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Additive Mappings between Directed Wedges with the Riesz Interpolation Property

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Abstract

In this article we provide generalized versions of two well-known theorems from the vector lattice theory, Kantorovich extension theorem and Riesz-Kantorovich theorem, for directed wedges. In order to achieve our goal, we study generalized order structures by considering the set of all increasing, non-empty, downward-directed, lower bounded subsets of a directed, ordered vector space with the Riesz decomposition property. This set becomes a partially ordered generalized wedge, and we discuss the behavior of certain types of mappings (linear mappings and order bounded mappings) that can be defined between such wedges. We further prove a Kantorovich type theorem for additive mappings , and finally, as an application, we prove a Riesz-Kantorovich type theorem for order bounded mappings.

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1 Introduction

In this note we formulate and investigate certain types of mappings between partially ordered, directed, generalized wedges. In particular, we make use of certain associated spaces of a partially ordered vector space with the Riesz decomposition property. This property makes it possible to assign a meaning to expressions like $|x|, x^+$ and x^-, where x is an element of the initial vector space. Given this consideration, we are able to prove generalized versions of the well-known Kantorovich theorem (cf. [1] Theorem 1.15) as well as the Riesz-Kantorovich theorem (cf. [3] Theorem 2.6.1 or [1] Theorem 1.16) for mappings between generalized ordered structures. For certain purposes, towards our goal, it will be necessary to work with objects known as generalized wedges. Unlike vector spaces, addition in generalized wedges need not satisfy the cancellation law (i.e., $v + w = u + w$ implies $v = u$), and these objects also need not be embeddable in vector spaces. That is, we shall consider structures that are closed to the usual vector spaces, and whose elements do not necessarily have additive inverses.

For an extensive presentation of generalized wedges the reader may refer to [2, 4, 7]. In addition, approximation theory based on generalized wedges is provided in [6], where the authors make use of the term ordered cone instead of the term generalized wedge used in the present work. Unless otherwise stated, the terminology throughout the paper is taken from [1, 3, 4, 7].

2 Preliminary Notes

Our definition of a generalized wedge follows the one provided in [7]; in particular, a *generalized wedge* is a set E equipped with two operations (denoted by $+, \cdot$) such that: (a) $(E, +)$ is a commutative semi-group with zero (i.e., the mapping $(x, y) \mapsto x + y : E \times E \to E$, is associative, commutative and admits a unique neutral element), and (b) the mapping $(\lambda, x) \mapsto \lambda \cdot x : \mathbb{R}_+^* \times E \to E$ satisfies the following axioms for all $\lambda, \mu \in \mathbb{R}_+^*, v, w \in E$:

(i) $\lambda \cdot (v + w) = \lambda \cdot v + \lambda \cdot w$

(ii) $(\lambda + \mu) \cdot v = \lambda \cdot v + \mu \cdot v$

(iii) $\lambda \cdot (\mu \cdot v) = (\lambda\mu) \cdot v$

(iv) $1 \cdot v = v$.

For all $\lambda \in \mathbb{R}_+^*$, the equality $\lambda \cdot 0 = 0$ is a consequence of the above axioms. Indeed, $\lambda \cdot 0 + x = \lambda \cdot 0 + (\lambda\lambda^{-1}) \cdot x = \lambda \cdot 0 + \lambda \cdot (\lambda^{-1} \cdot x) = \lambda \cdot (0 + \lambda^{-1} \cdot x) = \lambda \cdot (\lambda^{-1} \cdot x) = x$ and by the uniqueness of the neutral element it follows that $\lambda \cdot 0 = 0$. Also, as in [2, 6], in the previous definition of the generalized wedge we assume, for each $x \in E$, the additional axiom

$$0 \cdot x = 0,$$

in order to extend the scalar multiplication to $\lambda = 0$.

Example 2.1. Most common examples of generalized wedges are:

(1) Any ordinary wedge contained in a vector space.

(2) The extended non-negative real numbers i.e., $\overline{\mathbb{R}}_+ = \mathbb{R}_+ \cup \{+\infty\}$ equipped with the usual linear order, addition and scalar multiplication. The order, addition and multiplication are extended to $+\infty$ in the usual way. In particular, we define $0 \cdot +\infty = 0$.

(3) The set of convex subsets of a vector space, where the two operations (addition and scalar multiplication) have their conventional meaning.

A *partially ordered generalized wedge* is a generalized wedge E with a partial order $(\leq)$ compatible with the algebraic structure of E in the sense that it satisfies the following two axioms, for all $\lambda \in \mathbb{R}_+, v_1, v_2, w \in E$:

(i) $v_1 \leq v_2$ implies $v_1 + w \leq v_2 + w$

(ii) $v_1 \leq v_2$ implies $\lambda \cdot v_1 \leq \lambda \cdot v_2$

In what follows, the notation λv stands for the operation $\lambda \cdot v, \lambda \in \mathbb{R}_+, v \in E$.

Example 2.2. Consider the positive cone E_+ of any partially ordered vector space E. We define an order $(\preccurlyeq)$ on E_+ by $x \preccurlyeq y$ if an only if $x, y > 0$ and $x \leq y$ or else $x = y = 0$. Then $(E_+, \preccurlyeq)$ is a partially ordered generalized wedge.

We shall consider partially ordered generalized wedges E possessing the *Riesz interpolation property* (i.e., for each $a, b, c, d \in E$ with $a, b \leq c, d$ there exists an element $v \in E$ such that $a, b \leq v \leq c, d$). Recall that in partially ordered vector spaces the *Riesz decomposition property* (i.e., if $v, w_1, w_2 \in E, v, w_1, w_2 \geq 0$ with $v \leq w_1 + w_2$ then there exist $v_1, v_2 \in E$ such that $v = v_1 + v_2$ and $0 \leq v_1 \leq w_1, 0 \leq v_2 \leq w_2$) is equivalent with the Riesz interpolation property. These properties are not equivalent for a partially ordered generalized wedge (cf. [7] Example 3 in p.2 as well as the comments of p.3). For notions on ordered spaces and ordered subspaces with the Riesz decomposition property or the Riesz interpolation property, that are not defined here, the reader is referred to [3, 5].

A subset C of a partially ordered set S is *upward- (resp. downward-)directed* if for every pair c_1, c_2 of elements of C there exists an element $c \in C$ such that $c_1, c_2 \leq c$ (*or* $c \leq c_1, c_2$, *respectively*). If C is both upward and downward directed then C is called *directed*. A subset A of S is *increasing* if $a \in A$ and $a \leq b$ implies $b \in A$. In what follows, we shall refer to any partially ordered, directed, generalized wedge as *directed wedge*.

Let V, W be directed wedges. A mapping $T : V \to W$ is called *order isomorphism* if it preserves the order structure as well as the algebraic structure. That is, for all $v, w \in V$ and each $\lambda \in \mathbb{R}_+$, the following properties hold (*i*) $T(v+w)=T(v)+T(w)$ (*additivity*),

(*ii*) $T(\lambda v)=\lambda T(v)$ (*positive homogeneity*),

(*iii*) $T(v)\geq 0$ if and only if $v\geq 0$.

A mapping $T:V\rightarrow W$ is called *linear*, if it is additive and positive homogeneous.

Assume that W is a directed wedge possessing the Riesz interpolation property. According to [7], the directed wedge W can be embedded in another directed wedge, that we shall denote by $U(W)$, which also possesses the Riesz interpolation property. In addition, $U(W)$ is an upper semi-lattice (i.e., the supremum of each two elements of $U(W)$ exists). We illustrate the details of the construction of $U(W)$:

Let $U(W)$ be the set of all increasing, non-empty, downward-directed, lower bounded subsets of W. A partial order ($\leq$) on $U(W)$ is defined by

$$A\leq B \textit{ if and only if } B\subseteq A, \textit{ for all } A,B\in U(W).$$

For each $A,B\in U(W)$ it follows that the set $A\cap B$ is increasing. Also, since A,B are downward-directed and W has the Riesz interpolation property we obtain that $A\cap B$ is downward-directed. It is easy to see that $A\cap B$ is lower bounded and non-empty since, if $a\in A, b\in B$ from the directedness of W there exists some $z\in W$ with $a,b\leq z$, but A,B are increasing so $z\in A, z\in B$. Hence, $A\cap B$ is an element of $U(W)$ and an easy argument shows that $A\cap B$ is the least upper bound of the sets A,B. In fact, $U(W)$ is a directed upper semi-lattice that possesses the Riesz interpolation property. The symbol $\triangledown$ stands for the semi-lattice operation on $U(W)$ and for each $A,B\in U(W)$ it holds that

$$A\triangledown B=A\cap B.$$

For each $A,B\in U(W)$ the following two operations can be defined in $U(W)$, namely

$$A+B=\{w\in W|w\geq a+b, \textit{ for some } a\in A, b\in B\}$$

and

$$\lambda A=\{\lambda a|a\in A\},\ \lambda\in\mathbb{R}_{+}^{*}\ ,\ 0\cdot A=\{w\in W|w\geq 0\}.$$

Under these two operations $U(W)$ becomes a directed wedge.

For each $x\in W$ we shall denote by $\check{x}$ the set $\{w\in W|w\geq x\}$. Then the mapping

$$\check{}: x\mapsto\check{x}$$

is an order isomorphism from W onto a subset of $U(W)$. For each $A\in U(W)$, one has $A\subseteq\bigcup_{a\in A}\check{a}$.

Let us denote by $\mathcal{C}$ a subset of $U(W)$. If $A \leq B$ (or $B \leq A$) for each $A \in \mathcal{C}$ and some $B \in U(W)$, we write $\mathcal{C} \leq B$ (or $B \leq \mathcal{C}$, respectively) and say that B is an *upper bound* (*lower bound*, respectively) of $\mathcal{C}$. In such a case the subset $\mathcal{C}$ of $U(W)$ is called *upper bounded* (*or lower bounded*, respectively). Assume that E is a directed (partially) ordered vector space with the Riesz decomposition property and let us define the following sets

$$U(E^+) := \{A \in U(E) | a \geq 0, \ \textit{for each } a \in A, \}, U(E)^+ := \{A \in U(E) | \check{0} \leq A\},$$

then by [4, Proposition 6] it holds that $U(E^+) = U(E)^+$. Also, for each $x \in E$ the following elements of $U(E)$ can be defined,

(i) $x^+ = \check{x} \triangledown \check{0} = \check{x} \cap \check{0}$

(ii) $x^- = (-x)\check{} \triangledown \check{0} = (-x)\check{} \cap \check{0}$

(iii) $|x| = \check{x} \triangledown (-x)\check{} = \check{x} \cap (-x)\check{}$

The following example clarifies that x^+, x^- are not, necessarily, of the form $\check{a}$ for some $a \in E$.

Example 2.3. Let $E = \mathbb{R}^2$ be ordered by the cone

$$C = \{(x, y) \in \mathbb{R}^2 | x, y > 0\} \cup \{0\}.$$

Recall that (E, C) is a partially ordered vector space satisfying the Riesz decomposition property. Set $x = (1, -1)$.

Then

$$x^+ = \{(x, y) \in \mathbb{R}^2 | x > 1, y > 0\},$$

which is not an element of the form $\check{a}$, for some $a \in \mathbb{R}^2$, as figure 1 shows.

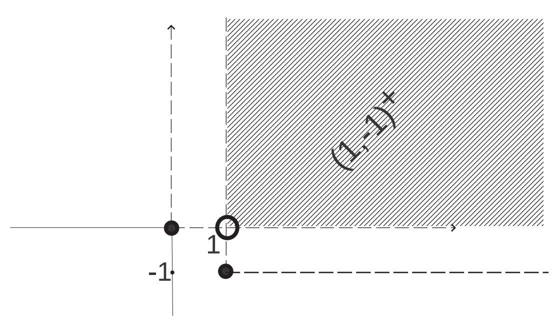

Figure 1: Example 2.3

Proposition 2.4. *[7, Propositions 2.2 and 2.4] Let E be a directed, (partially) ordered vector space with the Riesz decomposition property. Then the following identities hold, in $U(E)$,*

(i) $(A \triangledown B) + C = (A + C) \triangledown (B + C)$, *for all* $A, B, C \in U(E)$.

(ii) $\lambda(A \triangledown B) = (\lambda A) \triangledown (\lambda B)$, *for all* $A, B \in U(E), \lambda \in \mathbb{R}_+$.

(iii) $\check{x} + x^- = x^+$, *for all* $x \in E$.

3 A Kantorovich type theorem

This section deals with mappings between directed wedges with the Riesz interpolation property. In the following, we shall apply the construction of the directed wedge $U(E)$ to a directed, (partially) ordered vector space E with the Riesz decomposition property.

First, we shall discuss some features about a well-known class of vector spaces and their affect in the validity of the cancellation law in the directed wedge $U(E)$. Recall that a (partially) ordered vector space E is said to be *Archimedean* if $nx \leq y$, for all $n \in \mathbb{N}$ implies $x \leq 0$.

One of the least satisfactory features of the directed wedge $U(E)$ is the extent to which it fails to satisfy the cancellation law (i.e., for each $A, B \in U(E)$ the expression $A + B = A + C$ implies $B = C$). For example, let us consider the Archimedean vector space $E = \mathbb{R}^{\mathbb{R}}$ of all real-valued functions on $\mathbb{R}$ with the pointwise order (i.e., $f \leq g$ if and only if $f(x) \leq g(x)$, for each $x \in \mathbb{R}$), then E is a vector lattice, hence it has the Riesz decomposition property. Also, consider the the elements A, B of $U(E)$ defined by

$$A = \{f \in E^+ : f(t) = 0, \textit{ for at most finitely many values of } t\},$$

$$B = \{f \in E^+ : f(t) = 0, \textit{ for at most countably many values of } t\},$$

then $A + B = A = A + A$ and obviously $A \neq B$.

Also, if E is an arbitrary directed, ordered vector space with the Riesz decomposition property then, since the following identities hold for all $x \in E, A \in U(E)$,

1. $\check{x} + (-x)\check{} = \check{0}$,

2. $A + \check{0} = A$

the cancellation law $A + B = A + C$ implies $B = C$ holds if $A = \check{x}$, for some $x \in E$.

On the other hand, this is not true if A is arbitrary and both B, C are of this form. For example, let $E = \mathbb{R}^2$ with the lexicographic order $(\leq)$ i.e.,

$$(x_1, y_1) \leq (x_2, y_2) \textit{ if and only if } x_1 < x_2 \textit{ or } x_1 = x_2 \textit{ and } y_1 \leq y_2,$$

and consider the directed wedge $U(E)$. Recall, that E with the lexicographic order is not Archimedean(cf. [3] p.15), and if

$$A = \{(x, y) \in \mathbb{R}^2 | y > 0\}$$

then $A + (0,0)\check{} = A + (1,0)\check{}$, but $(0,0)\check{} \neq (1,0)\check{}$.

The following result for Archimedean vector spaces can be stated.

Proposition 3.1. *[7, Theorem 2.10] Let E be an Archimedean, directed, ordered vector space with the Riesz decomposition property and let $x, y \in E, A \in U(E)$. Then $\check{x} + A \leq \check{y} + A$ implies that $\check{x} \leq \check{y}$. Hence $\check{x} + A = \check{y} + A$ implies $\check{x} = \check{y}$.*

Now, we shall be concerned for mappings between directed wedges. In particular, we assume that E, F are directed, ordered vector spaces with the Riesz decomposition property and we prove the existence of linear extensions $S : U(E) \to U(F)$ of additive mappings $T : U(E)^+ \to U(F)^+$, as described in theorem 15 of [4]. The key,in order to do that, is the assumption of the directedness of the initial space E, that allow us the expression of each element $x \in E$ as a difference of two elements $x_1, x_2 \in E^+$.

We shall start with the definition of an important class, for our study, of mappings those that carries elements of the form $\check{x}, x \in E$ of $U(E)$ to elements of the same form of $U(F)$.

Definition 3.2. Let E, F be directed, ordered vector spaces with the Riesz decomposition property. A mapping $T : U(E) \to U(F)$ is called $\vee$-*preserving* if it carries subsets of the form $\check{x}$ of $U(E)$ to subsets of the same form of $U(F)$(i.e., for each $T(\check{x}) \in U(F), \check{x} \in U(E)$ there exists an element $a \in F$ such that $T(\check{x}) = \check{a}$).

According to the well-known Kantorovich theorem (cf. [1] Theorem 1.15) for ordered spaces, if X is a directed, ordered vector space and Y is an Archimedean, ordered vector space then every additive mapping $T : X^+ \to Y^+$ can be extended to a linear mapping from X to Y. The Kantorovich theorem is an important tool for the proof of the Riesz-Kantorovich theorem (cf., [3] Theorem 2.6.1). So we prove an analogue of the Kantorovich theorem for mappings between directed wedges in order to prove, in section 4, a Riesz-Kantorovich type theorem.

Proposition 3.3. *[4, Theorem 5] Let E, F be directed, ordered vector spaces with the Riesz decomposition property with F be an Archimedean space. Suppose that $T : U(E)^+ \to U(F)^+$ is a $\vee$-preserving mapping and for all $A, B \in U(E)^+$ it holds $T(A+B) = T(A) + T(B)$. Then if $S : U(E) \to U(F)$ is $\vee$-preserving mapping and for each $\check{x} \in U(E)$, S is given by the formula*

$$S(\check{x}) + T(x^-) = T(x^+)$$

then the following statements hold,

(i) $S(\check{x} + \check{y}) = S(\check{x}) + S(\check{y})$, *for all* $\check{x}, \check{y} \in U(E)$,

(ii) *if* $\check{x} \in U(E)$ *then there exists* $\check{y}, \check{z} \in U(E)^+$ *such that* $S(\check{x}) + T(\check{y}) = T(\check{z})$,

(iii) $S(\lambda\check{x}) = \lambda S(\check{x})$, *for all* $\lambda \in \mathbb{R}^+, \check{x} \in U(E)$,

(iv) $S(\check{x}) = T(\check{x})$, *for all* $\check{x} \in U(E)^+$.

In view of Proposition 3.3, it is only left to show that such a mapping S exists. Under the notations of the above theorem, we prove the existence of a mapping S that satisfies $(i) - (iv)$ of Proposition 3.3. In other words we provide a solution of the equation $S(\check{x}) + T(x^-) = T(x^+)$.

Theorem 3.4. *Let E, F be directed, ordered, vector spaces satisfying the Riesz decomposition property with F be an Archimedean space. Suppose that $T : U(E)^+ \to U(F)^+$ is a $\vee$-preserving mapping and for all $A, B \in U(E)^+$ it holds $T(A+B) = T(A) + T(B)$. For each $x \in E$, we consider the mapping $\widetilde{S}$ which is defined by the formula*

$$\widetilde{S}(\check{x}) = (b-a)\check{},$$

where $x = x_1 - x_2, x_1, x_2 \in E^+$ and $\check{b} = T(\check{x}_1), \check{a} = T(\check{x}_2), a, b \in F$. Then, the following statements hold

(i) *$\widetilde{S}$ is well defined, $\vee$-preserving and it satisfies the equation*

$$\widetilde{S}(\check{x}) + T(\check{x}_2) = T(\check{x}_1),$$

(ii) *the mapping $\widetilde{S}$ satisfies the equation $\widetilde{S}(\check{x}) + T(x^-) = T(x^+)$,*

(iii) *$\widetilde{S}(\check{x}) = S(\check{x})$, where S is the mapping of Proposition 3.3.*

Proof. (i) First we shall prove that the mapping $\widetilde{S}$ is well defined. Indeed, let $x, y \in E$ with $\check{x} = \check{y}$ then $x = y$ and there exist $x_1, x_2, y_1, y_2 \in E^+$ such that $x_1 - x_2 = x = y = y_1 - y_2$ hence $(x_1 + y_2)\check{} = (y_1 + x_2)\check{}$ and since the mapping $\check{}: x \mapsto \check{x}$ is an order isomorphism we have that $\check{x}_1 + \check{y}_2 = \check{y}_1 + \check{x}_2$. The additivity property holds for T in $U(E)^+$ thus we have

$$T(\check{x}_1) + T(\check{y}_2) = T(\check{y}_1) + T(\check{x}_2). \tag{1}$$

Assume that $\check{a} = T(\check{x}_1), \check{d} = T(\check{y}_2), \check{c} = T(\check{y}_1), \check{b} = T(\check{x}_2)$ where $a, b, c, d \in F$. From (1) it follows that

$$\check{a} + \check{d} = \check{c} + \check{b},$$

thus

$$\check{a} + \check{d} + (-a)\check{} + (-c)\check{} = \check{c} + \check{b} + (-c)\check{} + (-a)\check{},$$

therefore

$$\check{b} + (-a)\check{} = \check{d} + (-c)\check{},$$

and since the mapping $\check{}: x \mapsto \check{x}$ is an order isomorphism, it easily follows that $(d-c)\check{} = (b-a)\check{}$. It is evident from the last equality that $\widetilde{S}(\check{x}) = \widetilde{S}(\check{y})$. Finally, it holds that

$$\widetilde{S}(\check{x}) + T(\check{x}_2) = (b-a)\check{} + T(\check{x}_2) = (b-a)\check{} + \check{a} = \check{b} = T(\check{x}_1),$$

and it is easy to see that $\widetilde{S}$ is $\vee$-preserving.

(ii) For each $x \in E$ the following relations are valid :

$$\check{x} + x^- = x^+ \textit{ and } \check{x} + \check{x}_2 = \check{x}_1.$$

So we have

$$\check{x} + x^- + \check{x}_2 = x^- + \check{x}_1 \textit{ and } x^+ + \check{x}_2 = x^- + \check{x}_1.$$

Since $x^+, x^-, \check{x}_1, \check{x}_2 \in U(E)^+$ and T is additive in $U(E)^+$, it is implied that

$$T(x^+) + T(\check{x}_2) = T(x^-) + T(\check{x}_1),$$

hence

$$T(x^+) + \widetilde{S}(\check{x}) + T(\check{x}_2) = \widetilde{S}(\check{x}) + T(x^-) + T(\check{x}_1)$$

and from (i) it follows that

$$T(x^+) + T(\check{x}_1) = \widetilde{S}(\check{x}) + T(x^-) + T(\check{x}_1).$$

It is evident, from $\check{b} = T(\check{x}_1)$, that

$$T(x^+) + \check{b} + (-b)\check{} = \widetilde{S}(\check{x}) + T(x^-) + \check{b} + (-b)\check{}$$

and then

$$T(x^+) + \check{0} = \widetilde{S}(\check{x}) + T(x^-) + \check{0}.$$

But $\check{0}$ is the neutral element of the directed wedge $U(F)$, and thus we have the proof of (ii).

(iii) Since $S(\check{x}) + T(x^-) = T(x^+)$ and $\widetilde{S}(\check{x}) + T(x^-) = T(x^+)$, it follows that

$$S(\check{x}) + T(x^-) = \widetilde{S}(\check{x}) + T(x^-)$$

and

$$\check{c} + T(x^-) = (b-a)\check{} + T(x^-), \ \textit{where } S(\check{x}) = \check{c}, c \in F.$$

From Proposition 3.1 it is implied that $\check{c} = (b-a)\check{}$, thus $S(\check{x}) = \widetilde{S}(\check{x})$. □

We shall discuss a notion of disjointness as in [5, Definition 8]. Let E be a partially ordered vector space, the set $[x, y] = \{z \in E | x \leq z \leq y\}$ is the order interval in E which is defined by the elements $x, y \in E$.

We say that two positive elements $x, y \in E, x, y \neq 0$ are *disjoint* if

$$[0, x] \cap [0, y] = \{0\}.$$

According to [7, Proposition 2.3(iii)] it is evident that x, y are *disjoint* in E (in the above sense) if and only if $\check{x} \vee \check{y} = \check{x} + \check{y}$. Consider the notations of Theorem 3.4, then the following corollary inform us that for each pair of disjoint elements $x, y \in E$ we are able to determine the extension mapping S, of T, at the element $\check{x} \vee \check{y} \in U(E)$ by its action on the elements $\check{x}, \check{y}$. Note that $\check{x} \vee \check{y} \in U(E)^+$ does not necessarily imply that $\check{x}$ or $\check{y}$ or both of them belong to $U(E)^+$.

Corollary 3.5. *Let E, F be directed, ordered, vector spaces satisfying the Riesz decomposition property with F be an Archimedean space. Assume that, $T : U(E)^+ \to U(F)^+$ is a $\vee$-preserving mapping and for all $A, B \in U(E)^+$ it holds $T(A + B) = T(A) + T(B)$. If x, y are disjoint elements of E, then it holds $S(\check{x} \vee \check{y}) = S(\check{x}) + S(\check{y})$ and for each $x, y \in E^+$ with x, y disjoint it holds $S(\check{x} \vee \check{y}) = T(\check{x} \vee \check{y})$.*

Proof. Suppose that x, y are disjoint elements of E, then by Proposition 3.3 (i) we have $S(\check{x} \vee \check{y}) = S(\check{x} + \check{y}) = S(\check{x}) + S(\check{y})$. Also, if x, y are disjoint elements of E^+ then $S(\check{x} \vee \check{y}) = S((x + y)\check{}) = T((x + y)\check{}) = T(\check{x} \vee \check{y})$. □

4 A Riesz-kantorovich type theorem

In the present section we shall discuss a generalization of the Riesz-Kantorovich theorem (cf., [3] Theorem 2.6.1) for order bounded mappings between directed wedges. In particular, it will be proved that the validity of the Riesz decomposition property on $U(E)$ is a sufficient condition for the existence of the positive part f^+ of an order bounded mapping between directed wedges.

In what follows, E, F shall denote directed, ordered vector spaces satisfying the Riesz decomposition property and $U(E), U(F)$ shall denote the directed wedges that come from E, F, respectively. A mapping $P : X \to U(E)$ between a vector space X and $U(E)$ is called *subadditive* if $P(x+y) \leq P(x) + P(y)$, for each $x, y \in X$ and *positive homogeneous* if $P(rx) = rP(x)$, for each $r \geq 0$ and each $x \in X$. We shall say that a mapping from X to $U(E)$ is *sublinear*, if it is subadditive and positive homogeneous. Let $x, y \in E$, then the set

$$[\check{x}, \check{y}] = \{A \in U(E) | \check{x} \leq A \leq \check{y}\}$$

is the *corresponding order interval* in $U(E)$, of the order interval $[x, y]$ which is defined by the elements x, y.

For each $u, v \in E^+$ we define the sublinear mapping $P : \mathbb{R}^2 \to U(E)$ with

$$P(t, s) = \frac{1}{2}((|t| + t)\check{u} + (|s| + s)\check{v}). \tag{2}$$

Also, for each $C \in [\check{0}, (u+v)\check{\ }]$ we define a mapping $T : \mathbb{R}^+ \to U(E)$ with

$$T(\lambda) = \lambda C, \text{ for each } \lambda \geq 0. \tag{3}$$

Definition 4.1. The directed wedge $U(E)$ has the *(E)-property (with respect to* (P, T)*)* if there exists an additive mapping $A : \mathbb{R}^2 \to U(E)$ such that $\check{0} \leq A(1, 0), A(0, 1)$ and

$$A(t, s) \leq P(t, s), \text{ for each } t, s \in \mathbb{R}, \text{ and } A(z, z) = T(z), \text{ for each } z \in \mathbb{R}^+.$$

Example 4.2. Assume that $E = \mathbb{R}$ and consider the generalized wedge $U(E)$. An easy argument shows that the generalized wedge $U(E)$ consists of those intervals of $\mathbb{R}$ which have one of the forms $(x, +\infty)$ or $[x, +\infty)$, for some $x \in \mathbb{R}$. Let $u, v \in \mathbb{R}^+$ and $P : \mathbb{R}^2 \to U(E)$ is the sublinear mapping defined in (2), and for each $C \in [\check{0}, (u+v)\check{\ }]$ let us consider the mapping $T : \mathbb{R}^+ \to U(E)$ as defined in (3). Since $C \in [\check{0}, (u+v)\check{\ }]$ it follows that $T(z) = z \cdot C \leq z(u+v)\check{\ } = P(z, z)$, for each $z \geq 0$.

Case 1: Suppose that $C = (a, +\infty)$, then since $C \in [\check{0}, (u+v)\check{\ }]$ we have that $0 \leq a \leq u + v$, we can also assume that $u \leq a$. We define the mapping $A : \mathbb{R}^2 \to U(E)$ such that

$$A(t, s) = \begin{cases} (tu + (a-u)s, +\infty) & , s \geq 0 \\ (tu + (a+u)s, +\infty) & , s < 0 \end{cases}$$

for each $t \in \mathbb{R}$. It easily follows that $\check{0} \leq A(1, 0), A(0, 1)$. Also, A is additive and it holds

$$A(z, z) = (zu + (a-u)z, +\infty) = z(a, +\infty) = T(z), \text{ for each } z \geq 0.$$

Suppose that $s \geq 0$, then

$$P(t,s) = [\frac{|t|+t}{2}u + sv, +\infty)$$

and since $a \leq u + v$ it holds $tu + (a-u)s \leq \frac{|t|+t}{2}u + sv$ which implies that $A(t,s) \leq P(t,s)$ for each $s \geq 0$ and $t \in \mathbb{R}$. On the other hand, if $s < 0$ then

$$P(t,s) = [\frac{|t|+t}{2}u, +\infty)$$

and it easily follows that $tu + (a+u)s \leq \frac{|t|+t}{2}u$, for each $s < 0$ and $t \in \mathbb{R}$. Thus, it holds $A(t,s) \leq P(t,s)$.
Case 2: Suppose that $C = [a, +\infty)$, then we perform the analysis as in the previous.
Therefore, the generalized wedge $U(\mathbb{R})$ has the (E)-property.

We say that a partially ordered generalized wedge W possesses the *Riesz decomposition property* if, whenever $0 \leq v, w_1, w_2$ in W and $v \leq w_1 + w_2$, then there exist $v_1, v_2 \in W$ such that $v = v_1 + v_2$ where $0 \leq v_i \leq w_i (i = 1, 2)$. This is equivalent (cf. [7] p.2) to requiring that the order intervals $[0, v]$ in W should be additive: that is, that

$$[0, v + w] = [0, v] + [0, w], \ \textit{for each } v, w \geq 0 \ \textit{in } W.$$

Proposition 4.3. *If $U(E)$ has the (E)-property, then $U(E)$ possesses the Riesz decomposition property.*

Proof. Let $v, v' \in E^+$ and consider the following mapping $P : \mathbb{R}^2 \to U(E)$:

$$P(t,s) = \frac{1}{2}((|t|+t)\check{v} + (|s|+s)(v')\check{}).$$

Then, it holds $P(1,1) = (v+v')\check{}$ and $P(0,0) = \check{0}$ hence

$$[\check{0}, (v+v')\check{}] = [P(0,0), P(1,1)].$$

It easily follows that P is positive homogeneous. Also, the mapping $\check{} : x \mapsto \check{x}$ is an order isomorphism and for each $t_i, s_i \in \mathbb{R}, i = 1, 2$ it holds $|t_1+t_2|v+(t_1+t_2)v+|s_1+s_2|v'+(s_1+s_2)v' \leq |t_1|v+|t_2|v+(t_1+t_2)v+|s_1|v+|s_2|v'+(s_1+s_2)v'$ therefore P is subadditive. Let

$$C \in [\check{0}, (v+v')\check{}] = [P(0,0), P(1,1)]$$

then $C \leq P(1,1)$ and for each $\lambda \in \mathbb{R}^+$ we have $\lambda C \leq \lambda P(1,1) = P(\lambda, \lambda)$. Let $T : \mathbb{R}^+ \to U(E)$ with $T(\lambda) = \lambda C$. It easily follows that T is additive, thus by the (E)-property there exists an additive mapping $A : \mathbb{R}^2 \to U(E)$ such that $A(t,s) \leq P(t,s)$, for each $t,s \in \mathbb{R}$ and $A(x,x) = T(x)$, for each $x \in \mathbb{R}^+$. It follows that $A(1,1) = T(1) = C$ so, if we set $Q = A(1,0), Q' = A(0,1)$ we have

$$Q = A(1,0) \leq P(1,0) = \check{v} \text{ and } Q' = A(0,1) \leq P(0,1) = (v')\check{}.$$

But $\check{0} \leq A(1,0), A(0,1)$ and the additivity of A implies that $Q+Q' = A(1,1) = C$. Thus $U(E)$ possesses the Riesz decomposition property. □

We shall denote by $L(U(E),U(F))$ the set of all linear mappings from $U(E)$ to $U(F)$, that is

$$L(U(E),U(F)) = \{T : U(E) \to U(F) | T \text{ linear mapping } \}.$$

The set $L(U(E),U(F))$ equipped with the pointwise operations of addition and scalar multiplication i.e.,

$$(T+S)(A) = T(A) + S(A), \quad (\lambda T)(A) = \lambda T(A), \quad A \in U(E), \lambda \in \mathbb{R}^+$$

is a generalized wedge since for two mappings $S,T \in L(U(E),U(F))$ the sum $T+S$ and λT, $\lambda \in \mathbb{R}^+$ are also elements of $L(U(E),U(F))$.

The generalized wedge $L(U(E),U(F))$ with the partial order $(\leq_L)$ defined by

$$f \leq_L g \text{ if and only if } f(A) \leq g(A) \text{ in } U(F), \text{ for each } A \in U(E),$$

becomes a partially ordered generalized wedge, since $U(F)$ is a partial ordered generalized wedge. Also, we shall denote by $\mathbf{0}$ the zero element of $L(U(E),U(F))$.

Definition 4.4. A mapping $T \in L(U(E),U(F))$ is called *order bounded* if for each $x,y \in E, x \leq y$ it maps the corresponding order interval $[\check{x},\check{y}]$ of $U(E)$ into an order interval $[\check{a},\check{b}]$ of $U(F)$ for some $a,b \in F$.

We shall denote by $L^b(U(E),U(F))$ the set of all linear, order bounded mappings from $U(E)$ to $U(F)$. In order to give a generalized version of the Riesz-Kantorovich theorem for the generalized wedge $L^b(U(E),U(F))$ we shall need a notion of order completeness.

Definition 4.5. The directed wedge $U(E)$ is called *order complete* if each upper bounded subset of $U(E)$ has a least upper bound $\check{u}$, for some $u \in E$.

Consider a mapping $T : U(E) \to U(F)$ and a subset $\mathcal{C}$ of $U(E)$, then we shall denote by $T(\mathcal{C})$ the subset of $U(F)$ that contains all those elements $T(K)$ of $U(F)$ where $K \in \mathcal{C}$. Also, if $\mathcal{C}, \mathcal{K}$ are subsets of $U(E)$ then we define an operation of addition between subsets of $U(E)$ by

$$\mathcal{C} + \mathcal{K} = \{A + B | A \in \mathcal{C}, B \in \mathcal{K}\}.$$

Now we are ready to provide a Riesz-Kantorovich type theorem for the generalized wedge $L^b(U(E),U(F))$. We start with an easy Lemma.

Lemma 4.6. *Let T be an additive mapping from $U(E)$ to $U(F)$. Then for each $x, y \in E$ it holds*

$$T([\check{0}, \check{x}] + [\check{0}, \check{y}]) = T([\check{0}, \check{x}]) + T([\check{0}, \check{y}]).$$

Proof. Let $W \in T([\check{0}, \check{x}] + [\check{0}, \check{y}])$ then $W = T(A+B)$ for some $A \in [\check{0}, \check{x}], B \in [\check{0}, \check{y}]$. Since T is additive it follows that $W = T(A) + T(B)$ while $T(A) \in T([\check{0}, \check{x}]), T(B) \in T([\check{0}, \check{y}])$, thus we have the inclusion $T([\check{0}, \check{x}] + [\check{0}, \check{y}]) \subseteq T([\check{0}, \check{x}]) + T([\check{0}, \check{y}])$. For the converse inclusion, consider $W \in T([\check{0}, \check{x}]) + T([\check{0}, \check{y}])$. Then, $W = C + D$ where $C \in T([\check{0}, \check{x}]), D \in T([\check{0}, \check{y}])$. Thus, $C = T(Q), D = T(P)$ for some $Q \in [\check{0}, \check{x}]$ and $P \in [\check{0}, \check{y}]$ which implies that $W = T(Q) + T(P) = T(Q + P)$. Therefore $W \in T([\check{0}, \check{x}] + [\check{0}, \check{y}])$ and the converse inclusion is proved. □

We shall denote by $f \vee_b g$ the supremum (if it exists) of two elements $f, g \in L^b(U(E), U(F))$.

Theorem 4.7. *Let $U(E)$ has the (E)-property and $U(F)$ is an order complete, directed wedge. Then, for each $f \in L^b(U(E), U(F))$ and each $x \in E^+$ we have*

(*i*) $f^+(\check{x}) = \sup f([\check{0}, \check{x}])$.

(*ii*) $(f \vee_b g)(\check{x}) = \sup\{f(\check{u}) + g((x-u)\check{\ })|\check{0} \leq \check{u} \leq \check{x}\}$.

Proof. (*i*) Let $f \in L^b(U(E), U(F))$ be given; for each $x \in E^+$, we define a mapping g with

$$g(\check{x}) = \sup f([\check{0}, \check{x}]).$$

The $\sup f([\check{0}, \check{x}])$ exists in $U(F)$ since f is order bounded and $U(F)$ is order complete.

We shall extend g to a mapping $\widetilde{g}$ where for each $x = x_1 - x_2, x_1, x_2 \in E^+$ it holds

$$\widetilde{g}(\check{x}) = (b-a)\check{\ }, \check{b} = g(\check{x_1}), \check{a} = g(\check{x_2}).$$

In view of the proof of Theorem 3.4(*i*), it is enough to show that g is a $\vee$-preserving mapping and for each $u, v \in E^+$ it holds $g(\check{u} + \check{v}) = g(\check{u}) + g(\check{v})$. Indeed, g is a $\vee$-preserving mapping since, f is an order bounded mapping and $U(F)$ is an order complete directed wedge. In order to prove that $g(\check{u} + \check{v}) = g(\check{u}) + g(\check{v})$, we start by pointing out that according to Proposition 4.3 the assumption of the (E)-property in $U(E)$ implies that $[\check{0}, \check{u}+\check{v}] = [\check{0}, \check{u}]+[\check{0}, \check{v}]$, so using Lemma 4.6 it holds $g(\check{u}+\check{v}) = \sup f([\check{0}, \check{u}+\check{v}]) = \sup(f([\check{0}, \check{u}])+f([\check{0}, \check{v}]))$. Then, we set $g(\check{u} + \check{v}) = \check{b}$, $g(\check{u}) = \check{d}$, $g(\check{v}) = \check{c}$, $b, d, c \in F$, therefore $A \leq \check{d}$, for each $A \in f([\check{0}, \check{u}])$ and $B \leq \check{c}$, for each $B \in f([\check{0}, \check{v}])$. Thus, $A + B \leq \check{d} + \check{c}$, for all $A \in f([\check{0}, \check{u}]), B \in f([\check{0}, \check{v}])$ which implies that $\check{b} \leq \check{d} + \check{c}$. Suppose that $A + B \leq \check{x}$, for each $A \in f([\check{0}, \check{u}])$ and $B \in f([\check{0}, \check{v}])$. Then, $A + \check{b} \leq \check{x}$, for each

$\check{b} \in f([\check{0}, \check{v}])$ so $A \leq (x-b)\check{}$, for each $A \in f([\check{0}, \check{u}])$. It follows that $\check{d} \leq (x-b)\check{}$ hence $\check{b} + \check{d} \leq \check{x}$. Thus, for each $u, v \in E^+$ it holds $g(\check{u} + \check{v}) = g(\check{u}) + g(\check{v})$.

We show that $f^+(\check{x}) = \widetilde{g}(\check{x})$ for each $x \in E$; indeed, $f^+(\check{x}) \leq \widetilde{g}(\check{x})$ for each $x \in E^+$, and if $\mathbf{0} \leq_L h$ is an order bounded mapping on $U(E)$ such that $x \in E^+$ it is implied that $f(\check{x}) \leq h(\check{x})$, then $f(\check{y}) \leq h(\check{y}) \leq h(\check{x})$ for all $\check{y} \in [\check{0}, \check{x}]$, which shows that $\widetilde{g}(\check{x}) = \sup f([\check{0}, \check{x}]) \leq h(\check{x})$ whenever $x \in E^+$. Finally, it is easy to see that by the definition of $\widetilde{g}$ we have that $\widetilde{g}$ is order bounded.

(ii) Define $k : U(E)^+ \to U(F)^+$ by $k(\check{x}) = \sup\{f(\check{u}) + g((x-u)\check{})|\check{0} \leq \check{u} \leq \check{x}\}$. Similar arguments, as in the proof of (i), can be used in order to prove that the mapping k can be extended to an order bounded mapping $\widetilde{k}$ from $U(E)$ to $U(F)$ such that $\widetilde{k}(\check{x}) = (f \vee_b g)(\check{x})$, for each $x \in E$. □

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