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# Frequent Hypercyclicity for Cowen-Douglas Operators

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### Abstract

In this paper, we first establish that every operator $T \in B_n(\Omega)$ is frequently hypercyclic. Second, we show that the inverse $T^{-1}$ of such operators is also frequently hypercyclic when $T$ satisfies mild spectral conditions (e.g., its spectrum excludes the origin). Third, we characterize the frequent hypercyclicity of operators in the commutant of $T \in B_n(\Omega)$ different from $\lambda I$. This work advances the theory of frequent hypercyclicity by establishing a unified framework for operators in $B_n(\Omega)$, their inverses, and their multiples, thus complementing the incomplete studies in literature.

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**Keywords:** Frequent Hypercyclicity, Cowen-Douglas operators

# 1 Introduction

We consider a separable Hilbert space $H$. Given a domain (nonempty connected open subset) $\Omega \in \mathbb{C}$ and a positive integer $n$, Cowen and Douglas [1] introduced the operator class $B_n(\Omega)$, consisting of operators $T$ on $H$ that satisfy

1. $\Omega \subset \sigma(T) = \{\omega \in \mathbb{C} : \ T - \omega I \ is \ not \ invertible\}$;
2. $ran\,(T - \omega) = H$ for $\omega$ in $\Omega$;
3. $\bigvee ker_{\omega \in \Omega}\,(T - \omega) = H$; and

4. *dim ker* $(T-\omega)=n$ for $\omega$ in $\Omega$.

Cowen and Douglas have linked the investigation of such operators to the specific problems in complex geometry. For an operator $T$ in $B_n(\Omega)$, the mapping $\omega \longmapsto \ker(T-\omega)$ creates a rank $n$ holomorphic Hermitian vector bundle over $\Omega$, denoted here as $E_T$. They have specified [1] that the invariants of $T$ can be uncovered by examining their geometric counterparts within $E_T$.

However, the class $B_n(\Omega)$ includes several key types of operators-such as the shift and the multiplication operators-and the characterization and reduction of their subspaces constitute an intriguing area in operator theory. Therefore, the Cowen–Douglas class $B_n(\Omega)$ has drawn the interest of numerous researchers (see references [1][2][3][4]).

Hereafter, we investigate the frequent hypercyclicity of operators $T \in B_n(\Omega)$ from the perspective of dynamical systems theory. Though such operators demonstrate both chaos and topological strong mixing under appropriate conditions, not all Devaney chaotic and topologically strongly mixing operators inherently possess frequent hypercyclicity. This distinction has been substantiated by Menet's recent discovery [5] of a chaotic operator on $l^1$-space that fails to exhibit frequent hypercyclicity. Therefore, the systematic investigation of frequent hypercyclicity of $T \in B_n(\Omega)$ has both non-trivial and theoretical significance, since frequent hypercyclicity cannot be presumed here.

Let $X$ be an infinite-dimensional separable $F$-space, and let $T \in L(X)$ denote a bounded linear operator on $X$. Thus, the pair $(X,T)$ can be referred to as a linear system. $T$ is termed a **hypercyclic operator** if there exists a vector $x \in X$ such that its orbit under $T$ is dense in $X$: particularly, $\overline{Orb(x,T)} = X$. In such a scenario, $x \in X$ is termed a hypercyclic vector for $T$. $T$ is said to be **topologically transitive** if, for any pair $(U,V)$ of nonempty open subsets of $X$, there exists an integer $n$ where $T^n(U)\cap V$ is nonempty.

Given that the space $X$ is assumed to be separable, a straightforward Baire category argument shows that $T$ is topologically transitive if and only if $T$ is hypercyclic. Hypercyclic operators contain Birkhoff translation operators and MacLane differentiation operators on $H(\mathbb{C})$. Moreover, they encompass scalar multiples $\omega B$ of the backward shift $B$ on $l^p(1\le p<+\infty)$ or $c_0$, where $|\omega|>1$ (see [6, 7, 8]).

Bayart and Grivaux [9] have proposed a concept in hypercyclic operator theory, viz., frequently hypercyclic operators, from the perspective of topological dynamics and ergodic theory. This serves to quantify the frequency of iteration of a given hypercyclic vector to visit any nonempty open set.

**Definition 1.1** ([9, Definition 1.2]). *An operator $T$ on $X$ is said to be frequently hypercyclic provided there exists a vector $x\in X$ such that for every nonempty open subset $U$ of $X$, the set of integers $n$ such that $T^n x$ belongs to*

*$U$ has positive lower density. Therefore,*

$$\underline{dens}\,\{n \in \mathbb{N},\ T^n x \in U\} > 0.$$

*In this case, $x$ is called a frequently hypercyclic vector for $T$, and the set of frequently hypercyclic vectors will be denoted by $FHC\,(T)$.*

Leveraging the theoretical framework of Hypercyclicity Criteria, Bayart and Grivaux [9] have developed a computationally verifiable sufficient condition to guarantee frequent hypercyclicity. Grosse and Erdmann refined this approach by introducing a strengthened formulation featuring stricter quantitative constraints.

**Theorem 1.2** ([10, Theorem 2.1]**(Frequent Hypercyclicity Criterion)**). *Let $T$ be an operator on a separable $F$-space $X$. Suppose that there is a dense subset $X_0$ of $X$ and a mapping $S : X_0 \to X_0$, such that*

1. *$\sum_{n=1}^{\infty} T^n x$ converges unconditionally, for all $x \in X_0$;*
2. *$\sum_{n=1}^{\infty} S^n x$ converges unconditionally, for all $x \in X_0$;*
3. *$TSx = x$ for all $x \in X_0$.*

*Then, $T$ is frequently hypercyclic.*

According to these results, several hypercyclic operators qualify as frequently hypercyclic. Furthermore, they encompass Birkhoff's translation operator, MacLane's differentiation operator on $H(\mathbb{C})$, and scalar multiples $\omega B$ (with $|\omega| > 1$) of the backward shift $B$ on $l^p$ ($1 \leq p < +\infty$) or $c_0$. However, all hypercyclic operators do not exhibit frequent hypercyclicity. Bayart and Grivaux [9] have constructed a counterexample of a hypercyclic operator lacking frequent hypercyclicity: the simple weighted backward shift on $l^2$ with weight sequence $\omega_n = \sqrt{1 + \frac{1}{n}}$ for $n \in \mathbb{N}$.

Nevertheless, within this framework, the investigation of frequently hypercyclic operators focuses on the "individual" behavior of an orbit, whereas classical elements in hypercyclicity theory become ambiguous when transposed into the frequent hypercyclicity context. For instance, in the hypercyclicity setting, we have that

1. [11, Proposition 2.23] If $T$ is an invertible operator, then $T$ is hypercyclic if and only $T^{-1}$ is;

2. [12, Theorem 2.3] $T \oplus T$ is hypercyclic on $X \oplus X$ when $T$ satisfies the well-known Hypercyclic Criterion.

But this is still mysterious in the frequent hypercyclicity setting.

We obtain the following results in this work. The first one below says that every operator in the commutant of $T$, different from $\lambda I$, has a frequently hypercyclic multiple, a chaotic multiple, and a topological mixing multiple. This indicates that there exists $m \in \mathbb{C}$ such that $mA$ is frequently hypercyclic, chaotic, and topological mixing, respectively, thereby strengthening the previous result of González and León-Saavedra [13].

The second one below indicates that every operator in the Cowen–Douglas operator class $B_n(\Omega)$ is frequently hypercyclic under a suitable condition.

Further, we prove the frequent hypercyclicity of the inverse of $T \in B_n(\Omega)$.

## 2 Preliminary Notes

**Definition 2.1** ([9]). *Let $T$ be a bounded operator on a complex separable Banach space $X$ and $\sigma$ a probability measure on the unit circle $\mathbb{T} = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$. We state that $T$ has a $\sigma$-spanning set of $\mathbb{T}$-eigenvectors if for every $\sigma$-measurable subset $A$ of $\mathbb{T}$ with $\sigma(A) = 1$, the eigenspaces ker $(T - \lambda I)$, with $\lambda \in A$, span a dense subspace of $X$. If the measure $\sigma$ can be chosen to be continuous (i.e., $\sigma(\lambda) = 0$ for every $\lambda \in \mathbb{T}$), then we claim that $T$ has a perfectly spanning set of $\mathbb{T}$-eigenvectors.*

**Theorem 2.2** ([9]). *If $T$ has a perfect spanning set of eigenvectors associated with unimodular eigenvalues, then $T$ is frequently hypercyclic, and particularly, $T$ is hypercyclic. It is even weakly topologically mixing: if $U$ and $V$ are two nonempty open subsets of $X$, there exists a sequence $(n_k)$ of integers of density 1 such that $T^{n_k}(U) \cap V$ is nonempty for every $k$.*

We describe an alternative definition of the spanning eigenvector field associated to unimodular eigenvalues, for a convenient usage.

**Definition 2.3** ([11, *Definition* 9.21]). *Let $T$ be an operator on a complex Fréchet space $X$. Then, a collection of functions $E_j : \mathbb{T} \longrightarrow X$, $j \in J$, where $J$ is a nonempty index set, is called a spanning eigen-vector field associated with unimodular eigen values, if $E_j(\lambda) \in$ ker $(\lambda I - T)$ for any $\lambda \in \mathbb{T}$, $j \in J$. Furthermore, the vector field is said to be continuous (or $C^2$) if each function $E_j : \mathbb{T} \longrightarrow X$, $j \in J$, is continuous (or $C^2$, respectively).*

Therefore, a function $E : \mathbb{T} \longrightarrow X$, is called $C^2$ if it is twice continuously differentiable, where the differentiation is defined as in the scalar-valued case.

Therefore, we can describe the eigenvalue criterion for frequent hypercyclicity.

**Theorem 2.4** ([11, *Theorem* 9.22]). *Let $T$ be an operator on a complex separable Fréchet space.*

(a) *If $T$ has a spanning continuous eigenvector field associated with unimodular eigenvalues, then it is mixing and chaotic.*

(b) *If $T$ has a spanning $C^2$-eigenvector field associated with unimodular eigenvalues, then it is frequently hypercyclic.*

We have known certain properties of these operators from [1]. We list some of them below.

**Proposition 2.5** ([1])**.** *Let $T \in \mathcal{B}_n(\Omega)$, and $\omega_0 \in \Omega$. Then, $\bigvee ker_{\omega\in\Omega}(T-\omega) = H$ is equivalent to $\bigvee_{k=1}^{\infty} ker\,(T-\omega_0)^k = H$.*

**Proposition 2.6** ([1, *Corollary* 1.13])**.** *If $\Omega_0 \subseteq \Omega$ are bounded and connected open subset of $\mathbb{C}$, then $\mathcal{B}_n(\Omega) \subseteq \mathcal{B}_n(\Omega_0)$.*

**Proposition 2.7** ([1, *Corollary* 1.12])**.** *For $T$ in $\mathcal{B}_n(\Omega)$ the mapping $\omega \to ker(T-\omega)$ defines a complex bundle $E_T$ over $\Omega$.*

Approximately, this indicates that the mapping $\omega \longrightarrow ker(T-\omega)$ define a rank $n$ holomorphic vector bundle $E_T$ over $\Omega$ for $T \in \mathcal{B}_n(\Omega)$.

# 3 Frequent Hypercyclicity of the Cowen–Douglas class $B_n(\Omega)$

Hou et al. [14] have established that the operators within the Cowen–Douglas class exhibit both Devaney chaos and strong mixing. However, this equivalence does not hold universally: not all Devaney-chaotic operators are necessarily frequently hypercyclic. This distinction gains concrete illustration through Menet's recent work [5], where a chaotic operator is constructed on the Banach space $l^1$ that notably fails to possess frequent hypercyclicity. These results motivate a critical inquiry: studying the conditions for the emergence of frequent hypercyclicity for operators $T$ in the Cowen–Douglas class $\mathcal{B}_n(\Omega)$.

**Theorem 3.1.** *Let $T \in \mathcal{B}_n(\Omega)$. Suppose $\Omega \cap \mathbb{T} \neq \emptyset$. Then, $T$ is frequently hypercyclic.*

**Proof of Theorem 3.1**. In the following, we use the eigenvalue criterion for the frequent hypercyclicity (Theorem 2.4) to prove $T$ to be frequently hypercyclic, which has been employed above. Since $\Omega \cap \mathbb{T} \neq \emptyset$, we will define an eigenvector field associated to unimodular eigenvalues for $T$. Owing to the proposition 2.7, the mapping $\omega \longrightarrow ker(T-\omega)$ defines a complex bundle $E_T$ over $\Omega$, and take any point $0 \neq x_\omega \in ker(T-\omega)$. Then, the mapping $\widehat{x} : \Omega \cap \mathbb{T} \longrightarrow ker(T-\omega)$ defined by $\widehat{x}(\omega) = x_\omega$ for $\omega \in \Omega$ is holomorphic on $\Omega \cap \mathbb{T}$. Indeed, this mapping $\widehat{x}$ constitutes all that we need.

We can define an eigenvector field $E_T : \mathbb{T} \longrightarrow H$ associated with the unimodular eigenvalues from $\mathbb{T}$ to $H$, such that $E_T(\omega)$ satisfying $TE_T(\omega) = \omega E_T(\omega)$ for $\omega \in \Omega \cap \mathbb{T}$. In fact, let $E_T(\omega) = \widehat{x}(\omega)$ for $\omega \in \Omega \cap \mathbb{T}$, i.e. $E_T$ is the restriction of $\widehat{x}$ on $\Omega \cap \mathbb{T}$. Thus $E_T$ is a holomorphic function.

Finally, the denseness of the subspace

$$span\left\{E_T(\omega) : \ TE_T(\omega) = \omega E_T(\omega) \ for \ \omega \in \Omega \cap \mathbb{T}\right\}$$

follows from the identity theorem. Therefore, the operator $T$ has a perfectly spanning set associated with the unimodular eigenvalues by Theorem 2.4. Further, $T$ is frequently hypercyclic. This completes the proof. □

**Remark 3.2.** *This condition $\Omega \cap \mathbb{T} \neq \emptyset$ had appeared in [14, Theorem 2.3]. Concomitantly, the authors show that $\Omega \cap \mathbb{T} \neq \emptyset$ is not a necessary condition for Devaney chaos for Cowen–Douglas operators. The weighted backward shift operator $B_\omega$ with weight sequence $\left\{\omega_n = \frac{n+1}{n}\right\}_{n=1}^{\infty}$, is a Devaney-chaotic Cowen–Douglas operator. However, the largest connected open domain for $B_\omega$, which admits $B_\omega \in \mathcal{B}_1(\Omega)$, is the unit open disk, and therefore, disjoint with $\mathbb{T}$.*

The same analogue above holds for a frequently hypercyclic Cowen–Douglas operator, i.e., $\Omega \cap \mathbb{T} \neq \emptyset$ is not a necessary condition for frequent hypercyclicity for Cowen–Douglas operators.

**Theorem 3.3** ([15, Theorem 3])**.** *Let $p \in [1, \infty)$ and let $\omega = (\omega_n)_{n\in\mathbb{Z}}$ be a bounded sequence of positive real numbers. The following assertions are equivalent.*

1. *$B_\omega$ is frequently hypercyclic on $l^p(\mathbb{Z})$;*
2. *$B_\omega$ is $U$-frequently hypercyclic on $l^p(\mathbb{Z})$;*
3. *The series $\sum_{n\geq 1} \frac{1}{(\omega_1\cdots\omega_n)^p}$ and $\sum_{n<0} (\omega_{-1}\cdots\omega_n)^p$ are convergent.*

**Example 3.4.** *Let $B_\omega : l^2(\mathbb{N}) \longrightarrow l^2(\mathbb{N})$ be the unilateral weighed backward shift with weight sequence $\left\{\omega_n = \frac{n+1}{n}\right\}_{n=1}^{\infty}$. Then, $B_\omega$ is frequently hypercyclic. However, the largest connected open domain for $B_\omega$, which admits $B_\omega \in \mathcal{B}_1(\Omega)$, is the unit open disk and therefore disjoint with $\mathbb{T}$.*

*Proof.* First, we prove that $B_\omega$ is frequently hypercyclic. Since $\omega_n = \frac{n+1}{n}$ for each $n \in \mathbb{N}$, it implies that $\omega_1\omega_2\cdots\omega_n = n+1$. Further, it follows that the series

$$\sum_{n\geq 1} \frac{1}{(\omega_1\omega_2\cdots\omega_n)^2} = \sum_{n\geq 1} \frac{1}{(n+1)^2}$$

is convergent. Therefore, owing to the Theorem 3.3, we have that $B_\omega$ satisfies the Frequently Hypercyclic Criterion (Theorem 1.2), particularly, $B_\omega$ is frequently hypercyclic on $l^2(\mathbb{N})$. Second, from [13, Theorem 3.3], we know that

$$r(B_\omega) = \liminf_{n\to\infty} (\omega_1\omega_2\cdots\omega_n)^{\frac{1}{n}} = \liminf_{n\to\infty} (n+1)^{\frac{1}{n}} = 1.$$

Therefore, we obtain that

$$\{\lambda \in \mathbb{C},\ |\lambda| < r(B_\omega)\} = \mathbb{D} \subseteq \sigma_p(B_\omega)$$

and consists of simple eigenvalues of $B_\omega$. This is the largest connected open domain for $B_\omega$, which admits $B_\omega \in \mathcal{B}_1(\Omega)$, which is disjoint with $\mathbb{T}$. Therefore, $\Omega \cap \mathbb{T} \neq \emptyset$ is only a sufficient condition for $B_\omega \in \mathcal{B}_1(\Omega)$ to be frequently hypercyclic. This completes the proof. □

In the above example, we require that $p = 2$. Here, $l^2(\mathbb{N})$ is an infinite-dimensional separable Hilbert space, and $B_\omega \in \mathcal{B}_1(\Omega)$ belongs to the category of Cowen–Douglas class. We investigate the case of $p = 1$. First, we give a characterization of $B_\omega$'Devaney chaos on $l^p$ or $c_0$, where $1 \leq p < \infty$.

**Theorem 3.5** ([11, Example 4.9])**.** *$B_\omega$ is Devaney chaotic on $l^p$ or $c_0$ if and only if the series $\sum_1^\infty \frac{1}{|\omega_1\omega_2\cdots\omega_n|^p}$ is convergent.*

Therefore, we can express our result as follows.

**Example 3.6.** *Let $B_\omega : l^1(\mathbb{N}) \longrightarrow l^1(\mathbb{N})$ be the unilateral weighed backward shift with weight sequence $\left\{\frac{n+1}{n}\right\}_{n=1}^\infty$. Then, $B_\omega$ is neither a Devaney-chaotic operator nor frequently hypercyclic. However, if the weight sequence becomes $\left\{-\frac{n+1}{n}\right\}_{n=1}^\infty$, then $B_\omega$ is not Devaney chaotic.*

*Proof.* For the first part, $\omega_n = \frac{n+1}{n}$ for each $n \in \mathbb{N}$ is a positive weight. This implies the following: $\omega_1\omega_2\cdots\omega_n = n+1$, and the series

$$\sum_{n\geq 1} \frac{1}{\omega_1\omega_2\cdots\omega_n} = \sum_{n\geq 1} \frac{1}{|\omega_1\omega_2\cdots\omega_n|} = \sum_{n\geq 1} \frac{1}{n+1}$$

is divergent. Therefore, according to Theorem 3.3 and Theorem 3.5, we find that $B_\omega$ is neither a Devaney-chaotic operator nor a frequently hypercyclic. For the second part, since $\omega_n = -\frac{n+1}{n}$ for each $n \in \mathbb{N}$ is a negative weight, this implies that $\omega_1\omega_2\cdots\omega_n = (-1)^n(n+1)$. It follows that the series

$$\sum_{n\geq 1} \frac{1}{|\omega_1\omega_2\cdots\omega_n|} = \sum_{n\geq 1} \frac{1}{n+1}$$

is divergent. Therefore, according to Theorem 3.5, $B_\omega$ is not Devaney chaotic. This completes the proof. □

**Remark 3.7.** *We find that the series*

$$\sum_{n\geq 1}\frac{1}{\omega_1\omega_2\cdots\omega_n}=\sum_{n\geq 1}(-1)^n\,\frac{1}{n+1}$$

*is conditionally convergent to* $(\ln 2)-1$*, instead of unconditional convergence, and the weights in Theorem 3 [15] have been required to be positive. Therefore, we cannot use Theorem 3.3 to assert that* $B_\omega$ *is frequently hypercyclic.*

Next, we consider the problem of the frequent hypercyclicity of the inverse of $T \in B_n(\Omega)$. We have Theorem 3.8, which says that $T^{-1}$ is also frequently hypercyclic when $T$ is invertible and frequently hypercyclic for $T \in B_n(\Omega)$.

**Theorem 3.8.** *Let* $T \in B_n(\Omega)$*. Suppose that* $0 \notin \sigma(T)$ *and* $\Omega \cap \mathbb{T} \neq \emptyset$*. Then,* $T^{-1}$ *is also frequently hypercyclic.*

**Proof of Theorem 3.8**. The proof is effortless. The condition $0 \notin \sigma(T)$ implies that $T^{-1}$ exists. Furthermore, from Theorem 3.1 above, when $\Omega\cap\mathbb{T} \neq \emptyset$, it implies that $T$ has a perfectly spanning set associated with unimodular eigenvalues. Therefore, $T^{-1}$ also has perfectly spanning set associated with unimodular eigenvalues. Particularly, $T^{-1}$ is frequently hypercyclic. This completes the proof. □

# 4 The frequent hypercyclicity of operators in the commutant of $T \in B_n(\Omega)$ different from $\lambda I$

Godefroy and Shapiro [16] have shown that every convolution operator—i.e., continuous linear operators on $H(\mathbb{C})$ that commute with differential operators and are not a scalar multiple of identity—is mixing and chaotic on $H(\mathbb{C})$. Later, Herzog and Schmoeger [17] have extended the results of Godefroy and Shapiro's for the general operators $T$ on Banach spaces. They have obtained the hypercyclicity of operators $\varphi(T)$ under spectral conditions on $T$. Bermúdez and Miller [18] have proved that any operator in the commutant of a generalized backward shift (acting on certain Banach spaces) different from $\lambda I$ is supercyclic.

Following the foundational work of Godefroy–Shapiro, Herzog–Schmoeger, and Bermúdez–Miller, González and León-Saavedra [13] have established that every nontrivial operator in the commutant algebra of $T$, i.e., operators different from scalar multiples $\lambda I$, possesses hypercyclic multiples.

**Theorem 4.1** ([13, Theorem 2.1]). *Let* $T$ *be a bounded operator on a (complex) Banach space* $X$*. Suppose that the spectrum of* $T$ *contains a non-empty connected open subset* $U$ *such that the following conditions are satisfied:*

1. *Every* $\lambda \in U$ *is a simple eigenvalue of* $T$*,*
2. *span* $\{ker(T - \lambda I) : \lambda \in U\}$ *is dense in* $X$*, and*
3. *there exists a holomorphic function* $\widehat{x} : \lambda \in U \longrightarrow \widehat{x}(\lambda) := x_\lambda \in X$ *such that* $0 \neq x_\lambda \in ker(T - \lambda I)$*.*

*Hence, every operator in the commutant of* $T$ *that differs from* $\lambda I$ *has a hypercyclic multiple.*

Based on the seminal work of Blasco et al. [19], who demonstrated the frequent hypercyclicity of all convolution operators on the space $H(\mathbb{C})$, we now establish our central result in Theorem 4.2. Hereafter, the frequently hypercyclic multiple, chaotic multiple, and topological mixing multiple imply that each operator in the commutant different from $\lambda I$ has a frequently hypercyclic multiple, chaotic multiple, and topological mixing multiple, respectively.

**Theorem 4.2.** *Let* $T$ *be a bounded operator on a (complex) Banach space* $X$*. Suppose that the spectrum of* $T$ *contains a nonempty connected open subset* $U$ *such that the following conditions are satisfied:*

1. $U \cap \mathbb{T} \neq \emptyset$*,*
2. *every* $\lambda \in U$ *is a simple eigenvalue of* $T$*,*
3. *span* $\{ker(T - \lambda I) : \ \lambda \in U\}$ *is dense in* $X$*,*
4. *there exists a holomorphic function* $\widehat{x} : \lambda \in U \longrightarrow \widehat{x}(\lambda) = x_\lambda \in X$ *such that* $0 \neq x_\lambda \in ker(T - \lambda I)$*.*

*Thus, every operator in the commutant of* $T$ *different from* $\lambda I$ *has a frequently hypercyclic multiple, chaotic multiple, and mixing multiple.*

**Proof of Theorem 4.2**. Since $U \cap \mathbb{T} \neq \emptyset$, given a non-empty open subset $W \subset U \cap \mathbb{T}$, it follows that $W$ admits an accumulation point in $U$. First, we prove that the set

$$span\,\{ker(T - \lambda I) : \ \lambda \in W\}$$

remains dense in $X$. Let $x^* \in X^*$, such that $x^*(\widehat{x}(\lambda)) = 0$ for all $\lambda \in W$, where $\widehat{x}(\lambda) \in ker(T - \lambda I)$. By employing the identity theorem in complex analysis, we obtain that $x^*(\widehat{x}(\lambda)) = 0$ for all $\lambda \in U$, and further utilizing the condition "*span* $\{ker(T - \lambda I) : \ \lambda \in U\}$ is dense in $X$", we get that $x^* = 0$. Therefore, *span* $\{ker(T - \lambda I) : \ \lambda \in W\}$ is dense in $X$.

Let $A \in \mathcal{L}(X)$ in the commutant of $T$ with $A \neq \mu I$. For each $\lambda \in U$ we have $Tx_\lambda = \lambda x_\lambda$. Hence, $TAx_\lambda = \lambda Ax_\lambda$. Further, by the second condition, we have $Ax_\lambda = a(\lambda)x_\lambda$ for some complex number $a(\lambda)$. Therefore, we get a function

$a : U \longrightarrow \mathbb{C}$. According to open mapping theorem in complex analysis, $a(\lambda)$ is non-constant holomorphic function on a domain $U$, which implies that $a$ is an open mapping. Therefore, $a(U)$ is an open subset of the complex plane $\mathbb{C}$. Thus, for some $m \in \mathbb{C}$ the set $m \cdot a(U)$ intersects the unit circle, i.e., $m \cdot a(U) \cap \mathbb{T} \neq \emptyset$.

Next, we prove that $mA$ is frequently hypercyclic by applying the eigenvalue criterion (Theorem 2.4). We can define an eigenvector field $E_{mA} : \mathbb{T} \longrightarrow X$ associated to unimodular eigenvalues for $mA$ by $E_{mA}(\mu) = x_\mu = \widehat{x}(\mu)$ if $\mu \in m \cdot a(U) \cap \mathbb{T}$. From the fourth condition, we find that $E_{mA} : \mathbb{T} \longrightarrow X$ is a $C^2$-function on $\mathbb{T}$ under our assumption.

Nevertheless, the set

$$\begin{aligned} span\,\{E_{mA}(\mu) :\ \mu \in m \cdot a(U) \cap \mathbb{T}\} &= span\,\{x_\mu :\ \mu \in m \cdot a(U) \cap \mathbb{T}\} \\ &= span\,\{ker(mA - \mu I) :\ \mu \in m \cdot a(U) \cap \mathbb{T}\} \\ &= span\,\{ker(T - \lambda I) : \lambda \in U, \mu = m \cdot a(\lambda)\} \end{aligned}$$

is dense in $X$ owing to the considerations at the beginning of this proof. Moreover, $mA$ is also chaotic and mixing through Theorem 2.4, since $E_{mA}(\lambda)$ is obviously continuous. This completes the proof. □

Finally, we provide an alternative proof of the Devaney chaos of the Cowen–Douglas class to complete this section. According to the above mentions, the authors Hou et al.[14] have given the proof of the Devaney chaos of $\mathcal{B}_n(\Omega)$. Their proofs depend on the topologically strong mixing, which implies hypercyclicity or topological transitive, and dense periodic points set. The Devaney chaos of $\mathcal{B}_n(\Omega)$ is self-evident under the following theorem according to Aron et al. in [20].

**Theorem 4.3** ([20, Theorem 2.1])**.** *Let $X$ be a separable Banach space and $T$ a bounded linear operator on $X$, besides $U \subset \sigma_p(T)$ being an open and connected subset of the point spectrum of $T$.*

*For all $\lambda \in U$ choose $x_\lambda \in X \setminus \{0\}$ with $Tx_\lambda = \lambda x_\lambda$. For $x^* \in X^*$ (i.e. the dual space of $X$), we define the function*

$$F_{x^*} : U \longrightarrow \mathbb{C} \text{ by } \lambda \mapsto \langle x^*,\ x_\lambda \rangle .$$

*If*

1. *$F_{x^*}$ is analytic in $U$ for all $x^* \in X^*$,*
2. *$F_{x^*} = 0$ if and oniy if $x^* = 0$, and*
3. *$U \cap \mathbb{T} \neq \emptyset$,*

*Thus $T$ is a chaotic operator.*

With reference to Theorem 4.3, the Devaney chaos of $T \in \mathcal{B}_n(\Omega)$ is obvious. Therefore, we ought to check the conditions of Theorem 4.3.

**Theorem 4.4.** *Let $T \in \mathcal{B}_n(\Omega)$. Suppose $\Omega \cap \mathbb{T} \neq \emptyset$. Then, $T$ is Devaney chaotic on $H$.*

*Proof.* Almost all conditions are satisfied. The only one that needs to be considered is the function

$$F_{x^*} \ : \ \Omega \longrightarrow \mathbb{C}.$$

By Proposition 2.7, the mapping $\omega \longrightarrow ker(T-\omega)$ defines a complex bundle $E_T$ over $\Omega$. Take any point $y_\omega \in ker(T-\omega)$. Then, the mapping $\widehat{y} : \Omega \longrightarrow ker(T-\omega)$ defined by $\widehat{y}(\omega) = y_\omega$ for $\omega \in \Omega$ is holomorphic on $\Omega$. Therefore, we can define the function $F_{x^*}$ as a composition of the following two mapping.

$$\Omega \xrightarrow{\widehat{y}} X \xrightarrow{x^*} \mathbb{C} \ for \ x^* \in X^*.$$

That is, $F_{x^*} = x^* \circ \widehat{y}$ for $x^* \in X^*$. Exactly, $F_{x^*}(\lambda) = x^* \circ \widehat{y}(\lambda) = x^*(y_\lambda) = \langle x^*, \ y_\lambda \rangle$. Hence, all conditions of Theorem 4.3 have been satisfied. Therefore, $T \in \mathcal{B}_n(\Omega)$ is chaotic on $H$. This completes the proof. □

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