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Finite Branched Coverings in a Generalized Inverse Mapping Theorem

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Abstract. Let $U \subset I\!R^n$ be a nonempty open connected set and let $f : U \to I\!R^n$ be an open continuous map such that, for all $y \in f(U)$, $f^{-1}(y)$ is a discrete set. Given $x \in U$, there exist: **(a)** an open connected neighborhood V of x with $\mathrm{cl}(V)$ compact and contained in U, and **(b)** an integer $\ell \geq 1$ and an ℓ–fold branched covering $f_{|_V} : V \to f(V)$ which is a proper map. Moreover if, for all $x \in U$, $|\deg(f, x)| = 1$, then f is a local homeomorphism.

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1. Introduction

The local inversion of maps is one of the most important subjects in Topology and Analysis. With respect to this we have the fundamental Inverse Mapping Theorem for maps from $I\!R^n$ to $I\!R^n$ which requires regularity and continuity of the derivative, of the map, around the given point. We shall now mention some results that use weaker assumptions. F. H. Clarke [7], using the concept of generalized gradient, proves the Inverse Mapping Theorem under a condition that need the map be differentiable. In [20], S. Radulescu and M. Radulescu, state some Inverse Mapping Theorems for mappings in Banach spaces, where the derivative is regular, around the given point, but may no vary continuously. Our result generalizes, in the finite dimensional case, the Inverse Mapping Theorem of [4], [5] (see also [21], [18]). Some articles somehow related to our work are those of [3], [6], [10], [11], [12], [13], [14], [16], [17], [19], [22].

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We say that $f : X \to Y$ is it discrete if for all $y \in f(X)$, $f^{-1}(y)$ is a discrete set.

Let ℓ be a positive integer and let Z, B be open connected subsets of $I\!R^n$; we say that a surjective open proper continuous map $f : Z \to B$ is an *$\ell-$fold branched covering* if no point in B has more than ℓ pre-images, $U = \{y \in B \,|\, \#f^{-1}(y) = \ell\}$ is an open and dense connected subset of B, $\dim(B \setminus U) \leq n-2$ and

$$f : f^{-1}(U) \to U$$

is an $\ell-$fold covering map. Notice that if B_f denotes the closed set made up of the points $x \in Z$ such that f is not a local homeomorphism at x then $f(B_f) = B \setminus U$, $\dim(Z \setminus f^{-1}(U)) \leq n-2$ and so $f^{-1}(U)$ is a connected open and dense subset of Z.

We state below the main result of this article which generalizes the Inverse Mapping Theorem:

Theorem A. Let $U \subset I\!R^n$ be a nonempty open connected set and let $f : U \to I\!R^n$ be an discrete open continuous map. Given $x \in U$, there exist:

(a) An open connected neighborhood Z of x with $\mathrm{cl}(Z)$ compact and contained in U and such that $f(Z)$ is a ball;
(b) An integer $\ell \geq 1$ and an ℓ–fold branched covering $f_{|Z} : Z \to f(Z)$.

Černavskii's Theorem implies that the $\mathrm{cl}(V) \setminus V$, $n-2$

As a consequence of Theorem above, we will obtain

Corollary. Let $U \subset I\!R^n$ be an open set and let $f : U \to I\!R^n$ be a discrete, continuous map. If, for all $x \in U$, $|\deg(f, x)| = 1$, then f is a local homeomorphism.

A particular case of corollary above is when f is a differentiable (not necessarily C^1) map such that, for all $x \in U$, the derivative $Df(x)$ is non–singular.

Section 2 and 3 are devoted to prove our main result. The section 4 is dedicated to state some corollaries of Theorem A. In Section 3, we provide an alternative proof of a Černavskii's Theorem (Černavskii [4], [5], Väisälä [21]).

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2. Preliminary results

When $A \subset B \subset Y$ and Y is a topological space, then $\mathrm{int}_B\, A$ denotes the interior of A relative to B, and similarly $\mathrm{cl}_B\, A$ and $\partial_B A$ denote the relative closures and relative boundaries of the set A in B.

Proposition 2.1. *Let $f : X \to Y$ be an open surjective continuous map with finite point-inverses. If X and Y are metrizable and Y is a Baire space, then the set $\bigcup \mathrm{int}(E_n)$ is dense in Y, where*

$$E_n = \{y \in Y : \#f^{-1}(y) = n\}$$

Proof: Let $F = \mathrm{cl}\big(\bigcup_{n=1}^{\infty} \mathrm{int}(E_n)\big)$ and suppose by contradiction that

$$(*) \qquad U = Y \setminus F \neq \emptyset.$$

Given $n \in \mathbb{N}$, denote by

$$O_n(U) = \{y \in U : \#f^{-1}(y) \geq n\}.$$

As U is an open set and f is an open map, for all n, $O_n(U)$ is an open set. Certainly, $O_1(U) = U$ and – as U is a Baire space (open subsets of Baire spaces are Baire spaces too) – there exists $j \geq 1$ such that

$$U = \mathrm{cl}_U(O_1(U)) = \mathrm{cl}_U(O_2(U)) = \cdots = \mathrm{cl}_U(O_j(U)),$$

but $\mathrm{cl}_U(O_{j+1}(U))$ is properly contained in U. Therefore,

$$\mathrm{int}_U(E_j(U)) \neq \emptyset,$$

where $E_j(U) = \{y \in U : \#f^{-1}(y) = j\}$. As U is an open set,

$$\emptyset \neq \mathrm{int}_U(E_j(U)) = \mathrm{int}(E_j(U)).$$

However, this is not possible because

$$\mathrm{int}(E_j(U)) \subset \mathrm{int}(E_j), \ \ \mathrm{int}_U(E_j(U)) \subset U$$

and

$$\mathrm{int}(E_j) \cap U = \emptyset.$$

This gives a contradiction with (*) and proves the proposition. ■

In the following, given a locally compact, Hausdorff, topological space W, we shall denote by $W^\infty = W \cup \{\infty\}$ the one-point Alexandroff compactification of W. Observe that when W is compact, ∞ is an isolated point of W^∞. In this way, as W is a subspace of W^∞, given a closed subset X of W we may consider X^∞ as a subspace of W^∞.

In this article, the considered homologies and cohomologies will be that of Čech having $\mathbb{K} \in \{\mathbb{Z}, \mathbb{Z}_2\}$ as the coefficient group.

Lemma 2.2. *Let W be a connected topological n-manifold $(n > 0)$ and X be a closed subset of W. Suppose that $\emptyset \subsetneqq X \subsetneqq W$. Then $H^n(X^\infty) = 0$, under either of the following assumptions:*

(a) *the coefficient group is $\mathbb{Z}_2$ or*
(b) *the coefficient group is $\mathbb{Z}$ and W is orientable. $\mathbb{K} \in \{\mathbb{Z}, \mathbb{Z}_2\}$.*

Proof: Let $I\!\!K$ denote either $\mathbb{Z}_2$ or $\mathbb{Z}$ according to the assumptions (a) or (b) of the lemma.

Considering the cohomology exact sequence of the triple $(W^\infty, X^\infty, \infty)$, we get

$$(a1) \qquad H^n(W^\infty, X^\infty) \to H^n(W^\infty, \infty) \to H^n(X^\infty, \infty) \to 0.$$

Now observe that as W is connected and $W \setminus X \neq \emptyset$,

$$(a2) \qquad H_0(W \setminus X) \xrightarrow{i*} H_0(W) \cong I\!\!K$$

is onto, where $i : W \setminus X \to W$ is the inclusion.

As $W^\infty \setminus X^\infty = W \setminus X$ and $W^\infty \setminus \{\infty\} = W$, we may use Alexandroff-Čech Duality to obtain

$$H^n(W^\infty, X^\infty) \cong H_0(W^\infty \setminus X^\infty) \cong H_0(W \setminus X)$$

and

$$(a3) \qquad H^n(W^\infty, \infty) \cong H_0(W^\infty \setminus \{\infty\}) \cong H_0(W).$$

Using this and (a2), the exact sequence (a1), up to isomorphisms, can be rewritten as

$$(a4) \qquad H_0(W \setminus X) \xrightarrow{i*} H_0(W) \cong I\!\!K \to H^n(X^\infty, \infty) \to 0.$$

Hence, as i^* is onto, we obtain that $0 = H^n(X^\infty, \infty) = H^n(X^\infty)$, which implies the Lemma. ■

Lemma 2.3. *Let $f : (X, A) \to (Y, B)$ be a continuous proper map, where X, Y are locally compact Hausdorff topological spaces, and A and B are closed subsets of X and Y, respectively. If $H^n(X^\infty) \cong 0$ and $f^* : H^{n-1}(B^\infty) \to H^{n-1}(A^\infty)$ is onto, then $\tilde{f}^* : H^n(Y^\infty, B^\infty) \to H^n(X^\infty, A^\infty)$ is onto too. Here, $\tilde{f} : (X^\infty, A^\infty) \to (Y^\infty, B^\infty)$ denotes the continuous extension of f which takes ∞ to ∞; also, the considered cohomologies have coefficient group $I\!\!K \in \{\mathbb{Z}_2, \mathbb{Z}\}$.*

Proof: Consider the following diagram where rows are exact:

$$\begin{array}{ccccc}
H^{n-1}(B^\infty) & \longrightarrow & H^n(Y^\infty, B^\infty) & \longrightarrow & H^n(Y^\infty) \\
\downarrow \tilde{f}^* & & \downarrow \tilde{f}^* & & \downarrow \tilde{f}^* \\
H^{n-1}(A^\infty) & \longrightarrow & H^n(X^\infty, A^\infty) & \longrightarrow & H^n(X^\infty) \cong 0.
\end{array}$$

The assumptions imply our claim: the middle vertical arrow of the diagram is a surjective map. ■

Proposition 2.4. *Let W be a connected topological n-manifold ($n > 0$) and Y be a locally compact Hausdorff space. Let $\emptyset \subsetneq A \subsetneq X \subsetneq W$ and $\emptyset \subsetneq B \subsetneq Y$ consist of closed subsets of W and Y, respectively. Suppose that $X \setminus A$ and $Y \setminus B$ are topological n-manifolds and $Y \setminus B$ is connected. If $f : (X, A) \to (Y, B)$ is* *a proper continuous map such that $f_{|A} : A \to B$ is a homeomorphism, then $X \setminus A$ is connected.*

Proof: In this proof, the considered homologies and cohomologies have coefficient group $\mathbb{Z}_2$. Since $Y \setminus B = Y^\infty \setminus B^\infty$ is connected,

$$H^n(Y^\infty, B^\infty) = H_0(Y \setminus B) \cong \mathbb{Z}_2.$$

Moreover, as $X^\infty \setminus A^\infty = X \setminus A \neq \emptyset$,

$$H^n(X^\infty, A^\infty) = H_0(X \setminus A) \neq 0.$$

Now, as by Lemma 2.2, $H^n(X^\infty) \cong 0$, Lemma 2.3 implies that $\tilde{f}^* : H^n(Y^\infty, B^\infty) \to H^n(X^\infty, A^\infty)$ is onto. Therefore, $H_0(X \setminus A) = H^n(X^\infty, A^\infty) \cong \mathbb{Z}_2$ which implies that $X \setminus A$ is connected. ■

Now, we shall recall the definition of the degree of a continuous proper map $f : X \to Y$ between the oriented connected topological $n-$manifolds X and Y, with $n \geq 1$. Now, we will consider cohomologies which have coefficient group $\mathbb{Z}$. Observe that $H^n_C(Y) \cong H^n(Y^\infty) \cong \mathbb{Z} \cong H^n_C(X) \cong H^n(X^\infty)$, where H^*_C denotes the Čech-cohomology with compact support. Given the cohomological fundamental classes $\alpha \in H^n_C(Y)$ and $\beta \in H^n_C(X)$, the *degree* of f is the integer number $\deg(f)$ such that $f^*(\alpha) = \deg(f) \cdot \beta$, where $f^* : H^n_C(Y) \to H^n_C(X)$. In this way, when f^* is an isomorphism, $|\deg(f)| = 1$.

Proposition 2.5. *Let X and Y be locally compact, connected, metric spaces. Let A (resp. B) be a closed subset of X (resp. of Y). Suppose that $Y \setminus B$ and $X \setminus A$ are connected, oriented, topological n-manifolds. Also, assume that $f : X \to Y$ is a proper surjective map such that $f^{-1}(B) = A$ and $f|_A$ a homeomorphism of A onto B. If $H^n(X^\infty) = 0$, where the coefficient group is $\mathbb{Z}$, then $f\colon X \setminus A \to Y \setminus B$ is a continuous proper map and $|\deg(f|_{X\setminus A})| = 1$.*

Proof: It is obvious that $f\colon X \setminus A \to Y \setminus B$ is a continuous proper map. Observe that

(a1) $H^n(X^\infty, A^\infty) \cong H^n_C(X \setminus A) \cong H_0(X \setminus A) \cong \mathbb{Z}$,
(a2) $H^n(Y^\infty, B^\infty) \cong H^n_C(Y \setminus B) \cong H_0(Y \setminus B) \cong \mathbb{Z}$.

Since $H^n(X^\infty) = 0$ and $(f|_A)^*$ is an isomorphism, it follows from Lemma 2.3 (with the notations introduced there) that $\tilde{f}^*\colon H^n(Y^\infty, B^\infty) \to H^n(X^\infty, A^\infty)$ is an epimorphism. Using this, (a1) and (a2), we have that

$$(f|_{X\setminus A})^* : H^n_C(Y \setminus B) \to H^n_C(X \setminus A)$$

is an epimorphism. As $H^n_C(X \setminus A) \cong \mathbb{Z}$ and $H^n_C(Y \setminus B) \cong \mathbb{Z}$, we conclude that $(f|_{X\setminus A})^*$ is an isomorphism. Thus $|\deg(f|_{X\setminus A})| = 1$. ■

3. Main Theorem

To prove Theorem A we shall use the following

Lemma 3.1. *Given $x \in U$, there exists a connected open neighborhood V of x such that $V \subset \mathrm{cl}(V) \subset U$ and*

$$f : V \to f(V)$$

is a proper map.

Proof: Let W be a small disc centered at x such that $\mathrm{cl}(W) \subset U$. As f is a open mapping, $f(\partial W)$ is a closed set having empty interior. As f is discrete we way take W so that $f(x) \notin f(\partial W)$. Let V be the connected component of $W \setminus (f^{-1}(f(\partial W)))$ containing x. Since $W \setminus (f^{-1}(f(\partial W)))$ is open, V is an open connected neighborhood of x and $f(V) \subset f(W) \setminus f(\partial W)$ is an open connected neighborhood of $f(x)$. As $f(\partial V) \subset f(\partial W)$, $f(\partial V) \cap f(V) = \emptyset$. □

Lemma 3.2. *Suppose that f is also a proper map. Given $y \in f(U)$ and $\{x_1, x_2, \cdots, x_k\} = f^{-1}(y)$, there exists $\epsilon > 0$ such that if $0 < r < \epsilon$ and B denotes the open ball centered at y, having radius r, then $f^{-1}(B)$ is the finite union of pairwise disjoint open connected sets $V_1, V_2, \cdots, V_k$, such that, for all i, $x_i \in V_i$, $f(V_i) = B$ and*

$$f : f^{-1}(B) \to B$$

is a proper map.

Proof: Take pairwise disjoint connected open subsets $W_1, W_2, \cdots, W_k$, of U such that, for all i, W_i is a neighborhood of x_i. Let B be any ball centered at y such that $\mathrm{cl}(B) \subset \cap_{i=1}^{k} f(W_i)$. Let V_i be the (open) connected component of $f^{-1}(B)$ which contains x_i. As f is a proper map, $f^{-1}(\mathrm{cl}(B))$ is a compact set. Moreover if Z is an open connected component of $f^{-1}(B)$, then $f(Z)$ is an open connected subset of B such that $f(\partial Z) \subset \partial B$; this implies that $f(Z) = B$ and $f(\partial Z) = \partial B$. Therefore, for some i, $Z = V_i$ and $f : V_i \to B$ is a proper map. This proves the lemma. ■

Lemma 3.3. *Suppose that $B \subset f(U)$ is an open ball and that $f : f^{-1}(B) \to B$ is a proper map. Then $B \setminus f(B_f)$ is a connected set.*

Proof: Let E be a connected component of $B \setminus f(f^{-1}(B) \cap B_f)$. Denote by $G_2(E)$ the set of points $x \in f(f^{-1}(B) \cap B_f)$ such that, for some open ball D, $x \in D \subset B$ and $(D \setminus f(f^{-1}(B) \cap B_f)) \subset E$. As E is an open connected set, $E \cup G_2(E)$ is also a open connected set. Let G_2 be the union of all sets $G_2(E)$ such that E is a connected component of $B \setminus f(f^{-1}(B) \cap B_f)$. Let $G_1 = f(f^{-1}(B) \cap B_f) \setminus G_2$. We conclude that

(a) if C is a connected component of $B \setminus G_1$ then there exists exactly one connected component E of $B \setminus f(f^{-1}(B) \cap B_f)$ such that $C = E \cup G_2(E)$.

Suppose, by contradiction, that

(b) $B \setminus f(f^{-1}(B) \cap B_f)$ is not connected.

This and (a) imply that

(c) $B \setminus G_1$ is not connected.

By Lemma 3.2, as f is an open map and by Proposition 2.1 applied to the map $f : f^{-1}(G_1) \to G_1$, we obtain that

(d) there exists an open ball $Z \subset B$, with $\mathrm{cl}(Z) \subset B$, such that $Z \cap G_1 \neq \emptyset$ and if $Y_1, Y_2, \cdots, Y_t$ are the connected components of $f^{-1}(Z)$ then, for all $i \in \{1, 2, \cdots, t\}$,

$$h_i : h_i^{-1}(Z \cap G_1) \to Z \cap G_1$$

is a homeomorphism, where $h_i = f|_{Y_i}$.

Observe that by definition of G_1, the set $Z \setminus G_1$ is not connected. Let C be a connected component of $Z \setminus G_1$ and let $i \in \{1, 2, \cdots, t\}$.

As Y_i is connected,

$$\emptyset \subsetneqq h_i^{-1}(\mathrm{cl}_Z(C)) = \mathrm{cl}_{Y_i}(h_i^{-1}(C)) \subsetneqq Y_i,$$

and $h_i^{-1}(\mathrm{cl}_Z(C))$ is a closed subset of the open set Y_i, we may apply Lemma 2.2 to obtain that

(e) $H^n(h_i^{-1}(\mathrm{cl}_Z(C))) \cong 0$, where the considered cohomology has coefficient group $\mathbb{Z}$.

Also, as Y_i is connected (see (d)), by Proposition 2.4 applied to the map

$$h_i : (\mathrm{cl}_{Y_i}(h_i^{-1}(C)), \partial_{Y_i}(h_i^{-1}(C))) \to (\mathrm{cl}_Z(C)), \partial_Z(C))),$$

we obtain that

(f) $h_i^{-1}(C)$ is connected.

Therefore, by (e) and by Proposition 2.5 applied to the map

$$h_i : (\mathrm{cl}_{Y_i}(h_i^{-1}(C)), \partial_{Y_i}(h_i^{-1}(C))) \to (\mathrm{cl}_Z(C)), \partial_Z(C))),$$

it follows that h_i takes homeomorphically $h_i^{-1}(C)$ onto C. Since h_i is an open map and

$$\# h_i^{-1} : Z \to \mathbb{N}$$

is a lower semi-continuous map, we conclude that h_i takes homeomorphically $\mathrm{cl}_{Y_i}(h_i^{-1}(C))$ onto $\mathrm{cl}_Z(C))$.

As $\mathrm{cl}_Z(C)$ is the closure of an arbitrary connected component C of $Z \setminus G_1$ we obtain that, for all $i \in \{1, 2, \cdots, t\}$,

(g) h_i takes homeomorphically Y_i onto Z.

This implies that $Z \cap G_1 = \emptyset$. This contradiction with (d) proves that $B \setminus f(B_f) = B \setminus f(f^{-1}(B) \cap B_f)$ is connected. ■

The Lemma above is equivalent to that of [5, page 471] The proof of the following result can be found in [15].

Proposition 3.4. *Let Y be a compact metric space with finite topological dimension* $\dim(Y)$ *and let p be a positive integer. Then* $\dim(Y)$ *is less or equal than p if, and only if, for all closed subset C of Y, $H^{p+1}(Y,C)=0$.*

Corollary 3.5. *Let Y be a locally compact separable metric space with finite topological dimension and let p be a positive integer. Then,* $\dim(Y)\le p$ *if, and only if, for all closed subset C of Y, $H^{p+1}(Y,C)=0$.*

Proof: Consider the one-point Alexandroff Compactification $Y^\infty = Y\cup\{\infty\}$ of Y. Let $C\subset Y$ be closed; then $C = C\cup\{\infty\}$ is a compact subset of Y^∞. In this way, by Proposition 3.4, $H^{p+1}(Y,C)=H^{p+1}(Y^\infty,C^\infty)=0$. ∎

The following proposition is a sort of converse of a theorem due to Marzukiewicz [9, Ch. 1, § 8, Theorem 1.8.19]

Proposition 3.6. *Let X be an open subset of connected topological manifold of dimension $n>0$. Let A be a proper subset of X which is closed in X. Suppose that, for every $x\in A$ and for every small disc $V\subset X$ centered at x, $V\setminus A$ is a nonempty connected set. Then* $\dim(A)\le n-2$.

Proof: Let $C\subset A$ be closed in A. Observe that as $X\setminus A\neq\emptyset$,

$$H^n(A^\infty/C^\infty)\cong H^n(A^\infty,C^\infty)\cong H_0(X\setminus C, X\setminus A)=0 \tag{3.1}$$

Also, as $n>0$,

$$H^n(X^\infty/C^\infty, A^\infty/C^\infty)\cong H^n(X^\infty/C^\infty)\cong \mathbb{Z}. \tag{3.2}$$

In fact, $H^n(X^\infty/C^\infty)\cong H^n(X^\infty,C^\infty)\cong H_0(X^\infty\setminus C^\infty)\cong H_0(X\setminus C)=\mathbb{Z}$. Similarly, $H^n(X^\infty/C^\infty, A^\infty/C^\infty)\cong H_0(X\setminus A)=\mathbb{Z}$.

Now, the assumptions related to the discs V implies that the homomorphism induced by inclusion $i_*: H_1(X\setminus A)\to H_1(X\setminus C)$ is onto. Hence, as the following diagram is commutative,

$$\begin{array}{ccc} H^{n-1}(X^\infty/C^\infty, A^\infty/C^\infty) & \to & H^{n-1}(X^\infty/C^\infty) \\ \updownarrow & & \updownarrow \\ H_1(X\setminus A) & \to & H_1(X\setminus C), \end{array}$$

where the vertical arrows are isomorphisms, we obtain that

$$H^{n-1}(X^\infty/C^\infty, A^\infty/C^\infty)\to H^{n-1}(X^\infty/C^\infty) \qquad \text{is onto.} \tag{3.3}$$

Now, using (3.1) - (3.3) above and considering the exact sequence associated to $C^\infty\subset A^\infty\subset X^\infty$:

$$\begin{array}{ccccc} H^{n-1}(X^\infty/C^\infty, A^\infty/C^\infty) & \to & H^{n-1}(X^\infty/C^\infty) & \to & H^{n-1}(A^\infty/C^\infty) \quad \to \\ H^{n}(X^\infty/C^\infty, A^\infty/C^\infty) & \to & H^{n}(X^\infty/C^\infty) & \to & H^{n}(A^\infty/C^\infty), \end{array}$$

we obtain $H^{n-1}(A^\infty/C^\infty)=H^{n-1}(A^\infty,C^\infty)=0$. This together with Theorem 3.4 imply this theorem. ∎

Proposition 3.7. $\dim\big(f(B_f)\big)\le n-2$.

Proof. Suppose that V is an open subset of U such that $f_{|_V} : V \to f(V)$ is a proper map. Notice that $V \cap B_f$ is a closed subset in V and, as $f_{|_V}$ is open, $f(V \cap B_f)$ is closed in $f(V)$. Therefore, is follows from Lemmas 3.2, 3.3 and Proposition 3.6 applied to the map $f_{|_V} : V \to f(V)$ that $\dim(f(V \cap B_f)) \leq n-2$.

By Lemma 3.1, we may find a countable open covering $\{V_i\}_{i=1}^{\infty}$ of U such that $f : V_i \to f(V_i)$ is a proper map. Notice that each V_i is a contable union of closed sets and that B_f is closed. Therefore, as $\cup_{i=1}^{\infty} f(V_i \cap B_f) = f(B_f)$, it follows that $\dim(f(B_f)) \leq n-2$. ■

Proof of Theorem A:

The existence of an open ball B and an open connected set Z such that $f_{|_Z} : Z \to B$ is a proper surjective map, follows from Lemmas 3.1 and 3.2. Since $\dim(f(B_f)) \leq n-2$, $Z \setminus B_f$ and $B \setminus f(Z \cap B_f)$ are dense connected open subsets of Z and B, respectively. In particular

$$\# f_{|_Z}^{-1} : B \setminus f(Z \cap B_f) \to \mathbb{N}$$

is a constant map equal to $\ell \in \mathbb{N}$. Moreover, as $\# f_{|_Z}^{-1} : B \to \mathbb{N}$ is a lower semicontinuous map, we conclude that for all $y \in f(Z \cap B_f)$, $\#(f_{|_Z})^{-1}(y) \leq \ell$ and so, $B \setminus f(Z \cap B_f) = \{y \in B : \# f^{-1}(y) = \ell\}$.

Therefore $f : Z \mapsto B$ is an ℓ-fold branched covering. ■

As an application we provide an alternative proof of a Černavskii's Theorem.

Theorem 3.8. *(Černavskii). Let $U \subset \mathbb{R}^n$ be an open set and let $f : U \to \mathbb{R}^n$ be an open, discrete, continuous map. Then* $\dim(B_f) \leq n-2$.

Proof: It follows directly from Proposition 3.7 and from the fact that f is a discrete map. ■

Remark 3.9. *Theorems A and 3.8 are also true if f is replaced by a map between generalized manifolds (for definition and basic proprieties, see* [1], [2], [23]*. Specifically, for theorems of duality, see* [1, Theorem 7.2], [2, Theorem 9.2 and Theorem 16.30] *).*

4. Local Inverse Mapping Theorems

Recall, the following well known result [8, Ch. VIII, Proposition 4.5]

Proposition 4.1. *Let X and Y be connected oriented topological manifolds of dimension n and let $f \colon X \to Y$ be a proper continuous map. Then,* $\deg(f, y)$ *does not depend on $y \in Y$ and* $\deg(f)$ *equals to* $\deg(f, y)$.

As a consequence of Theorem 3.1 and Proposition 4.1, we obtain the following theorems

Theorem 4.2. *Let $U \subset \mathbb{R}^n$ be an open set and let $f : U \to \mathbb{R}^n$ be a discrete, continuous map. If, for all $x \in U$, $|\deg(f, x)| = 1$, then f is a local homeomorphism.*

Theorem 4.3. *Let $U \subset \mathbb{R}^n$ be an open set and let $f : U \to \mathbb{R}^n$ be differentiable (not necessarily C^1) without critical points. Then f is a local homeomorphism.*

This last result allow us to extend a classical Darboux Theorem as follows

Theorem 4.4. *Let $U \subset \mathbb{R}^n$ be an open connected set and let $f : U \to \mathbb{R}^n$ be differentiable (not necessarily C^1). Given $\alpha \in \mathbb{R}$ e $x_0, x_1 \in U$ such that* $\det(Df(x_0) - \alpha I) < 0$ *and* $\det(Df(x_1) - \alpha I) > 0$*, Then, there exists $x_2 \in U$ such that* $\det(Df(x_2) - \alpha I) = 0$ *(i.e. α is an eigenvalue of $Df(x_2)$.*

Proof: Let I denote the identity map of $\mathbb{R}^n$. We may apply Theorem 4.3 to the map $g(x) = f(x) - \alpha x$ to obtain that the sign of $\det(Dg(x) - \alpha I$ is locally constant. Therefore, this theorem follows by observing that U is pathwise connected. ∎

References

[1] A. Borel, *The Poincaré duality in generalized manifolds,* Michigan Math. J. 4 (1957), 227–239.

[2] G. E. Bredon, *Sheaf Theory,* Grad. Texts in Math. 170, Springer-Verlag, New York, 1997.

[3] L.A. Campbell, *Characteristic values of the jacobian matrix and global invertibility* Annales Polonici Mathematici, **76** (2001), No. 1-2, 11–20.

[4] A. V. Černavskii, *Finite-to-one open mappings of manifolds.* Mat. Sbornik **65** (1964), 357–369.

[5] A. V. Černavskii, *Addendum to the paper "Finite-to-one open mappings of manifolds".* Mat. Sbornik **66** (1965), 471–472.

[6] M. Chamberland and G. Meisters, *A mountain pass to the Jacobian Conjecture*, Canadian Math. Bull. **41** (1998), 442–451.

[7] F. H. Clarke, *On the Inverse Function Theorem* , Pacific Journal **64** (1976), 97–102.

[8] A. Dold, *Lectures on Algebraic Topology* , Die Grundlehren der mathematischen Wissenschaften, Band 200, Springer - Verlag, 1972.

[9] R. Engelking, *Dimension Theory* , PWN - North Holand, 1978.

[10] A. Fernandes e C. Gutierrez and R. Rabanal, *Global asymptotic stability for differentiable vector fields of $\mathbb{R}^2$.* Journal of Differential Equations, **206** (2004), n. 2, 470-482.

[11] A. Fernandes e C. Gutierrez and R. Rabanal, *On local diffeomorphisms of $\mathbb{R}^n$ that are injective.* Qualitative Theory of Dynamical Systems, **4** (2003), n. 2, 255-262.

[12] B. Pires, C. Gutierrez and R. Rabanal, *Asymptotic Stability at Infinity for Differentiable Vector Fields of the Plane.* To appear in Journal of Differential Equations. doi:10.1016/j.jde-2006.07.025

[13] C. Gutierrez and R. Rabanal, *Injectivity of differentiable maps $\mathbb{R}^2 \to \mathbb{R}^2$ at infinity* , To appear in Bulletim of the Braz. Math. Soc. New Series, **37** (2), (2006).

[14] C. Gutierrez and Nguyen Van Chau, *A remark on an eigenvalue condition for the global injectivity of differentiable maps of $\mathbb{R}^2$.*, To appear in Discrete & Continuous Dynamical Systems.

[15] W. Hurewicz and H. Wallman, *Dimension Theory.* Princeton University Press, 1948.

[16] S. Nollet and F. Xavier, *Global inversion via the Palais-Smale Condition,* Discrete and Cont. Dyn. Sys. **8**, No. 1, (01/2002), 17–28.

[17] S. Pinchuck, *A counterexample to the strong Jacobian conjecture*, Math. Z. **217** (1994), 1–4.

[18] R. Plastock, *Homeomorphisms between Banach spaces,* Trans. Amer. Math. Soc. **200** (1974), 169–183.

[19] B. Smyth, F. Xavier, *Injectivity of local diffeomorphisms from nearly spectral conditions,* J. Diff. Equations 130, (1996), 406–414.

[20] S. Radulescu e M. Radulescu, *Local Inversion theorems without assuming continuous differenciability,* J. Math. Anal. Appl. **138** (1989), No. 2, 581–590.

[21] J. Väisälä, *Discrete open mappings on manifolds* , Annales Academiae Scientiarum Fennicae. Ser. A I No. **392** (1966), 10 pages.

[22] G. Vidossich, *Two remarks on the stability of ordinary differential equations,* Nonlinear Analysis, Theory Methods & Applications **4**, No. 5, 967–974.

[23] R. L. Wilder, *Topology of Manifolds,* Amer. Math. Soc. Colloq. Publ. 32, Amer. Math. Soc., New York, 1949.

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