Int. Journal of Math. Analysis, Vol. 1, 2007, no. 16, 775 - 782

Strictly Webbed Convenient Locally Convex Spaces

Thomas E. Gilsdorf

Department of Mathematics
University of North Dakota
Grand Forks, ND 58202-8376, USA
thomas_gilsdorf@und.nodak.edu

Abstract

We study convenient locally convex spaces within the context of spaces that have ordered, strict webs. The main result is that if a space having an ordered, strict web satisfies: (i) : Every sequentially continuous seminorm is continuous and (ii): Property $\mathcal{K}$, then it is a convenient locally convex space. Conversely, if a strictly webbed convenient locally convex space satisfies the Mackey convergence condition, then it satisfies (i) and (ii).

Mathematics Subject Classification: Primary: 46A08; Secondary: 46F30, 46A17

1 Introduction and terminology.

In the 1980s J.-F. Colombeau (see [C]) defined algebras of generalized functions in order to solve L. Schwartz's "impossibility result" ([S]) of multiplication of distributions. Colombeau's theory is now part of an active area of nonlinear distribution theory- see for example, [GFKS]. A fundamental part of this theory is that of a convenient vector space, which for the case of locally convex spaces is equivalent to the space being both bornological and locally complete. Most of the main ideas of how convenient vector spaces are used in this analysis can be found in [KrM]. Individually, both concepts of bornological spaces and locally complete spaces are well-understood, see for example [PCB]. Inspired by [AB], we prove some sequential results regarding convenient locally convex spaces.

Throughout the paper (E, τ) denotes a Hausdorff locally convex space over $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$). Most of the time this will be called simply 'space'. Given any subset A of E, we let $lin(A)$ denote the linear space spanned by A. For an absolutely convex set B we denote $lin(B)$ by E_B and we use μ_B as the notation for the Minkowski seminorm associated with B. (E_B, μ_B) will denote the space E_B endowed with the topology generated by μ_B. If B is closed and bounded, then (E_B, μ_B) is a normed space.

A bounded, absolutely convex, closed set $D \subset E$, called a *disk*, is a *Banach (Baire) disk* if (E_D, μ_D) is a Banach (normed Baire) space. If every bounded set $A \subset E$ is contained in a Banach disk (Baire disk) we say that E is *locally complete (locally Baire)*. Recall that a space is *bornological* if every absolutely convex set that absorbs bounded sets is a zero neighborhood. A space is *ultrabornological* if every absolutely convex set that absorbs all Banach disks is a zero neighborhood. We will use the notation τ^{ub} to denote the associated ultrabornological topology of a space (E, τ).

The definition below of a convenient locally convex space comes from 2.14, p. 20 of [KrM]:

Definition 1 *A convenient locally convex space, abbreviated here as clcs, is a locally complete bornological space.*

Clearly, any *clcs* is ultrabornological, and hence also barrelled.

Throughout most of this paper, we will assume the space E is *strictly webbed* with web $\mathcal{W}$. Background information regarding webbed spaces can be found in [R], Appendix 1 of [RR], §35 of Chapter 7 of [K], Chapter 5 of [J], and of course in [DeW] of DeWilde, who originally developed the concept of webs on topological vector spaces. A useful property of a web is that of being ordered, as defined by Valdivia in [V, p. 150]: The web $\mathcal{W}$ is *ordered* if, given any $k, r_1, r_2, ..., r_k, s_1, s_2, ..., s_k \in \mathbb{N}$, such that $r_j \leq s_j$ for $j = 1, 2, ..., k$, we have $W_{r_1,..,r_k} \subset W_{s_1,..,s_k}$. We will follow the conventions and notation of W. Robertson, [R]: Each strand of a web $\mathcal{W}$ will be denoted by $(W_k) = \{W_k : k \in \mathbb{N}\}$. In addition, given any strand (W_k), we assume that for each $k \in \mathbb{N}$, $W_{k+1} \subset \frac{1}{2} W_k$. The web $\mathcal{W}$ is *compatible* if given any zero neighborhood U in E, and given any strand (W_k) of $\mathcal{W}$, there is an $n \in \mathbb{N}$ such that $W_k \subset U$, for all $k \geq n$. Unless otherwise stated, we will assume that a given web on a space is compatible.

For convenience we recall here the specific definitions of being (strictly) webbed. Let (W_k) be a given strand in a web $\mathcal{W}$ on E. Consider $x_n \in W_n, n \in \mathbb{N}$ and the series $\sum_{n\in\mathbb{N}} x_n$. The space (E, τ) is *webbed* if $\sum_{n\in\mathbb{N}} x_n$ is $\tau-$ convergent in E for any strand (W_n) and any choice of $x_n \in W_n$. E is *strictly webbed* if $\sum_{k=n+1}^{\infty} x_k$ converges to an element of W_n for every $n \in \mathbb{N}$ and any choice of $x_k \in W_k$.

A space E satisfies the *Mackey convergence condition*, *(MCC)*, if every null sequence in E is a null sequence in (E_B, μ_B), for some disk B. Results regarding the (MCC) can be found in [PCB] and [J]. It is well - known that every metrizable space satisfies the (MCC).

Finally, a locally convex space E satisfies *property* $\mathcal{K}$ if each null sequence (x_n) has a subsequence (x_{n_k}) for which $\sum_{k\in\mathbb{N}} x_{n_k}$ converges in E.

2 Main Result.

Theorem 1 *Assume the space E has an ordered, strict web. Then E is a convenient locally convex space if it satisfies the following:*

(i) Every sequentially continuous seminorm on E is continuous;

(ii) E satisfies property $\mathcal{K}$.

Conversely, if E is a convenient locally convex space and satisfies the Mackey convergence condition, then (i) and (ii) hold.

Proof: Suppose (i) and (ii). The main result of [AB] is that *(i)* and *(ii)* imply E is bornological (and barrelled). In [G2], it is shown that any space satisfying property $\mathcal{K}$ is locally Baire. We will show that a space that has an ordered, strict web and is locally Baire must be locally complete. Let A be any bounded subset of E and let B denote a disk such that $A \subset B$ and (E_B, μ_B) is a Baire space. Denote an ordered, strict web on E by $\mathcal{W}$. For a given strand $\omega = (W_k)$ of $\mathcal{W}$, put

$$F_\omega = \bigcap\{lin(W_k) : k \in \mathbb{N}\}.$$

A basis for a locally convex topology τ_ω on F_ω is given by

$$\{F_\omega \bigcap \frac{1}{k}W_k : k \in \mathbb{N}\}.$$

By [V, 7.21, p. 164], (F_ω, τ_ω) is a Fréchet space, and τ_ω is finer than the topology inherited from E. As (E_B, μ_B) is a Baire space, and the injection $(E_B, \mu_B) \hookrightarrow E$ is continuous, we may apply [V, Thm 7.6, p. 164], concluding that there exists a strand ω of $\mathcal{W}$ such that $(E_B, \mu_B) \hookrightarrow (F_\omega, \tau_\omega)$ is continuous along with the aforementioned $(F_\omega, \tau_\omega) \hookrightarrow E$. The conclusion now follows from the completeness of (F_ω, τ_ω).

Conversely, assume that the *clcs* E satisfies the (MCC). Then E satisfies (i) because every (ultra)bornological already does. To finish the proof we will show that a strictly webbed *clcs* satisfying the (MCC) also satisfies (ii). Let (x_n) be any null sequence in E. By assumption, E is locally complete and strictly webbed, so we may apply [G1, Thm 18, p. 481], concluding that there exists a strand (W_k) of $\mathcal{W}$ such that for each $k \in \mathbb{N}$, there is an $N_k \in \mathbb{N}$ for which $x_n \in W_k$ for all $n \geq N_k$. In particular, for each $k \in \mathbb{N}$, there is a term $x_{n_k} \in W_k$. The web $\mathcal{W}$ is completing, and this implies that the series

$$\sum_{k \in \mathbb{N}} x_{n_k}$$

converges in E. □

3 Three examples.

To reduce wordiness, let us denote the property of having an ordered strict web, by (iii). The examples below show that no two of the assumptions $(i), (ii)$, and (iii) are sufficient to conclude that a space is a *clcs*.

(a): Let E denote $(l_1, \sigma(l_1, l_\infty))$. Then E has an ordered, strict web: Use the single strand given by $\{\frac{1}{n}B : n \in \mathbb{N}\}$, where B is the closed unit ball of l_1. Because weakly convergent sequences converge in norm (Shur's Theorem), E satisfies property $\mathcal{K}$. Though locally complete, E is not bornological. Thus, *(ii) and (iii) do not imply clcs.*

(b): There exists a metrizable, hence bornological, space that satisfies property $\mathcal{K}$, yet is incomplete; see [LL, Theorem 2, p. 94]. Because a metrizable space is locally complete if and only if it is complete, we conclude that *(i) and (ii) do not imply clcs.*

(c): Let E be a non-regular (LB)- space such as Köthe's classic example, as given in [PCB; 7.3.6, pg. 211]. Writing $E = indlim_n(E_n)$, an ordered, strict web $\mathcal{W}$ on E is formed in the following way (see [V] for more details if needed): Define each strand (W_k) of $\mathcal{W}$ by

$$W_k = \frac{1}{k} B_n, \qquad k, n \in \mathbb{N},$$

where B_n is the closed unit ball in E_n. Then E is ultrabornological, has an ordered, strict web, but is not locally complete. Hence, *(i) and (iii) do not imply clcs.*

4 Mackey convergence and *clcs*.

If a space is not bornological, then a natural response is to check the associated ultrabornological topology τ^{ub} , for local completeness. Surprisingly, even though an ultrabornological, strictly webbed space would seem to be at least locally complete, it need not be; witness Example (c) above of a non-regular $(LB)-$ space. On the other hand, the proof of the second part of Theorem 1 insinuates that the (MCC) can be useful in determining if a space is a *clcs*, and we will apply it below for the case of τ^{ub}.

A space for which the terms of a null sequence are eventually elements of a strand of a compatible web is called sequentially webbed. See [G1]. Specifically, a space E having a compatible web $\mathcal{W}$ is called *sequentially webbed* if for each null sequence (x_n) in E, there is a finite collection of m strands of $\mathcal{W}$,

$$\left\{ \left(W_k^{(1)} \right), \cdots, \left(W_k^{(m)} \right) \right\},$$

such that for each $k \in \mathbb{N}$, there is an $N_k \in \mathbb{N}$ for which

$$x_n \in \bigcup_{i=1}^{m} W_k^{(i)},$$

for all $n \geq N_k$.

Metrizable spaces are sequentially webbed, and every sequentially webbed space satisfies the (MCC). More properties and details can be seen in [G1].

Theorem 2 *If E is any sequentially webbed space having an ordered strict web $\mathcal{W}$, then $\left(E, \tau^{ub}\right)$ is a clcs. Moreover, $\left(E, \tau^{ub}\right)$ has an ordered, strict web $\hat{\mathcal{W}}$ and satisfies property $\mathcal{K}$.*

Proof: Let (x_n) be any null sequence in E. Denote the assumed web on E by $\mathcal{W}$. By the sequentially webbed supposition, it is easy to see that there is one strand (W_k) of $\mathcal{W}$ and a subsequence (x_{n_k}) of (x_n) such that for each $k \in \mathbb{N}$, $x_{n_k} \in W_k$. We assumed the web $\mathcal{W}$ to be strict, and this implies that the series $\sum_{k\in\mathbb{N}} x_{n_k}$ converges in E. We have established property $\mathcal{K}$ for E, and by the proof of Theorem 1, we know that this means (E, τ) is locally complete. (E, τ) and (E, τ^{ub}) have the same Banach disks; in other words, (E, τ^{ub}) is a *clcs*.

Jarchow's proof (using Robertson's notation) of Powell's Theorem (see [J; Thm 3, p. 277] and [P; Thm 7.1, p. 405]), applies to our situation like this: Given an ordered, strict web $\mathcal{W}$ on E, define the web $\hat{\mathcal{W}}$ on E as having strands given by $(\hat{W}_k)$, where $\hat{W}_k = W_{2k-1}, k \in \mathbb{N}$. Clearly, $\hat{\mathcal{W}}$ is ordered if $\mathcal{W}$ is, and Powell's theorem states that $\left(E, \tau^{ub}\right)$ is strictly webbed with respect to $\hat{\mathcal{W}}$. If $x_n \to 0$ in E, then the first part of this proof indicates that there is a strand (W_k) and a subsequence (x_{n_k}) of (x_n) such that for each $k \in \mathbb{N}$, $x_{n_k} \in W_k$. Hence,

$$x_{2n_k-1} \in W_{2k-1} = \hat{W}_k,$$

from which we conclude that $\left(E, \tau^{ub}\right)$ satisfies property $\mathcal{K}$. Notice that we can also deduce that $\left(E, \tau^{ub}\right)$ satisfies property $\mathcal{K}$ by observing that $\left(E, \tau^{ub}\right)$ satisfies the (MCC), allowing us to apply Theorem 1. $\square$

Acknowledgements. Part of the research for this paper was done while I was on a Fulbright Scholarship at the Instituto Tecnológico Autónomo de México (ITAM), Mexico City, 2006. I am also grateful to the University of North Dakota for Developmental Leave support.

REFERENCES

[AB] Antosik, P., Burzyk, J., *Sequential conditions for barrelledness and bornology*, Bull. Polish Acad. Sci. Math., **35**, no. 7-8, (1987), 457 - 459.

[C] Colombeau, J-F., New Generalized Functions and Multiplication of Distributions, North-Holland, (1984).

[DeW] DeWilde, M., Closed Graph Theorems and Webbed Spaces, Pitman, (1978).

[G1] Gilsdorf, T. E., *The Mackey convergence condition for spaces with webs*, Int. J. Math. & Math. Sci., **11**, no. 3, (1988), 473 - 484.

[G2] Gilsdorf, T. E., *Regular inductive limits of $\mathcal{K}$ spaces*, Collect. Math., **42**, no. 1, (1991), 45 - 49.

[GFKS] Grosser, M., Farkas, E., Kunzinger, M., Steinbauer, R., *On the foundations of nonlinear generalized functions I and II*, Mem. AMS, **153**, no. 729, (2001).

[J] Jarchow, H., Locally Convex Spaces, B. G. Teubner Stuttgart, (1981).

[K] Köthe, G., Topological Vector Spaces, II, Springer, (1979).

[KrM] Kriegl, A. and Michor, P., The Convenient Setting of Global Analysis, AMS, (1997).

[LL] Labuda, I. and Lipecki, Z., *On subseries convergent series and m-quasi-bases in topological linear spaces*, Manuscr. Math. **38**, (1982), 87-98.

[PCB] Pérez Carreras, P., Bonet, J., Barrelled Locally Convex Spaces, North-Holland, (1987).

[P] Powell, M. H., *On Kōmura's closed graph theorem*, Trans. AMS., **211**, (1975), 391 - 426.

[R] Robertson, W., *On the closed graph theorem and spaces with webs*, Proc. London Math. Soc. (3), **24**, (1972), 692 - 738.

[RR] Robertson, A. P., Robertson, W., Topological Vector Spaces, Cambridge Univ. Press, (1973).

[S] Schwartz, L., *Sur l' impossibilité de la multiplication des distributions*, Comp. Rend. Acad. Sci., **239**, (1954), 847 - 848.

[V] Valdivia, M., *Quasi - LB - spaces*, J. London Math. Soc., (2), **35**, (1987), 149 - 168.

Received: January 16, 2007