

Stochastic Persistence of Two Prey Patches and a Central-Habitat Predator¹

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Abstract

In this paper, we study a three-dimensional predator-prey model consisting of two communicating prey patches and a predator population associated with a central habitat. Predation uses Holling type II predation functional response, and environmental fluctuations combine a common Brownian component with independent residual components. We establish existence and uniqueness of global solutions, positivity for strictly positive initial states, and uniform moment bounds of every positive order. A sufficient persistence criterion is formulated using two boundary growth rates: the prey recovery rate at extinction and the predator invasion rate in the stationary predator-free environment.

Keywords: Stochastic persistence; Prey dispersal; Predator invasion; Environmental noise; Occupation measure

1 Introduction

Spatial feeding and environmental variation can influence the survival of interacting populations. Holling's functional-response framework connects feeding saturation with handling time [1]. Landscape studies of insectivorous bats

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motivate distinguishing feeding areas from a central resting or breeding habitat [2]. We represent two prey habitats and one predator population, including individuals temporarily away foraging.

Patch predator–prey models already include adaptive predator habitancy [3], and stochastic two-patch models with Ornstein–Uhlenbeck environments have been studied by Liu [4]. The present model fixes search allocation and uses common plus component-specific Brownian growth noise. Its precise target is almost-sure occupation persistence of all three original components.

Evans et al. [5] characterize low-density spatial growth through total abundance and composition. Hening, Nguyen and Yin [6] establish persistence under density regulation and conservative dispersal. Their theorem supplies the predator-free boundary law here, but trophic conversion cannot be treated as conservative prey migration in the full system. Following the average Lyapunov viewpoint [7], we identify both extinction environments and verify their drift signs. We give the needed occupation and stationary-balance arguments directly, including the transfer from total prey to individual prey patches.

2 Model and persistence definition

Let $Z = (X_1, X_2, Y)$ denote biomasses divided by one reference biomass. Write $A(x) = \alpha_1 x_1 + \alpha_2 x_2$. For $i = 1, 2$, $j = 3 - i$, consider

$$\begin{aligned} dX_i &= \left[X_i(r_i - b_i X_i) - \frac{\alpha_i X_i Y}{1 + hA(X)} + m_{ji} X_j - m_{ij} X_i \right] dt + X_i dE_i, \\ dY &= Y \left[\frac{eA(X)}{1 + hA(X)} - \delta - cY \right] dt + Y dE_3, \end{aligned} \quad (2.1)$$

where $E_i = \sigma_c W_0 + \sigma_i W_i$. The four W_i are independent standard Brownian motions on a filtered probability space satisfying the usual conditions. Initial states are deterministic and nonnegative. Throughout,

$$r_i \in \mathbb{R}, \quad b_i, m_{12}, m_{21}, \alpha_i, h, \delta, c > 0, \quad 0 < e \leq 1, \quad \sigma_c \geq 0, \quad \sigma_1, \sigma_2, \sigma_3 > 0.$$

Here r_i, b_i describe prey net growth and crowding; m_{ij} is movement from i to j ; α_i includes fixed search allocation; h is the common handling scale; and e, δ, c describe conversion, mortality and predator crowding. With dimensionless states, all drift coefficients except e, h have units day^{-1} , h has units day and noise amplitudes have units $\text{day}^{-1/2}$.

The time budget $N_i = \alpha_i x_i T_s$, $T = T_s + h(N_1 + N_2)$ gives intake $N_i/T = \alpha_i x_i / (1 + hA)$. This assumes fixed allocation and negligible travel delays. Predator crowding and equal common-noise loadings are modelling assumptions, not empirical estimates or conclusions of a fast-foraging limit.

The growth covariance is positive definite:

$$\Sigma = \sigma_c^2 \mathbf{1}\mathbf{1}^\top + \text{diag}(\sigma_1^2, \sigma_2^2, \sigma_3^2).$$

The state covariance is $\text{diag}(Z)\Sigma\text{diag}(Z)$ and degenerates on extinction faces. Prey exchange is $D^\top X$, where

$$D = \begin{pmatrix} -m_{12} & m_{12} \\ m_{21} & -m_{21} \end{pmatrix}; \quad D\mathbf{1} = 0.$$

This matrix is irreducible. Set $S = X_1 + X_2$, $R = X_1/S$ for $S > 0$, and $L = S + Y/e$. For $u \in [0, 1]$, define

$$\begin{aligned} \bar{r}(u) &= r_1 u + r_2(1 - u), & B(u) &= b_1 u^2 + b_2(1 - u)^2, \\ a(u) &= \alpha_1 u + \alpha_2(1 - u), & v(u) &= \sigma_c^2 + \sigma_1^2 u^2 + \sigma_2^2(1 - u)^2, \end{aligned}$$

and $v_P = \sigma_c^2 + \sigma_3^2$, $d_P = \delta + v_P/2$. The logarithmic drifts are

$$g_S = \bar{r}(u) - sB(u) - \frac{ya(u)}{1 + hsa(u)} - \frac{v(u)}{2}, \quad g_P = \frac{esa(u)}{1 + hsa(u)} - cy - d_P. \quad (2.2)$$

Use the enlarged space and continuous projection

$$\mathcal{M} = [0, \infty)^2 \times [0, 1], \quad \Phi(s, y, u) = (su, s(1 - u), y).$$

The ratio has no continuous extension at zero prey abundance. Instead, extend the transformed coefficients to define $U = (S, Y, R)$:

$$\begin{aligned} dS &= S(g_S + v(R)/2)dt + S[\sigma_c dW_0 + \sigma_1 R dW_1 + \sigma_2(1 - R)dW_2], \\ dY &= Y(g_P + v_P/2)dt + Y dE_3, \end{aligned} \quad (2.3)$$

$$\begin{aligned} dR &= \left\{ m_{21}(1 - R) - m_{12}R + R(1 - R) \left[r_1 - r_2 - S\{b_1 R - b_2(1 - R)\} \right. \right. \\ &\quad \left. \left. - \frac{Y(\alpha_1 - \alpha_2)}{1 + hSa(R)} + \sigma_2^2(1 - R) - \sigma_1^2 R \right] \right\} dt \\ &\quad + R(1 - R)(\sigma_1 dW_1 - \sigma_2 dW_2). \end{aligned} \quad (2.4)$$

The g 's in (2.3) are evaluated at U . Common noise cancels directly from (2.4). Put $\mathcal{M}_0 = \{(s, y, u) \in \mathcal{M} : sy = 0\}$ and $\mathcal{M}_+ = \mathcal{M} \setminus \mathcal{M}_0$. For smooth f , write $\mathcal{L}f$ for its Itô drift, $\Gamma f = \mathcal{L}(f^2) - 2f\mathcal{L}f$, and $\mu f = \int f d\mu$. Define

$$\Pi_t^{(z)}(G) = t^{-1} \int_0^t \mathbf{1}_G(Z_v) dv, \quad \hat{\Pi}_t = t^{-1} \int_0^t \delta_{U_v} dv, \quad S_\eta = \{z \geq 0 : \min_i z_i \leq \eta\}.$$

Definition 1. The process is stochastically persistent if, for every $z \in (0, \infty)^3$,

$$\mathbb{P}_z \left\{ \lim_{\eta \downarrow 0} \limsup_{t \rightarrow \infty} \Pi_t^{(z)}(S_\eta) = 0 \right\} = 1. \quad (2.5)$$

This occupation definition allows low-density excursions and does not assert a positive pathwise lower bound.

3 Probabilistic preliminaries

Lemma 1 (Martingale strong law). *A continuous local martingale M , with $M_0 = 0$ and $\mathbb{E}\langle M \rangle_t \leq C(1+t)$, is square integrable on finite intervals and satisfies $M_t/t \rightarrow 0$ almost surely. The same strong law holds for a discrete martingale starting at zero with uniformly bounded increments.*

Proof. The Corollary 3 of Protter [8] gives $\mathbb{E}M_t^2 = \mathbb{E}\langle M \rangle_t$ and the true-martingale property. Doob's inequality gives $\mathbb{P}(\sup_{s \leq 2^{j+1}} |M_s| > \varepsilon 2^j) = O(2^{-j})$; Borel–Cantelli and dyadic interpolation prove the claim [9]. Continuity extends Doob's inequality from finite grids. For discrete martingales, bounded increments give $\mathbb{E}M_n^2 = O(n)$ and the same argument applies. $\square$

Lemma 2 (Stationary drift balance). *Write $\mu f = \int f d\mu$, under a stationary law μ , suppose f is measurable and finite μ -almost everywhere, $f(U_t) - f(U_0) = \int_0^t h(U_s) ds + M_t$, $\mu|h| < \infty$, and $M_t/t \rightarrow 0$ in probability. Then $\mu h = 0$, without assuming $f \in L^1(\mu)$ or ergodicity.*

Proof. Birkhoff's theorem [9], applied to the stationary integrable sequence $\int_k^{k+1} h(U_s) ds$, gives an L^1 limit of $A_n = n^{-1} \int_0^n h(U_s) ds$. Stationarity and finiteness give $\mathbb{P}_\mu(|f(U_n)| > \varepsilon n) = \mu\{|f| > \varepsilon n\} \rightarrow 0$. The decomposition implies $A_n \rightarrow 0$ in probability, so its L^1 limit is zero. Since $\mathbb{E}_\mu A_n = \mu h$ for every n , we obtain $\mu h = 0$. $\square$

Lemma 3 (Occupation limits). *Let U be a continuous, time-homogeneous Markov process on a locally compact Polish space, with jointly C_b -Feller semigroup. Suppose $L \geq 0$ is continuous with compact sublevel sets, g is continuous, and, from every initial state,*

$$M_t^L = L(U_t) - L(U_0) - \int_0^t g(U_s) ds$$

is a true martingale satisfying $M_t^L/t \rightarrow 0$ almost surely. If $g \leq C_0 - kL^2$, where $C_0 \geq 0$ and $k > 0$, then almost surely $\limsup_{t \rightarrow \infty} \widehat{\Pi}_t L^2 \leq K := C_0/k$, the occupation measures are tight, and all their weak limits as $t \rightarrow \infty$ are invariant. The invariant laws form a nonempty weakly compact set $\mathcal{I}$, with $\sup_{\mu \in \mathcal{I}} \mu L^2 \leq K$. For continuous $f = O(1+L)$, $\mu_n f \rightarrow \mu f$ whenever $\mu_n \Rightarrow \mu$ in $\mathcal{I}$, or along an occupation subsequence with times tending to infinity on this full-probability event.

Proof. Apply the Hypotheses 2–3 and Theorem 2.2(ii) [7] with $W = L$, $\widetilde{W} = kL^2$ and $LW = g$. Tight occupation measures have a weak limit, giving nonemptiness. For the last claim, use the Lemma 9.4(ii) of [7] with $W = L^2$, $H = f$: the continuous ratio $L^2/(1+|f|)$ has compact sublevel sets, since it tends to infinity as $L \rightarrow \infty$. $\square$

4 Well-posedness and control at infinity

Put $r_+ = \max(0, r_1, r_2)$, $k_0 = \min(b_1, b_2, ce)/3 > 0$ and $v_* = \sigma_c^2 + \max_i \sigma_i^2$.

Theorem 1. *Equation (2.1) has a unique global nonnegative strong solution from every nonnegative state. Positive components remain positive, and $S_0 > 0$ implies $X_1(t), X_2(t) > 0$ for every $t > 0$. The face $Y = 0$ is absorbing. Equations (2.3)–(2.4) define a unique global process from every $U_0 \in \mathcal{M}$ with jointly C_b -Feller semigroup and $\Phi(U_t) = Z_t$, for $Z_0 = \Phi(U_0)$ and the same Brownian motions.*

Proof. Replacing $A(X)$ in denominators by $A(X^+)$ gives locally Lipschitz coefficients on Euclidean space; truncation gives a maximal local strong solution by their Theorem 5.2.1 [10]. Divide coordinate i by $H_i^0 = \exp(E_i - v_i t/2)$, where $v_i = \sigma_c^2 + \sigma_i^2$. Itô's formula yields a random ODE $\dot{q} = \mathcal{F}(t, q)$ with locally Lipschitz state dependence. On $q \geq 0$, $q_i = 0$ implies $\mathcal{F}_i(t, q) \geq 0$. On bounded stopped regions, $N = \sum_i (q_i^-)^2$ satisfies $\dot{N} \leq 2 \sum_i q_i^- |\mathcal{F}_i(t, q) - \mathcal{F}_i(t, q^+)| \leq C_T N$. Thus $N = 0$ by Gronwall, proving nonnegativity.

Set $f_i = r_i - b_i X_i - \alpha_i Y / (1 + hA) - m_{ij}$ and $\mathcal{E}_i(t) = \exp\{\int_0^t (f_i - v_i/2) dv + E_i(t)\}$. The product Itô formula gives

$$X_i(t) = \mathcal{E}_i(t) \left[X_i(0) + \int_0^t \frac{m_{ji} X_j(v)}{\mathcal{E}_i(v)} dv \right].$$

The predator equals $Y_0 \exp\{\int_0^t g_P(U_v) dv + E_3(t)\}$. These stopped representations imply positivity, its propagation by positive migration, and absorption of $Y = 0$.

Trophic terms and migration cancel in L , giving

$$\begin{aligned} \mathcal{L}L &= \sum_{i=1}^2 (r_i X_i - b_i X_i^2) - \delta Y/e - cY^2/e \leq r_+ L - k_0 L^2, \\ \Gamma L &= \sigma_c^2 L^2 + \sigma_1^2 X_1^2 + \sigma_2^2 X_2^2 + \sigma_3^2 (Y/e)^2 \leq v_* L^2. \end{aligned} \quad (4.6)$$

Indeed, with $z_3 = Y/e$, $X_1^2 + X_2^2 + z_3^2 \geq L^2/3$ and $cY^2/e = ce z_3^2$. For $n > L_0$, stop at $\tau_n = \inf\{t : L_t \geq n\}$. Stopped Dynkin and Gronwall yield $n\mathbb{P}(\tau_n \leq T) \leq \mathbb{E}L_{T \wedge \tau_n} \leq L_0 e^{r+T}$. Thus no finite explosion occurs, and the positive representations are global.

For $s_0 > 0$, take $S, Y, R = X_1/S$ from the original solution. Quotient Itô gives (2.4); its correction is $R(1 - R)[\sigma_2^2(1 - R) - \sigma_1^2 R]$. For $s_0 = 0$, solve the scalar predator equation and the auxiliary system

$$dQ_i = [(r_i - \alpha_i Y_t)Q_i + m_{ji}Q_j - m_{ij}Q_i]dt + Q_i dE_i, \quad Q(0) = (u_0, 1 - u_0).$$

After division by H_i^0 this is a linear random ODE with coefficients bounded on finite intervals. It has a global solution with $Q_1 + Q_2 > 0$. Its ratio solves (2.4)

with $S = 0$, constructing all added initial states. The covariance $d[S, R] = SR(1 - R)[\sigma_1^2 R - \sigma_2^2(1 - R)]dt$ cancels the ratio correction in $d(SR)$. Product Itô therefore verifies projection, also at $s = 0$. Local Lipschitz uniqueness applies to the lifted coefficients near compact subsets of $\mathcal{M}$.

The exit bound is uniform on compact initial sets. Coupling globally Lipschitz truncations gives $\mathbb{E} \sup_{v \leq T} |U_v^{u,n} - U_v^{u',n}|^2 \leq C_{n,T} |u - u'|^2$ by Doob, isometry and Gronwall. Removing truncation gives $U_{t_j}^{u_j} \rightarrow U_t^u$ in probability when $(t_j, u_j) \rightarrow (t, u)$. Hence $P_t f(u)$ is jointly continuous for $f \in C_b$. Uniqueness and independent Brownian increments give the Markov property. $\square$

Theorem 2. *For every $p > 0$ and $U_0 \in \mathcal{M}$, $\sup_{t \geq 0} \mathbb{E} L_t^p \leq C_p(1 + L_0^p)$. With $C_0 = 1 + r_+^2/(2k_0)$, $k = k_0/2$ and $K = C_0/k$, almost surely*

$$\limsup_{t \rightarrow \infty} \widehat{\Pi}_t L^2 \leq K, \quad \limsup_{t \rightarrow \infty} \widehat{\Pi}_t \{L > a\} \leq K/a^2 \quad (a > 0). \quad (4.7)$$

All conclusions of Lemma 3 hold on $\mathcal{M}$.

Proof. For $p \geq 1$, the chain rule and (4.6) give

$$\mathcal{L}(1 + L)^p \leq p(1 + L)^{p-1}(r_+ L - k_0 L^2) + \frac{1}{2}p(p-1)v_*(1 + L)^{p-2}L^2.$$

The negative term of order $p+1$ dominates, so this is at most $A_p - B_p(1 + L)^p$ for positive constants. Stopped exponential Itô and Fatou give $\mathbb{E}(1 + L_t)^p \leq e^{-B_p t}(1 + L_0)^p + (A_p/B_p)(1 - e^{-B_p t})$. Jensen handles $0 < p < 1$. The biomass martingale has coefficients $(\sigma_c L, \sigma_1 X_1, \sigma_2 X_2, \sigma_3 Y/e)$. Thus $\mathbb{E}\langle M^L \rangle_t \leq C_{U_0} t$ by the second-moment bound, and Lemma 1 applies. Since $\mathcal{L}L \leq C_0 - kL^2$, $k \int_0^t L_v^2 dv \leq L_0 + C_0 t + M_t^L - L_t \leq L_0 + C_0 t + M_t^L$. This proves (4.7). The sets $\{L \leq a\} \subset [0, a] \times [0, ea] \times [0, 1]$ are compact, so Lemma 3 applies. $\square$

5 Boundary growth and persistence

At $s = y = 0$, let ν be the invariant law of (2.4). Define

$$\lambda_0 = \int_0^1 [\bar{r}(u) - v(u)/2] \nu(du), \quad \lambda_P = e \int \frac{A(x)}{1 + hA(x)} \pi(dx) - d_P, \quad (5.8)$$

where π is defined by the next lemma only when $\lambda_0 > 0$.

Lemma 4. *Under the model assumptions and the positivity and projection conclusions of Theorem 1, the composition process at $S = Y = 0$ has a unique invariant law ν , with $\nu((0, 1)) = 1$. Set $\lambda_0 = \int_0^1 [\bar{r}(u) - v(u)/2] \nu(du)$. If $\lambda_0 > 0$, the prey-only system has a unique invariant law π on $(0, \infty)^2$, and all invariant laws on $\mathcal{M}_0 = \{sy = 0\}$ are $a\mu_0 + (1 - a)\mu_F$, $a \in [0, 1]$, where $\mu_0 = \delta_{(0,0)} \otimes \nu$, $\mu_F = T_\# \pi$ and $T(x) = (x_1 + x_2, 0, x_1/(x_1 + x_2))$.*

Proof. Positive dispersal makes D irreducible, and $\Sigma_F = \sigma_c^2 \mathbf{1}\mathbf{1}^\top + \text{diag}(\sigma_1^2, \sigma_2^2)$ is positive definite. Evans et al. [5] give ν by their Proposition 1; positivity at positive times excludes stationary endpoint mass.

For [6], take $a_i = r_i$ and $\beta_i(x) = b_i x$. Their Assumptions 2.1–2.3 hold, since the competition functions are locally Lipschitz, vanish at zero, and, for $x_1 + x_2 > 0$,

$$\frac{\sum_i x_i (b_i x_i - r_i)}{x_1 + x_2} \geq \frac{\min(b_1, b_2)}{2} (x_1 + x_2) - r_+.$$

For $w = (u, 1 - u)^\top$, $w^\top \Sigma_F w = v(u)$, so their growth rate (2.8) equals λ_0 ; their Theorem 2.1 gives π .

The blocks $\{s = y = 0\}$, $\{s > 0, y = 0\}$ and $\{s = 0, y > 0\}$ satisfy $P_t \mathbf{1}_B = \mathbf{1}_B$ by Theorem 1. On the last block, $\log Y_t \leq \log Y_0 - d_P t + E_3(t)$ implies $Y_t \rightarrow 0$ almost surely by Lemma 1. An invariant law there would contradict stationarity and bounded convergence of $Y_t/(1 + Y_t)$. The first block has the unique law μ_0 . On the prey block, positivity removes endpoint mass, and the inverse prey coordinate maps transfer uniqueness of π , giving μ_F . Normalized restrictions of invariant laws to these blocks remain invariant, proving the classification; conversely, each stated mixture is invariant. $\square$

Theorem 3. *If $\lambda_0 > 0$ and $\lambda_P > 0$, then for every $\epsilon > 0$ there is a deterministic $\eta_\epsilon > 0$, independent of the positive initial state, such that*

$$\mathbb{P}_z \{ \limsup_{t \rightarrow \infty} \Pi_t^{(z)}(S_{\eta_\epsilon}) \leq \epsilon \} = 1 \quad (z \in (0, \infty)^3). \quad (5.9)$$

Consequently (2.5) holds, and the original process has an invariant probability assigning full mass to $(0, \infty)^3$.

Proof. Choose $d_P/(\lambda_0 + d_P) < \theta < 1$, put $q = 1 - \theta$, and on $\mathcal{M}_+$ define

$$V = \log(1 + L) - \theta \log S - q \log(Y/e), \quad H = \frac{\mathcal{L}L}{1 + L} - \frac{\Gamma L}{2(1 + L)^2} - \theta g_S - q g_P.$$

Weighted AM–GM gives $S^\theta(Y/e)^q \leq \theta S + qY/e \leq L < 1 + L$, hence $V \geq 0$. The drift H extends continuously to $\mathcal{M}$ and $|H| \leq C(1 + L)$. The four noise coefficients of V are

$$-\frac{\sigma_c}{1 + L}, \quad \sigma_1 R \left(\frac{S}{1 + L} - \theta \right), \quad \sigma_2 (1 - R) \left(\frac{S}{1 + L} - \theta \right), \quad \sigma_3 \left(\frac{Y/e}{1 + L} - q \right).$$

They are bounded. Localized Itô, global positivity and Lemma 1 give

$$V(U_t) = V(U_0) + \int_0^t H(U_v) dv + M_t^V, \quad M_t^V/t \rightarrow 0 \quad \text{a.s.} \quad (5.10)$$

For any invariant law concentrated on $\mathcal{M}_+$, Lemma 2 gives $\mu H = 0$, since H is integrable by Theorem 2. No existence of such a law is presumed at this step. The same lemma applies to $\log S$ and $\log(1 + L)$ under μ_F : their drifts are integrable and their noises bounded. Thus

$$\mu_0 H = -\theta \lambda_0 + q d_P < 0, \quad \mu_F H = -q \lambda_P < 0.$$

Invariant restriction and Lemma 4 imply for every $\mu \in \mathcal{I}$ that $\mu H \leq 0$, with equality precisely when $\mu(\mathcal{M}_0) = 0$.

Fix a positive initial state and intersect the probability-one events for occupation tightness, invariant limits, the time-moment bound and the martingale law. For any $\hat{\Pi}_{t_n} \Rightarrow \mu$, Lemma 3 gives $\hat{\Pi}_{t_n} H \rightarrow \mu H$. Dividing (5.10) by t_n yields $\mu H = \lim_n V(U_{t_n})/t_n \geq 0$. Hence all occupation limits belong to $\mathcal{I}_+ = \{\mu \in \mathcal{I} : \mu H = 0\}$. This set is nonempty and compact because $\mathcal{I}$ is compact and $\mu \mapsto \mu H$ is continuous there by Lemma 3.

Let $F_\eta = \{(s, y, u) : \min(su, s(1-u), y) \leq \eta\}$, a closed set. For $u \in \mathcal{M}_+$ and $t > 0$, projection and positive migration give $P_t(u, F_0) = 0$, so $\mu(F_0) = 0$ for every $\mu \in \mathcal{I}_+$. If $\sup_{\mu \in \mathcal{I}_+} \mu(F_\eta) \not\rightarrow 0$, compactness provides $\eta_n \downarrow 0$, $\mu_n \Rightarrow \mu \in \mathcal{I}_+$ with $\mu_n(F_{\eta_n}) \geq a > 0$. For each fixed $r > 0$, eventual inclusion in F_r and closed-set Portmanteau imply $\mu(F_r) \geq a$. Continuity from above then contradicts $\mu(F_0) = 0$.

Choose η_ϵ so the supremum is below $\epsilon/2$. Any trajectory limsup above ϵ would have an occupation-limit subsequence; Portmanteau forces its limit to assign at least ϵ mass to F_{η_ϵ} , a contradiction. Since $\hat{\Pi}_t(F_\eta) = \Pi_t^{(z)}(S_\eta)$, this proves (5.9). Take $\epsilon = 1/n$, replace thresholds by $\min(1/n, \eta_1, \dots, \eta_n)$ and intersect countably many events to get (2.5). Finally, $P_t^U(f \circ \Phi) = (P_t^Z f) \circ \Phi$ makes $\Phi_\# \mu$, for $\mu \in \mathcal{I}_+$, invariant for Z ; $\mu(F_0) = 0$ gives its full interior mass. $\square$

6 Discussion

We have established an almost-sure stochastic persistence criterion for two dispersing prey populations coupled to a predator associated with a central habitat. The analysis proceeds from global positivity and uniform moment bounds to the exclusion of invariant extinction environments. Positive prey recovery and predator invasion rates are sufficient for persistence of each of the three components from every strictly positive initial state. More precisely, the upper limiting proportion of time for which any component lies below a density threshold tends to zero as that threshold decreases to zero. The same argument yields the existence of an invariant probability measure concentrated in the positive interior.

The criterion separates recovery of the prey system from invasion of the predator into its stationary food environment. It therefore provides a way to

assess the combined effects of resource availability, prey redistribution, and environmental variability without assuming convergence of the full system to a unique stationary law. The enlarged state space retains the prey composition needed to evaluate growth near extinction, while positive bidirectional dispersal connects persistence of total prey abundance to persistence in both patches. But the biological interpretation remains conditional on fixed predator search allocation, a shared handling-time response, predator self-regulation, and the prescribed covariance of environmental fluctuations.

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