Int. Journal of Math. Analysis, Vol. 2, 2008, no. 13, 629 - 638

The Hadamard's Inequality for s–Convex Function of 2-Variables on the Co-ordinates

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Abstract

In this paper the extension of Hadamard's type inequality for s–convex function and s–convex functions on the co-ordinates defined in 2-variables and some applications are given.

Keywords: s–Hadamard's inequality, s–Convex function, Jensen's inequality

1 Introduction

Let $f : I \subseteq \mathbf{R} \to \mathbf{R}$ be a convex mapping defined on the interval I of real numbers and $a, b \in I$, with $a < b$. The following double inequality:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2} \tag{1}$$

is known in the literature as Hadamard's inequality for convex mappings.

In [1] Hudzik and Maligrada considered among others the class of functions which are s–convex in the second sense. This class is defined in the following way: a function $f : [0,\infty) \to \mathbf{R}$ is said to be *s–convex in the second sense* if

$$f(\lambda x + (1-\lambda) y) \leq \lambda^s f(x) + (1-\lambda)^s f(y) \tag{2}$$

holds for all $x, y \in [0,\infty)$, $\lambda \in [0,1]$ and for some fixed $s \in (0,1]$. It can be easily seen that every 1–convex function is convex.

In [2] Dragomir and Fitzpatrick proved a variant of Hadamard's inequality which holds for s–convex functions in the second sense.

Theorem A

Suppose that $f:[0,\infty)\to[0,\infty)$ is an s–convex function in the second sense, where $s\in(0,1)$ and let $a,b\in[0,\infty)$, $a<b$. If $f\in L^1[0,1]$, then the following inequalities hold:

$$2^{s-1}f\left(\frac{a+b}{2}\right)\le\frac{1}{b-a}\int_a^b f(x)\,dx\le\frac{f(a)+f(b)}{s+1}\qquad(3)$$

the constant $k=\frac{1}{s+1}$ is the best possible in the second inequality in (1.3). The above inequalities are sharp.

In [3], Dragomir established the following similar inequality of Hadamard-type for co-ordinated convex mapping on a rectangle from the plane $\mathbf{R}^2$.

Precisely, if $f:[a,b]\times[c,d]\to\mathbf{R}$ is convex function one can define the following mapping on $[0,1]^2$ such as:

$$H(t,r)=\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f\left(tx+(1-t)\frac{a+b}{2},ry+(1-r)\frac{c+d}{2}\right)dydx$$

Theorem B

Suppose that $f:\Delta\to\mathbf{R}$ is co-ordinated convex on Δ. Then one has the inequalities

$$\begin{aligned}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)&\le\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx\\&\le\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\end{aligned}\qquad(4)$$

The above inequalities are sharp.

Also, one can has the following properties for H (see [3-5]):

(i) H is co–ordinated convex on $[0,1]^2$.
(ii) One has the bounds

$$\sup_{(t,r)\in[0,1]^2}H(t,r)=\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dxdy=H(1,1)$$

and

$$\inf_{(t,r)\in[0,1]^2} H(t,r) = f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) = H(0,0).$$

2 Hadamard's Inequality

First of all, let us start with the following definition:

Definition 2.1

Consider the bidimensional interval $\Delta := [a,b] \times [c,d]$ in $[0,\infty)^2$ with $a < b$ and $c < d$. The mapping $f : \Delta \to \mathbf{R}$ is s–convex on Δ if

$$f(\lambda x + (1-\lambda) z, \lambda y + (1-\lambda) w) \leq \lambda^s f(x,y) + (1-\lambda)^s f(z,w)$$

holds for all $(x,y), (z,w) \in \Delta$ with $\lambda \in [0,1]$ and for some fixed $s \in (0,1]$.

A function $f : \Delta \to \mathbf{R}$ is s–convex on Δ is called co-ordinated s–convex on Δ if the partial mappings $f_y : [a,b] \to \mathbf{R}$, $f_y(u) = f(u,y)$ and $f_x : [c,d] \to \mathbf{R}$, $f_x(v) = f(x,v)$, are s–convex for all $y \in [c,d]$ and $x \in [a,b]$ with some fixed $s \in (0,1]$.

Lemma 2.1

Every s–convex mapping $f : \Delta = [a,b] \times [c,d] \subseteq [0,\infty)^2 \to [0,\infty)$ is s–convex on the co-ordinates, but the converse is not true in general.

Proof.

Suppose that $f : \Delta = [a,b] \times [c,d] \subseteq [0,\infty)^2 \to [0,\infty)$ is s–convex on Δ. Consider the function $f_x : [c,d] \to [0,\infty)$, $f_x(v) = f(x,v)$. Then for $\lambda \in [0,1]$ and $v_1, v_2 \in [c,d]$, one has:

$$\begin{aligned} f_x(\lambda v_1 + (1-\lambda) v_2) &= f(x, \lambda v_1 + (1-\lambda) v_2) \\ &= f(\lambda x + (1-\lambda) x, \lambda v_1 + (1-\lambda) v_2) \\ &\leq \lambda^s f(x, v_1) + (1-\lambda)^s f(x, v_2) \\ &= \lambda^s f_x(x, v_1) + (1-\lambda)^s f_x(x, v_2)\ . \end{aligned}$$

Therefore, $f_x(v) = f(x,v)$ is s–convex on $[c,d]$.

The fact that $f_x : [a,b] \to [0,\infty)$, $f_y(u) = f(u,y)$ is also s–convex on $[a,b]$ for all $y \in [c,d]$ goes likewise and we shall omit the details.

In [3] Dragomir gave a mapping $f_0 : [0,1]^2 \to [0,\infty)$ defined by $f_0(x,y) = xy$ which is convex on the co–ordinates but is not convex. We consider the same function with $s = 1$ to prove that the s–convexity on the co–ordinates does not imply the s–convexity.

The following inequality is considered the mapping connected with the inequality (3).

Theorem 2.1

Suppose that $f : \Delta = [a,b] \times [c,d] \subseteq [0,\infty)^2 \to [0,\infty)$ is s–convex function on the co–ordinates on Δ. Then one has the inequalities:

$$\begin{aligned}
& f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\
& \quad \leq 2^{s-2}\left[\frac{1}{b-a}\int_a^b f\left(x, \frac{c+d}{2}\right)dx + \frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2}, y\right)dy\right] \\
& \leq \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx \qquad (5) \\
& \leq \frac{1}{2(s+1)}\left[\frac{1}{b-a}\int_a^b f(x,c)\,dx + \frac{1}{b-a}\int_a^b f(x,d)\,dx \right. \\
& \qquad \left. + \frac{1}{d-c}\int_c^d f(a,y)\,dy + \frac{1}{d-c}\int_c^d f(b,y)\,dy\right] \\
& \leq \frac{f(a,c) + f(b,c) + f(a,d) + f(b,d)}{(s+1)^2}.
\end{aligned}$$

Proof.

Since $f:\Delta\to\mathbf{R}$ is co-ordinated s–convex on Δ it follows that the mapping $g_x:[c,d]\to[0,\infty)$, $g_x(y)=f(x,y)$ is s–convex on $[c,d]$ for all $x\in[a,b]$. Then by s–Hadamard's inequality (3) one has:

$$2^{s-1}g_x\left(\frac{c+d}{2}\right)\leq\frac{1}{d-c}\int_c^d g_x(y)\,dy\leq\frac{g_x(c)+g_x(d)}{s+1},\quad\forall x\in[a,b]\,.$$

That is,

$$2^{s-1}f\left(x,\frac{c+d}{2}\right)\leq\frac{1}{d-c}\int_c^d f(x,y)\,dy\leq\frac{f(x,c)+f(x,d)}{s+1},\quad\forall x\in[a,b]\,.$$

Integrating this inequality on $[a,b]$, we have

$$\begin{aligned}\frac{2^{s-1}}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)dx &\leq \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx \qquad (6)\\ &\leq \frac{1}{s+1}\left[\frac{1}{b-a}\int_a^b f(x,c)\,dx+\frac{1}{b-a}\int_a^b f(x,d)\,dx\right].\end{aligned}$$

A similar arguments applied for the mapping $g_y:[a,b]\to[0,\infty)$, $g_y(x)=f(x,y)$, we get

$$\begin{aligned}\frac{2^{s-1}}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)dy &\leq \frac{1}{(d-c)(b-a)}\int_c^d\int_a^b f(x,y)\,dxdy \qquad (7)\\ &\leq \frac{1}{s+1}\left[\frac{1}{d-c}\int_c^d f(a,y)\,dy+\frac{1}{d-c}\int_c^d f(b,y)\,dy\right].\end{aligned}$$

Summing the inequalities (6) and (7), we get the second and the third inequalities in (5).

Therefore, by s–Hadamard's inequality (3), we also have:

$$f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\leq\frac{2^{s-1}}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)dy \qquad (8)$$

and

$$f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\leq\frac{2^{s-1}}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)dx \qquad (9)$$

which give, by addition the first inequality in (5).

Finally, by the same inequality we can also state:

$$\frac{1}{b-a}\int_a^b f(x,c)\,dx \le \frac{f(a,c)+f(b,c)}{s+1}$$

$$\frac{1}{b-a}\int_a^b f(x,d)\,dx \le \frac{f(a,d)+f(b,d)}{s+1}$$

$$\frac{1}{d-c}\int_c^d f(a,y)\,dy \le \frac{f(a,c)+f(a,d)}{s+1}$$

and

$$\frac{1}{d-c}\int_c^d f(b,y)\,dy \le \frac{f(b,c)+f(b,d)}{s+1}$$

which give, by addition the last inequality in (5).

Note 1: In (5) if $s=1$ then the inequality reduced to inequality (4) .

Now, for the mapping H we have the following result(s):

Theorem 2.2

Suppose that $f:\Delta\to\mathbf{R}$ is co-ordinated s–convex on Δ. Then for $H(t,r)$ one has the inequalities

(i) H is co–ordinated s–convex on $[0,1]^2$.

(ii) The mapping H is monotonic nondecreasing on the co-ordinates.

(iii) One has the bounds

$$\inf_{(t,r)\in[0,1]^2} H(t,r) = H(0,0),$$

and

$$\sup_{(t,r)\in[0,1]^2} H(t,r) = H(1,1)$$

Proof.

(i) Fix $r \in [0,1]$. Then for all $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$ and $t_1, t_2 \in [0,1]$, we have:

$$\begin{aligned}
H(\alpha t_1 + \beta t_2, r) &= \frac{1}{(b-a)(d-c)} \\
&\quad \times \int_a^b \int_c^d f\left((\alpha t_1 + \beta t_2)x + (1 - (\alpha t_1 + \beta t_2))\frac{a+b}{2}, ry + (1-r)\frac{c+d}{2}\right) dydx \\
&= \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f\left(\alpha\left(t_1 x + (1-t_1)\frac{a+b}{2}\right)\right. \\
&\quad \left. + \beta\left(t_2 x + (1-t_2)\frac{a+b}{2}\right), ry + (1-r)\frac{c+d}{2}\right) dydx \\
&= \alpha^s \cdot \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f\left(t_1 x + (1-t_1)\frac{a+b}{2}, ry + (1-r)\frac{c+d}{2}\right) dydx \\
&\quad + \beta^s \cdot \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f\left(t_2 x + (1-t_2)\frac{a+b}{2}, ry + (1-r)\frac{c+d}{2}\right) dydx \\
&= \alpha^s H(t_1, r) + \beta^s H(t_2, r).
\end{aligned}$$

Similarly, if $t \in [0,1]$ is fixed, then for all $r_1, r_2 \in [0,1]$ and $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$ we also have:

$$H(t, \alpha r_1 + \beta r_2) \leq \alpha^s H(t, r_1) + \beta^s H(t, r_2)$$

and the statement is proved.

(ii) Firstly, we will show that

$$H(t,r) \geq H(0,r), \forall\, t, r \in [0,1]^2. \tag{10}$$

By s–Hadamard's inequality (3), we have

$$\begin{aligned}
H(t,r) &\geq \frac{1}{d-c} \int_c^d f\left(\frac{1}{b-a}\int_a^b \left[tx + (1-t)\frac{a+b}{2}\right] dx, ry + (1-r)\frac{c+d}{2}\right) dy \\
&= \frac{1}{d-c} \int_c^d f\left(\frac{a+b}{2}, ry + (1-r)\frac{c+d}{2}\right) dy = H(0,r).
\end{aligned}$$

$\forall\, t, r \in [0,1]^2$.

Now, let $0 \leq t_1 < t_2 \leq 1$. By the s–convexity of the mapping $H(\circ, r)$ for all $r \in [0,1]$, we have

$$\frac{H(t_2, r) - H(t_1, r)}{t_2 - t_1} \geq \frac{H(t_1, r) - H(0, r)}{t_1} \geq 0.$$

Note that, for the last inequality we have used (10).

(iii) Since f is s–convex on the co–ordinates, we have by Jensen's inequlaity for integrals that:

$$\begin{aligned}
H(t,r) &= \frac{1}{b-a}\int_a^b \left[\frac{1}{d-c}\int_c^d f\left(tx + (1-t)\frac{a+b}{2}, ry + (1-r)\frac{c+d}{2}\right) dy\right] dx \\
&\geq \frac{1}{b-a}\int_a^b f\left(tx + (1-t)\frac{a+b}{2},\ \frac{1}{d-c}\int_c^d \left[ry + (1-r)\frac{c+d}{2}\right] dy\right) dx \\
&= \frac{1}{b-a}\int_a^b f\left(tx + (1-t)\frac{a+b}{2}, \frac{c+d}{2}\right) dx \\
&\geq f\left(\frac{1}{b-a}\int_a^b \left[tx + (1-t)\frac{a+b}{2}\right] dx, \frac{c+d}{2}\right) \\
&= f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\
&= H(0,0).
\end{aligned}$$

By the s–convexity of H on the co-ordinates, we have

$$\begin{aligned}
H(t,r) &= \frac{1}{b-a}\int_a^b \left[\frac{1}{d-c}\int_c^d f\left(tx + (1-t)\frac{a+b}{2}, y\right) dy \right. \\
&\qquad \left. + (1-r)\frac{1}{d-c}\int_c^d f\left(tx + (1-t)\frac{a+b}{2}, \frac{c+d}{2}\right) dy\right] dx \\
&\leq r \cdot \frac{1}{d-c}\int_c^d \left[t \cdot \frac{1}{b-a}\int_a^b f(x,y)\, dxdy \right. \\
&\qquad \left. + (1-t)\frac{1}{b-a}\int_a^b f\left(\frac{a+b}{2}, y\right) dx\right] dy \\
&\qquad + (1-r)\frac{1}{d-c}\int_c^d t \cdot \frac{1}{b-a}\int_a^b f\left(x, \frac{c+d}{2}\right) dx \\
&\qquad + (1-t) f\left(\frac{a+b}{2}, \frac{c+d}{2}\right)\Bigg] dy
\end{aligned}$$

$$= rt.\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx + r(1-t)\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)dy$$
$$+t(1-r)\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)dx + (1-r)(1-t)\,f\left(\frac{a+b}{2},\frac{c+d}{2}\right)$$

Therefore, by (6), (7), (8) and (9) we deduce that

$$\begin{aligned} & H(t,r) \le [rt + r(1-t) + t(1-r) + (1-t)(1-r)] \\ & \times \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx \\ & = \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx \\ & = H(1,1). \end{aligned}$$

Thus, the second bound in (iii) is proved.

Note 2: In Theorem 2.2 set $s = 1$ we get the result obtained in Theorem B.

Corollary 2.1

Suppose that $f : \Delta \to \mathbf{R}$ is co-ordinated s–convex on Δ. Define the mapping $h : [0,1] \to R$, $h(t) = H(t,t)$. Then h is convex monotonic nondecreasing on $[0,1]$ and one has the bounds:

$$\begin{aligned} \inf_{t\in[0,1]} h(t) &= h(0) \\ &= H(0,0), \end{aligned}$$

and

$$\begin{aligned} \sup_{t\in[0,1]} h(t) &= h(1) \\ &= H(1,1). \end{aligned}$$

Proof.

It's an immediate consequence of Theorem 2.2.

Comment(s):

In the next paper we will give the most main applications on these result(s) by obtaining some Hadamard–type inequality for co–ordinated s–convex functions in a rectangle from the plane.

Acknowledgement : The work here is supported by the Grant: UKM–GUP–TMK–07–02–107.

References

[1] H. Hudzik, L. Maligranda, Some remarks on s-convex functions, *Aequationes Math.,* **48** (1994), 100?–111.

[2] S.S. Dragomir, S. Fitzpatrick, The Hadamard's inequality for s-convex functions in the second sense, *Demonstratio Math.,* **32** (4) (1999), 687–?696.

[3] S. S. Dragomir, On Hadamard's inequality for convex functions on the co-ordinates in a rectangle from the plane, *Taiwanese Journal of Mathematics,* **5** (2001), 775–788.

[4] S. S. Dragomir, A mapping in connection to Hadamard's inequality, *An Ostro. Akad. Wiss. Math. -Natur (Wien),* **128** (1991), 17–20.

[5] S. S. Dragomir, Two mappings in connection to Hadamard's inequality, *J. Math. Anal. Appl.,* **167** (1992), 49–56.

Received: January 12, 2008