (3.10) that

$$\lim_{n\to\infty} \|x_n - S_{n+i}x_n\| = 0, i = 1, 2, \cdots, N. \tag{3.13}$$

Because $S_n = S_{nmodN}$, it is easy to see, for any $l = 1, 2, \ldots, N$, that

$$\lim_{n\to\infty} \|x_n - S_l x_n\| = 0. \tag{3.14}$$

then

$$\lim_{n\to\infty} \|x_n - T_l x_n\| = k\|x_n - S_l x_n\| = 0. \tag{3.15}$$

By (3.14) and lemma 2.2 we obtain that $\omega_w(x_n) \subset F$. this together with (3.4) and Lemma 2.5, guarantees the strong convergence of $\{x_n\}$ to $P_F x_0$. □

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