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A Fixed Point Theorem for Nonlinear Contractions in Menger PM Spaces of Hyperbolic Type

Silvja Çobani and Elida Hoxha

Department of Mathematics, Faculty of Natural Sciences
University of Tirana, Albania

Abstract

In this paper, we investigate the existence of fixed points in Menger probabilistic metric spaces of hyperbolic type. We establish a new fixed point theorem for non-self mappings defined on nonempty, closed and compact subsets of complete Menger PM-spaces of hyperbolic type. The main result is obtained under a nonlinear contractive condition involving φ and ψ functions and provides a new contribution to the study of nonlinear contractions in Menger PM-spaces of hyperbolic type.

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Keywords: Menger probabilistic metric space; hyperbolic type; fixed point; non-self mapping; φ- ψ contraction

1 Introduction

The Banach Contraction Principle [1] remains one of the cornerstones of nonlinear analysis and fixed point theory. Thanks to its wide range of applications, numerous extensions and generalizations have been developed in different mathematical settings. Among the most influential generalizations is the nonlinear contraction principle introduced by Boyd and Wong [2], in which a nonlinear control function replaces the classical linear contraction constant.

An important development in fixed point theory focused on cases where the mapping being studied is not necessarily a self-mapping of a closed subset. In this direction, Assad and Kirk [3] first investigated non-self mappings in complete metrically convex spaces (X, d). They proved that if $f : K \to X$ satisfies the Banach Contraction principle, then the boundary condition $f(\partial K) \subseteq K$ guarantees the existence of a fixed point of f. Their result initiated an active line of research, leading to numerous extensions and generalizations in both classic and generalized metric spaces.

To model uncertainty and nondeterministic phenomena, Menger [4] introduced the concept of probabilistic metric spaces, replacing the deterministic distance between two points with a distribution function. Later, Schweizer and Sklar [5] established the fundamental theory of Menger probabilistic metric spaces, including their topology, convergence structure, continuity of mappings, and completeness properties. Since then, probabilistic metric spaces have become an important framework for extending classical fixed point results. Sehgal and Bharucha-Reid [6] obtained the first fixed point theorem in this setting as a probabilistic version of Banach's contraction principle.

On the other hand, Takahashi [7] introduced convex structures in metric spaces. This concept was later extended to Menger probabilistic metric spaces by Hadžić [8], who introduced convex structures and obtained fixed point results in this more general setting. Ješić et al. [9] later defined strictly convex structures in Menger PM-spaces and established several fixed point and common fixed point theorems under various contractive assumptions.

Subsequently, several authors generalized nonlinear contraction principles to probabilistic metric spaces. In particular, Ćirić [10] developed probabilistic analogues of Boyd–Wong [2] type contractions, while Fang [11] introduced a broader class of nonlinear contractive conditions governed by a function φ, leading to significant extensions of fixed point theory in probabilistic and fuzzy metric spaces.

Nikolić et al. ([12],[13]) extended fixed point theory in Menger probabilistic metric spaces by combining nonlinear φ-contractive conditions with strictly convex structures, especially for self-mappings, common fixed points, and non-self mappings. Their work shows that classical fixed point results, including those inspired by Boyd and Wong [2], Fang [11], and Assad and Kirk [3], can be generalized to spaces with nondeterministic distances and richer geometric structure.

Motivated by the concept of hyperbolic metric spaces introduced by Kirk [14] and by fixed point results for nonlinear contractions in Menger probabilistic metric spaces due to Ćirić [10], Fang [11], Ješić et al. [9], and Nikolić et al. [13], we study fixed points of non-self mappings in complete Menger PM-spaces of hyperbolic type. We prove a fixed point theorem for mappings $g : K \to X$ satisfying a nonlinear contractive condition involving not one, but two control functions φ and ψ. The proof relies on the hyperbolic structure of the space, the notion of probabilistic diameter, and the convergence properties of nested subsets whose probabilistic diameter tends to zero. Our result extends several known fixed point theorems in strictly convex Menger PM-spaces and provides a probabilistic generalisation of corresponding results in metric spaces of hyperbolic type.

2 Preliminary Notes

Definition 2.1. [5] *A function $F : \mathbb{R} \to [0,1]$ is called a distance distribution function if it is non-decreasing, left-continuous and satisfies $F(0) = 0$ and $\sup_{t\in\mathbb{R}} F(t) = 1$.*

The family of all distance distribution functions will be denoted by D^+.

The maximal element of D^+ is the distribution function $\varepsilon_0(t)$ defined by:

$$\varepsilon_0(t) = \begin{cases} 0, & t \leq 0, \\ 1, & t > 0. \end{cases}$$

Definition 2.2. [5] *A binary operation $T : [0,1] \times [0,1] \to [0,1]$ is called a continuous t-norm if it satisfies the following conditions:*

(i) $T(a,1) = a$ *for every* $a \in [0,1]$*;*

(ii) $T(a,b) = T(b,a)$ *for all* $a, b \in [0,1]$*;*

(iii) $T(T(a,b),c) = T(a,T(b,c))$ *for all* $a,b,c \in [0,1]$*;*

(iv) $T(a,b) \leq T(c,d)$ *whenever* $a \leq c$ *and* $b \leq d$*;*

(v) T *is continuous.*

Typical examples of continuous t-norms are $T_M(a,b) = \min\{a,b\}, T_P(a,b) = ab, T_L(a,b) = \max\{0, a+b-1\}$.

Definition 2.3. [5] *A Menger probabilistic metric space (briefly, Menger PM-space) is a triple $(X, \mathcal{F}, T)$, where X is a nonempty set, T is a continuous t-norm and $\mathcal{F} \times X \to D^+, \mathcal{F}(x,y) = F_{x,y}$, such that for all $x,y,z \in X$ and $s,t \geq 0$ the following conditions hold:*

(i) $F_{x,y}(t) = \varepsilon_0(t)$ *if and only if* $x = y$*;*

(ii) $F_{x,y}(t) = F_{y,x}(t)$*;*

(iii) $F_{x,z}(t+s) \geq T\big(F_{x,y}(t), F_{y,z}(s)\big)$.

Remark 1. [6] Every metric space induces a Menger PM-space. Indeed, let (X, d) be a metric space and let $T = T_M$. Define: $F_{x,y}(t) = \varepsilon_0\big(t - d(x, y)\big), x, y \in X,\ t > 0$. Then $(X, \mathcal{F}, T_M)$ is a Menger PM-space induced by the metric d.

A natural topology on a Menger PM-space is the (ϵ, λ)-topology introduced by Schweizer and Sklar [5]. For a point $p \in X$, its family of neighbourhoods is given by:

$$\mathcal{N}_p = \{N_p(\epsilon, \lambda) : \epsilon > 0, \lambda \in (0, 1)\},$$

where

$$\{q \in X : F_{p,q}(\epsilon) > 1 - \lambda\}.$$

The resulting (ϵ, λ)-topology is Hausdorff.

Definition 2.4. [5] *Let $(X, \mathcal{F}, T)$ be a Menger PM-space and let x_n be a sequence in X.*

(i) The sequence x_n is said to converge to $x \in X$ if for every $\varepsilon > 0$ and $\lambda \in (0, 1)$ there exists $N \in \mathbb{N}$ such that $F_{x_n,x}(\varepsilon) > 1 - \lambda, n \geq N$.

(ii) The sequence x_n is called a Cauchy sequence if for every $\varepsilon > 0$ and $\lambda \in (0, 1)$ there exists $N \in \mathbb{N}$ such that $F_{x_n,x_m}(\varepsilon) > 1 - \lambda, n, m \geq N$.

(iii) The space $(X, \mathcal{F}, T)$ is said to be complete if every Cauchy sequence converges to a point of X.

(iv) A mapping $f \to X$ is continuous at $x_0 \in X$ if and only if $x_n \to x_0 \Longrightarrow f(x_n) \to f(x_0)$.

Definition 2.5. [5] *Let $(X, \mathcal{F}, T)$ be a Menger PM-space and let $A \subseteq X$. The closure of A is the union of the set itself with all the limits of sequences from A, and is denoted by $\overline{A}$.*

Definition 2.6. [5] *Let $(X, \mathcal{F}, T)$ be a Menger PM-space and let $A \subseteq X$. The set A is said to be compact if every sequence in A contains a subsequence that converges to a point of A.*

The notion of probabilistic boundedness was introduced by Egbert [15] and plays an important role in the study of probabilistic metric spaces.

Definition 2.7. [15] *Let $(X, \mathcal{F}, T)$ be a Menger PM-space and let $A \subseteq X$. The function*

$$\delta_A(t) = \sup_{\varepsilon < t} \inf_{x,y \in A} F_{x,y}(\varepsilon), \qquad t > 0,$$

is called the probabilistic diameter of the set A.

The diameter of A is defined by

$$\delta_A = \sup_{t>0} \delta_A(t).$$

If there exists $\lambda \in (0,1)$ such that $\delta_A = 1-\lambda$, then A is said to be probabilistically semi-bounded. If $\delta_A = 1$, then A is called probabilistically bounded.

Remark 2. [9] Every bounded subset of a metric space is probabilistically bounded when the metric space is regarded as the induced Menger PM-space. Thus, probabilistic boundedness extends the classical notion of metric boundedness to the probabilistic setting.

Theorem 2.8. [15] *Let $(X, \mathcal{F}, T)$ be a Menger PM-space and let A and B be nonempty subsets of X such that $A \subseteq B$. Then $\delta_A \geq \delta_B$.*

Theorem 2.9. [15] *Let $(X, \mathcal{F}, T)$ be a Menger PM-space and let A be a nonempty subset of X. Then the probabilistic diameter of A coincides with the probabilistic diameter of its closure, i.e., $\delta_A = \delta_{\overline{A}}$.*

Lemma 2.10. [19] *Let $(X, \mathcal{F}, T)$ be a Menger PM-space and let $A \subseteq X$. Then A is probabilistically bounded if and only if for every $\lambda \in (0,1)$ there exists $t > 0$ such that $F_{x,y}(t) > 1-\lambda$, for all $x, y \in A$.*

Theorem 2.11. [16] *Every compact subset of a Menger PM-space is probabilistically bounded.*

Theorem 2.12. [20] *Let $(X, \mathcal{F}, T)$ be a Menger PM-space and let $\{A_n\}_{n\in\mathbb{N}}$ be a decreasing sequence of nonempty closed subsets of X. If we have that $\delta_{A_n} \to \varepsilon_0$, as $n \to \infty$, then there is only one point $x \in \bigcap_{n=1}^{\infty} A_n$.*

Lemma 2.13. [20] *Let $(X, \mathcal{F}, T)$ be a Menger PM-space and let $\{A_n\}_{n\in\mathbb{N}}$ be a decreasing sequence of nonempty closed subsets of X. Then the sequence $\{A_n\}$ has probabilistic diameter zero if and only if $\delta_{A_n} \longrightarrow \varepsilon_0$, as $n \to \infty$.*

Definition 2.14. ([9],[17]) *A Menger PM-space $(X, \mathcal{F}, T)$ is said to be metrically convex if for every pair of distinct points $p, q \in X$ and every $\lambda \in (0,1)$, there exists a unique point $r \in X$ satisfying:*

$$F_{p,r}(t) = F_{p,q}\left(\frac{t}{1-\lambda}\right), \qquad F_{r,q}(t) = F_{p,q}\left(\frac{t}{\lambda}\right), \tag{1}$$

for all $t > 0$.

For $\lambda \in [0,1]$, define $r_\lambda = q$ for $\lambda = 0, r_\lambda = p$, for $\lambda = 1$, and $r_\lambda = r$, for $\lambda \in (0,1)$ where r is the unique point satisfying (1), for every $t > 0$. The set $\operatorname{seg}[p,q] = \{r_\lambda : \lambda \in [0,1]\}$ is called the metric segment joining p and q.

Definition 2.15. [8] *A metrically convex Menger PM-space $(X, \mathcal{F}, T)$ is said to be of hyperbolic type if for every $p, q \in X$ there exists a unique metric segment* $\operatorname{seg}[p,q]$ *and, for every $u \in X$, every $r \in \operatorname{seg}[p,q]$, that satisfies (1), and every $t > 0$:*

$$F_{u,r}(2t) \geq T\left(F_{u,p}\left(\frac{t}{\lambda}\right), F_{u,q}\left(\frac{t}{1-\lambda}\right)\right), \tag{2}$$

Lemma 2.16. [9] *Let $(X, \mathcal{F}, T)$ be a Menger PM-space of hyperbolic type. Assume that $\forall p, q, u \in X$, $\forall \lambda \in (0,1)$, $\forall t > 0$, and $\forall r \in \mathrm{seg}[p,q]$, we have that:*

$$F_{u,r}(t) > \min\{F_{u,p}(t), F_{u,q}(t)\}. \tag{3}$$

If there exists a point $x \in X$ such that:

$$F_{x,r}(t) = \min\{F_{x,p}(t), F_{x,q}(t)\}, \tag{4}$$

for all $t > 0$, then $r \in \{p, q\}$.

The following definition and results play an important role in the study of nonlinear contractive mappings in Menger PM-spaces.

Definition 2.17. *Let Φ be the class of all functions $\varphi : [0,\infty) \to [0,\infty)$ satisfying:*

(i) φ is continuous and nondecreasing,

(ii) $\varphi(t) < t$, for every $t > 0$.

Lemma 2.18. [18] *Let $\varphi \in \Phi$, as defined in Definition 2.17. Then, for every $t > 0$, the sequence of iterates $\varphi^n(t)_{n\in\mathbb{N}}$ converges to zero.*

Lemma 2.19. [19] *Let $(X, \mathcal{F}, T)$ be a Menger PM-space and let $\varphi \in \Phi$. Suppose that for some $x, y \in X$, we have $F_{x,y}\big(\varphi(t)\big) \geq F_{x,y}(t), t > 0$. Then $x = y$.*

Theorem 2.20. [12] *Let $(X, \mathcal{F}, T)$ be a complete Menger PM-space of hyperbolic type that satisfies (3). Let C be a nonempty, closed and probabilistically bounded subset of X, and let $f \to X$ be a non-self mapping satisfying:*

$$F_{f(x),f(y)}(\psi(t)) \geq F_{x,y}(t), \qquad x, y \in C, t > 0, \tag{5}$$

where $\psi : (0,\infty) \to (0,\infty)$ satisfies:

$$\forall, t > 0,\ \exists, r \geq t, \quad \text{such that} \quad \lim_{n\to\infty} \varphi^n(r) = 0.$$

If $f(\partial C) \subseteq C$, then f has a unique fixed point in C.

Theorem 2.21. [13] *Let $(X, \mathcal{F}, T_M)$ be a Menger PM-space of hyperbolic type that satisfies (3). Let K be a nonempty, closed and probabilistically bounded subset of X, and let $f, g : K \to X$ be two non-self mappings satisfying:*

$$F_{f(x),f(y)}(\varphi(t)) \geq \min\Big\{F_{g(x),g(y)}(t), F_{g(x),f(x)}(t), F_{g(y),f(y)}(t), F_{g(x),f(y)}(2t), F_{g(y),f(x)}(2t)\Big\} \tag{6}$$

for all $x, y \in K$ and every $t > 0$, where $\varphi \in \Phi$. Assume furthermore that:

(i) $\partial K \subseteq g(K)$ *and* $f(K) \cap K \subseteq g(K)$*;*

(ii) $g(x) \in \partial K$ *implies* $f(x) \in K$*;*

(iii) $g(K)$ *is closed in* X*.*

Then f *and* g *possess a coincidence point in* K*. Moreover, if* f *and* g *are coincidentally commuting, then they admit a unique common fixed point.*

3 Main Results

We are now ready to formulate our main theorem, which combines and extends all the above fixed point results to Menger PM-spaces of hyperbolic type.

Theorem 3.1. *Let* $(X, \mathcal{F}, T_M)$ *be a complete Menger PM-space of hyperbolic type, satisfying* (3)*. Let* K *be a nonempty, closed and probabilistic bounded subset of* X*, and let* $g : K \to X$ *satisfy the condition:*

$$F_{g(x),g(y)}(\varphi(t)) \geq \min\left\{F_{x,y}(t), F_{x,g(x)}(t), F_{y,g(y)}(t), F_{x,g(y)}\left(t+\psi(t)\right), F_{y,g(x)}\left(t+\psi(t)\right)\right\}, \tag{7}$$

for all $x, y \in K$ *and every* $t > 0$*, where* $\varphi \in \Phi$*, and* $\psi : [0, \infty) \to [0, \infty)$ *is a continuous function such that* $\psi(0) = 0$ *and* (5) *holds.*

If $f(\partial K) \subseteq K$*, then* g *has a unique fixed point.*

Proof. First of all, applying the triangle inequality in Definition 2.3 to the two last terms of condition (7) we get: $F_{x,g(y)}(t+\psi(t)) \geq \min\left\{F_{x,g(x)}(t), F_{g(x),g(y)}(\psi(t))\right\}$, and $F_{y,g(x)}(t+\psi(t)) \geq \min\left\{F_{y,g(y)}(t), F_{g(x),g(y)}(\psi(t))\right\}$.

Thus condition (7), becomes:

$$F_{g(x),g(y)}(\varphi(t)) \geq \min\left\{F_{x,y}(t), F_{x,g(x)}(t), F_{y,g(y)}(t), F_{g(x),g(y)}\left(\psi(t)\right)\right\}. \tag{8}$$

Now, we choose an arbitrary point $x_0 \in K$ and define $y_1 = g(x_0)$. If $y_1 \in K$, we set $x_1 = y_1$. Otherwise, if $y_1 \notin K$, then, we choose $x_1 \in \partial K \cap \operatorname{seg}[x_0, y_1]$. Proceeding inductively, we construct $\{x_n\}, \{y_n\} \in K$ such that:

$$y_{n+1} = g(x_n),$$
$$x_{n+1} = y_{n+1}, \text{if } y_{n+1} \in K, \text{ or } x_{n+1} \in \partial K \cap \operatorname{seg}[x_n, y_{n+1}], \text{if } y_{n+1} \notin K.$$

Let $A = \{x_n : \ x_n = y_n, \ n \in \mathbb{N}\}$ and let $A_n = \{x_{n+i} : i \in \mathbb{N} \cup \{0\}\}$. Since $A_n \subseteq K$ for every $n \in \mathbb{N}$ and K is probabilistically bounded, it follows from Theorems 2.8 and 2.9 that $\overline{A_n}$ is also probabilistically bounded. Next, we proceed to estimate the probabilistic diameter of $\overline{A_n}$.

Let $i, j \in \mathbb{N} \cup \{0\}$ be arbitrary and let us first evaluate $F_{x_{n+i},x_{n+j}}(\varphi(t))$, for $t > 0$ for each possible case.

Case 1. Let $x_{n+i}, x_{n+j} \in A_n$. Since $x_{n+i} = y_{n+i} = g(x_{n+i-1})$ and $x_{n+j} = y_{n+j} = g(x_{n+j-1})$, from (8), (5) and Definition 2.7 we obtain:

$$\begin{aligned} F_{x_{n+i},x_{n+j}}(\varphi(t)) &= F_{g(x_{n+i-1}),g(x_{n+j-1})}(\varphi(t)) \\ &\geq \min \left\{ \begin{matrix} F_{x_{n+i-1},x_{n+j-1}}(t), F_{x_{n+i-1},g(x_{n+i-1})}(t), \\ F_{x_{n+j-1},g(x_{n+j-1})}(t), F_{g(x_{n+i-1}),g(x_{n+j-1})}(\psi(t)) \end{matrix} \right\} \\ &\geq \min \left\{ \begin{matrix} F_{x_{n+i-1},x_{n+j-1}}(t), F_{x_{n+i-1},x_{n+i}}(t), \\ F_{x_{n+j-1},x_{n+j}}(t), F_{x_{n+i},x_{n+j}}(t) \end{matrix} \right\} \\ &\geq \delta_{A_{n-2}}(t). \end{aligned} \tag{9}$$

Case 2. Let $x_{n+i} \in A_n$ and $x_{n+j} \notin A_n$. Thus, $x_{n+i} = y_{n+i} = g(x_{n+i-1})$, $x_{n+j} \in \text{seg}[x_{n+j-1}, y_{n+j}] \cap \partial K$, and $x_{n+j-1} = y_{n+j-1} = g(x_{n+j-2})$, from (3) we obtain:

$$F_{x_{n+i},x_{n+j}}(\varphi(t)) > \min \left\{ F_{x_{n+i},x_{n+j-1}}(\varphi(t)), F_{x_{n+i},y_{n+j}}(\varphi(t)) \right\}.$$

Now we consider the two possibilities.
Case 2.1. Using (8), (5) and Definition 2.7 we get:

$$\begin{aligned} F_{x_{n+i},x_{n+j}}(\varphi(t)) &> F_{x_{n+i},x_{n+j-1}}(\varphi(t)) \\ &= F_{g(x_{n+i-1}),g(x_{n+j-2})}(\varphi(t)) \\ &\geq \min \left\{ \begin{matrix} F_{x_{n+i-1},x_{n+j-2}}(t), F_{x_{n+i-1},g(x_{n+i-1})}(t), \\ F_{x_{n+j-2},g(x_{n+j-2})}(t), F_{g(x_{n+i-1}),g(x_{n+j-2})}(\psi(t)) \end{matrix} \right\} \\ &\geq \min \left\{ \begin{matrix} F_{x_{n+i-1},x_{n+j-2}}(t), F_{x_{n+i-1},x_{n+i}}(t), \\ F_{x_{n+j-2},x_{n+j-1}}(t), F_{x_{n+i},x_{n+j-1}}(t) \end{matrix} \right\} \\ &\geq \delta_{A_{n-2}}(t). \end{aligned} \tag{10}$$

Case 2.2. Using (8) and (5) we get:

$$\begin{aligned} F_{x_{n+i},x_{n+j}}(\varphi(t)) &\geq F_{x_{n+i},y_{n+j}}(\varphi(t)) \\ &= F_{g(x_{n+i-1}),g(x_{n+j-1})}(\varphi(t)) \\ &\geq \min \left\{ \begin{matrix} F_{x_{n+i-1},x_{n+j-1}}(t), F_{x_{n+i-1},g(x_{n+i-1})}(t), \\ F_{x_{n+j-1},g(x_{n+j-1})}(t), F_{g(x_{n+i-1}),g(x_{n+j-1})}(\psi(t)) \end{matrix} \right\} \\ &\geq \min \left\{ \begin{matrix} F_{x_{n+i-1},x_{n+j-1}}(t), F_{x_{n+i-1},x_{n+i}}(t), \\ F_{x_{n+j-1},y_{n+j}}(t), F_{x_{n+i},y_{n+j}}(t) \end{matrix} \right\} \end{aligned} \tag{11}$$

First of all, Lemma 2.19 implies that $F_{x_{n+i},y_{n+j}}(t)$ cannot be the minimal element. Moreover, by applying Definition 2.17, (8), and (5) to the third term of (11), we have:

$$\begin{aligned}
F_{x_{n+j-1},y_{n+j}}(t) &\geq F_{g(x_{n+j-2}),g(x_{n+j-1})}(\varphi(t)) \\
&\geq \min\left\{\begin{matrix} F_{x_{n+j-2},x_{n+j-1}}(t), F_{x_{n+j-2},g(x_{n+j-2})}(t), \\ F_{x_{n+j-1},g(x_{n+j-1})}(t), F_{g(x_{n+j-2}),g(x_{n+j-1})}(\psi(t)) \end{matrix}\right\} \\
&\geq \min\left\{\begin{matrix} F_{x_{n+j-2},x_{n+j-1}}(t), F_{x_{n+j-2},x_{n+j-1}}(t), \\ F_{x_{n+j-1},y_{n+j}}(t), F_{x_{n+j-1},y_{n+j}}(t) \end{matrix}\right\} \\
&= F_{x_{n+j-2},x_{n+j-1}}(t).
\end{aligned} \tag{12}$$

Thus, for (11), based on Definition 2.7, we can conclude that:

$$\begin{aligned}
F_{x_{n+i},x_{n+j}}(\varphi(t)) &\geq \min\left\{F_{x_{n+i-1},x_{n+j-1}}(t), F_{x_{n+i-1},x_{n+i}}(t), F_{x_{n+j-2},x_{n+j-1}}(t)\right\} \\
&\geq \delta_{A_{n-2}}(t).
\end{aligned} \tag{13}$$

Case 3. Let $x_{n+i}, x_{n+j} \notin A_n$. Then, $x_{n+i} \in \operatorname{seg}[x_{n+i-1}, y_{n+i}] \cap \partial K, x_{n+j} \in \operatorname{seg}[x_{n+j-1}, y_{n+j}] \cap \partial K, x_{n+i-1} = y_{n+i-1} = g(x_{n+i-2})$, and $x_{n+j-1} = y_{n+j-1} = g(x_{n+j-2})$. Hence, from (3) we have that:

$$F_{x_{n+i},x_{n+j}}(\varphi(t)) > \min\left\{F_{x_{n+i-1},x_{n+j-1}}(\varphi(t)), F_{x_{n+i-1},y_{n+j}}(\varphi(t)), F_{y_{n+i},x_{n+j-1}}(\varphi(t)), F_{y_{n+i},y_{n+j}}(\varphi(t))\right\}. \tag{14}$$

Now we consider each possibility one by one.

Case 3.1. If $F_{x_{n+i},x_{n+j}}(\varphi(t)) > F_{x_{n+i-1},x_{n+j-1}}(\varphi(t))$, then we get once more the same results as in *Case 1*.

Case 3.2. If $F_{x_{n+i},x_{n+j}}(\varphi(t)) > F_{x_{n+i-1},y_{n+j}}(\varphi(t))$, by applying (8), (5), Lemma 2.19, and finally Definition 2.7, we get the following:

$$\begin{aligned}
F_{x_{n+i},y_{n+j}}(\varphi(t)) > F_{x_{n+i-1},y_{n+j}}(\varphi(t)) &= F_{g(x_{n+i-2}),g(x_{n+j-1})}(\varphi(t)) \\
&\geq \min\left\{\begin{matrix} F_{x_{n+i-2},x_{n+j-1}}(t), F_{x_{n+i-2},g(x_{n+i-2})}(t), \\ F_{x_{n+j-1},g(x_{n+j-1})}(t), F_{g(x_{n+i-2}),g(x_{n+j-1})}(\psi(t)) \end{matrix}\right\} \\
&\geq \min\left\{\begin{matrix} F_{x_{n+i-2},x_{n+j-1}}(t), F_{x_{n+i-2},x_{n+i-1}}(t), \\ F_{x_{n+j-1},y_{n+j}}(t), F_{x_{n+i-1},y_{n+j}}(t) \end{matrix}\right\} \\
&\geq \min\left\{F_{x_{n+i-2},x_{n+j-1}}(t), F_{x_{n+i-2},x_{n+i-1}}(t), F_{x_{n+j-1},y_{n+j}}(t)\right\} \\
&\geq \delta_{A_{n-2}}(t).
\end{aligned} \tag{15}$$

Case 3.3. If $F_{x_{n+i},x_{n+j}}(\varphi(t)) > F_{y_{n+i},x_{n+j-1}}(\varphi(t))$, then we follow the same reasoning and get the same results as in *Case 2.2*.

Case 3.4. If $F_{x_{n+i},x_{n+j}}(\varphi(t)) > F_{y_{n+i},y_{n+j}}(\varphi(t))$, then by following the same reasoning as in *Case 2.2* we get the same results.

Finally, after evaluating every possible case, we conclude that $F_{x_{n+i},x_{n+j}}(\varphi(t)) \geq \delta_{A_{n-2}}(t)$, for any arbitrary $i, j \in \mathbb{N} \cup \{0\}$. Hence, by Definition 2.7,

$$\delta_{A_n}(\varphi(t)) \geq \delta_{A_{n-2}}(t), \quad \forall t > 0. \tag{16}$$

We now show that the family $\overline{A}_n$ has probabilistic diameter zero. Let $m \in \mathbb{N}$. Then $A_m \subseteq K$ is probabilistically bounded. Therefore, by Lemma 2.10, for $\lambda \in (0,1)$ there exists $t_0 > 0$ such that $F_{x,y}(t_0) > 1-\lambda$, for all $x, y \in A_m$. Consequently, $\delta_{G_k}(t_0) \geq 1-\lambda$.

By Lemma 2.18, there exists $n_0 \in \mathbb{N}$ such that $\varphi^{n_0}(t_0) < t$. Choose an even integer $q > n_0$ and let $n = 2q + m$. Then, applying (16) repeatedly, we obtain:

$$\delta_{\overline{A}_n}(t) = \delta_{A_n}(t) \geq \delta_{A_n}(\varphi^{n_0}(t_0)) \geq \delta_{A_n}(\varphi^{q}(t_0)) \geq \delta_{A_{n-2q}}(t_0) = \delta_{A_m}(t_0) \geq 1-\lambda. \tag{17}$$

Therefore, the family $\overline{A}_n$ has probabilistic diameter zero and Theorem 2.12 implies that there exists a unique point $z \in \bigcap_{n=1}^{\infty} \overline{A}_n$.

We will show that $x_n \to z$. Indeed, let $\varepsilon > 0$ and $\lambda \in (0,1)$ be arbitrary. Since the family $\{A_n\}$ has probabilistic diameter zero, there exists $n_0 \in \mathbb{N}$ such that $F_{u,v}(\varepsilon) > 1-\lambda$, for all $u, v \in A_{n_0}$. Since $z \in \overline{A_{n_0}}$, there exists a sequence $\{x_k\} \subseteq A_{n_0}$ such that $x_k \to z$.

Hence, for $k > n_0$, we have:

$$F_{x_k,z}(\varepsilon) > 1-\lambda. \tag{18}$$

On the other hand, since also $x_k \in A_{n_0}$, it follows that:

$$F_{x_n,x_k}(\varepsilon) > 1-\lambda. \tag{19}$$

Using the Menger triangle inequality in Definition 2.3, for $n \geq n_0$, we get:

$$F_{x_n,z}(2\varepsilon) \geq \min\{F_{x_n,x_k}(\varepsilon), F_{x_k,z}(\varepsilon)\} > 1-\lambda. \tag{20}$$

Therefore, $x_n \to z$, when $n \to \infty$.

We now show that z is a fixed point of g, i.e. $g(z) = z$.

Let $\{x_{n_k}\} \subseteq \{x_n\}$ be such that $\{x_{n_k}\} \subset A$. Consequently, we have that $\lim_{n\to\infty} x_{n_k} = z$. By applying (8) and (5), we get:

$$\begin{aligned} F_{x_{n_k},g(z)}(\varphi(t)) &= F_{g(x_{n_k-1}),g(z)}(\varphi(t)) \\ &\geq \min\left\{F_{x_{n_k-1},z}(t), F_{x_{n_k-1},g(x_{n_k-1})}(t), F_{z,g(z)}(t), F_{g(x_{n_k-1}),g(z)}(\psi(t))\right\} \\ &\geq \min\left\{F_{x_{n_k-1},z}(t), F_{x_{n_k-1},x_{n_k}}(t), F_{z,g(z)}(t), F_{x_{n_k},g(z)}(t)\right\} \end{aligned} \tag{21}$$

By taking the limit of (21), when $n \to \infty$ for $t > 0$ we get:

$$F_{z,g(z)}(\varphi(t)) \geq \min\left\{F_{z,z}(t), F_{z,z}(t), F_{z,g(z)}(t), F_{z,g(z)}(t)\right\} \geq \min\left\{1, F_{z,g(z)}\right\} \tag{22}$$

Using Lemma 2.19, the last inequality implies that $g(z) = z$.

Finally, let us prove that z is a unique fixed point. Suppose to the contrary that $w \in K$ is such that $g(w) = w$. Then, from (8) and (5), it follows that:

$$\begin{aligned} F_{z,w}(\varphi(t)) &= F_{g(z),g(w)}(\varphi(t)) \geq \min\left\{F_{z,w}(t), F_{z,g(z)}(t), F_{w,g(w)}(t), F_{g(z),g(w)}(\psi(t))\right\} \\ &\geq \min\left\{F_{z,w}(t), F_{z,z}(t), F_{w,w)}(t), F_{z,w}(t)\right\} \geq \min\{F_{z,w}(t), 1\} \end{aligned} \tag{23}$$

Here, Lemma 2.19 implies that $z = w$. $\square$

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