International Journal of Mathematical Analysis
Vol. 20, 2026, no. 1, 55 - 62
HIKARI Ltd, www.m-hikari.com
https://doi.org/10.12988/ijma.2026.912598

A Note on Hermite-Hadamard Inequality for Harmonically Convex Functions

Feixiang Chen

School of Mathematics and Statistics
Chongqing Sanxia University of Science and Technology
Wanzhou, 404000, China

Abstract

In this paper, the Jensen's inequality for harmonically convex functions is established. Some refinements of Hermite-Hadamard inequality for harmonically convex functions are also obtained.

Mathematics Subject Classification: 26D15; 26D10

Keywords: Hermite-Hadamard inequality; Harmonically convex functions; Jensen's inequality

1 Introduction

If $f : I \to R$ is a convex function on the interval I, then for any $a, b \in I$ with $a \neq b$ we have the following double inequality

$$f\Big(\frac{a+b}{2}\Big) \leq \frac{1}{b-a}\int_a^b f(t)dt \leq \frac{f(a)+f(b)}{2}. \tag{1}$$

This remarkable result is well known in the literature as the Hermite-Hadamard inequality.

An account on the history of this inequality can be found in [15]. Surveys on various generalizations and developments can be found in [9] and [16]. Other improvements and extensions of the inequality can be found in the recent papers [1-14]. In [11], Farissi established a new generalization of the Hermite-Hadamard inequality as follows:

Theorem 1.1 *([11]) Assume that $f : I \to R$ is a convex function on I. Then for all $\lambda \in [0,1]$, we have*

$$f(\frac{a+b}{2}) \leq l(\lambda) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq L(\lambda) \leq \frac{f(a)+f(b)}{2}, \tag{2}$$

where

$$l(\lambda) = \lambda f\Big(\frac{\lambda b + (2-\lambda)a}{2}\Big) + (1-\lambda) f\Big(\frac{(1+\lambda)b + (1-\lambda)a}{2}\Big),$$

and

$$L(\lambda) = \frac{1}{2}\Big(f(\lambda b + (1-\lambda)a) + \lambda f(a) + (1-\lambda) f(b)\Big).$$

In [14], Iscan gave definition of harmonically convexity as follows:

Definition 1.2 *([14]) Let $I \subseteq R\backslash\{0\}$ be a real interval. A function $f : I \to R$ is said to be harmonically convex, if*

$$f\Big(\frac{xy}{tx + (1-t)y}\Big) \leq tf(y) + (1-t)f(x) \tag{3}$$

for all $x, y \in I$ and $t \in [0,1]$. If the inequality in (3) is reversed, then f is said to be harmonically concave.

The following Hermite-Hadamard inequality for harmonically convex functions holds.

Theorem 1.3 *([14]) Let $f : I \subseteq R\backslash\{0\} \to R$ be a harmonically convex function and $a, b \in I$ with $a < b$. If $f \in L(a,b)$ then we have*

$$f\Big(\frac{2ab}{a+b}\Big) \leq \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \leq \frac{f(a)+f(b)}{2}. \tag{4}$$

As in Theorem 1.1, we propose the following question: if f is a harmonically convex function on $[a,b]$, do there exist two real numbers l, L such that

$$f\Big(\frac{2ab}{a+b}\Big) \leq l \leq \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \leq L \leq \frac{f(a)+f(b)}{2}?$$

The aim of this paper is to give an affirmative answer to this question. The Jensen's inequality for harmonically convex functions is also obtained.

2 Lemmas

Lemma 2.1 *Let $[a,b] \subseteq R\backslash\{0\}$ be a real interval. Let f be an integrable function on $[a,b]$. Then we have*

$$\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx = \int_0^1 f\Big(\frac{ab}{(1-t)a+tb}\Big)dt = \int_0^1 f\Big(\frac{ab}{ta+(1-t)b}\Big)dt. \tag{5}$$

Proof. We use the change of variables $x = \frac{ab}{(1-t)a+tb}$ and $x = \frac{ab}{ta+(1-t)b}$ to prove the two identities, respectively.

Lemma 2.2 *Let $[a,b] \subseteq R\backslash\{0\}$ be a real interval. If a function $f : I \to R$ is harmonically convex on $[a,b]$, then $f \circ g$ is convex on $[\frac{1}{b}, \frac{1}{a}]$, where $g(x) = \frac{1}{x}$.*

Proof. Since f is a harmonically convex function on $[a,b]$, for all $x, y \in [a,b]$, we have

$$f\Big(\frac{xy}{tx+(1-t)y}\Big) \le tf(y) + (1-t)f(x).$$

So

$$f\left(\frac{1}{t\frac{1}{y}+(1-t)\frac{1}{x}}\right) \le tf(y) + (1-t)f(x),$$

that is

$$f \circ g\Big(t\frac{1}{y}+(1-t)\frac{1}{x}\Big) \le tf \circ g\Big(\frac{1}{y}\Big) + (1-t)f \circ g\Big(\frac{1}{x}\Big).$$

From $\frac{1}{y}, \frac{1}{x} \in [\frac{1}{b}, \frac{1}{a}]$, we have completed the proof.

Lemma 2.3 *(Jensen's Inequality) Let $[a,b] \subseteq R\backslash\{0\}$ be a real interval. If a function $f : I \to R$ is harmonically convex on $[a,b]$, then we have the following inequality*

$$f\left(\frac{1}{\sum_{i=0}^n \frac{\lambda_i}{x_i}}\right) \le \sum_{i=1}^n \lambda_i f(x_i), \tag{6}$$

where $x_i \in [a,b]$, $\lambda_i \ge 0$ and $\sum_{i=1}^n \lambda_i = 1$.

Proof. Since f is a harmonically convex function on $[a,b]$, then $f \circ g$ is convex on $[\frac{1}{b}, \frac{1}{a}]$, where $g(x) = \frac{1}{x}$. Applying Jensen's inequality for $f \circ g(x)$ on $[\frac{1}{b}, \frac{1}{a}]$, we obtain

$$f \circ g\left(\sum_{i=0}^n \frac{\lambda_i}{x_i}\right) = f\left(\frac{1}{\sum_{i=0}^n \frac{\lambda_i}{x_i}}\right) \le \sum_{i=0}^n \lambda_i f \circ g\Big(\frac{1}{x_i}\Big) = \sum_{i=1}^n \lambda_i f(x_i).$$

By $\frac{1}{x_i} \in [\frac{1}{b}, \frac{1}{a}]$, we have completed the proof.

Remark 1. Choosing $y_i = \frac{ab}{x_i} \in [a,b]$ to replace x_i in Lemma 2.3 gives

$$f\Big(\frac{ab}{\sum_{i=0}^n \lambda_i y_i}\Big) \le \sum_{i=1}^n \lambda_i f\Big(\frac{ab}{y_i}\Big). \tag{7}$$

3 Refinements of Hermite-Hadamard inequality for harmonically convex functions

These are the main results of the paper.

Theorem 3.1 *Let $f : I \subseteq R\backslash\{0\} \to R$ be a harmonically convex function and $a, b \in I$ with $a < b$. If $f \in L(a,b)$, then for $n \in N$, $\lambda_0 = 0$, $\lambda_{n+1} = 1$ and arbitrary $0 \le \lambda_1 \le \lambda_2 \le \cdots \lambda_n \le 1$, the following inequality holds:*

$$
\begin{aligned}
f\Big(\frac{2ab}{a+b}\Big) &\le l(\lambda_1, \cdots, \lambda_n) \\
&\le \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \\
&\le L(\lambda_1, \cdots, \lambda_n) \\
&\le \frac{f(a)+f(b)}{2},
\end{aligned}
$$

where

$$
l(\lambda_1, \cdots, \lambda_n) = \sum_{k=0}^{n}(\lambda_{k+1} - \lambda_k) f\Big(\frac{2ab}{(2-\lambda_k-\lambda_{k+1})b + (\lambda_k+\lambda_{k+1})a}\Big)
$$

and

$$
L(\lambda_1, \cdots, \lambda_n) = \sum_{k=0}^{n}(\lambda_{k+1} - \lambda_k)\frac{f\Big(\frac{ab}{(1-\lambda_k)b+\lambda_k a}\Big) + f\Big(\frac{ab}{(1-\lambda_{k+1})b+\lambda_{k+1}a}\Big)}{2}.
$$

Proof. Since f is a harmonically convex function on [a,b], we get

$$
f\Big(\frac{2ab}{a+b}\Big) \le \int_0^1 f\Big(\frac{ab}{(1-t)a+tb}\Big)dt \le \frac{f(a)+f(b)}{2}. \tag{8}
$$

Then we use $C = \frac{ab}{(1-\lambda_k)b+\lambda_k a}$ and $D = \frac{ab}{(1-\lambda_{k+1})b+\lambda_{k+1}a}$ to replace a and b, respectively, applying (8), for $k = 0, 1, \cdots, n$, we get

$$
\begin{aligned}
f\left(\frac{2CD}{C+D}\right) &= f\Big(\frac{2ab}{(2-\lambda_k-\lambda_{k+1})b + (\lambda_k+\lambda_{k+1})a}\Big) \\
&\le \int_0^1 f\Big(\frac{CD}{(1-t)C+tD}\Big)dt \\
&= \frac{ab}{(\lambda_{k+1}-\lambda_k)(b-a)}\int_{\frac{ab}{(1-\lambda_k)b+\lambda_k a}}^{\frac{ab}{(1-\lambda_{k+1})b+\lambda_{k+1}a}} \frac{f(x)}{x^2}dx \\
&\le \frac{f(C)+f(D)}{2} \\
&= \frac{f\Big(\frac{ab}{(1-\lambda_k)b+\lambda_k a}\Big) + f\Big(\frac{ab}{(1-\lambda_{k+1})b+\lambda_{k+1}a}\Big)}{2}.
\end{aligned}
$$

Multiplying each term above by corresponding $(\lambda_{k+1}-\lambda_k)$, and adding the resulting inequalities, we get

$$\begin{aligned}
&\sum_{k=0}^{n}(\lambda_{k+1}-\lambda_k)f\Big(\frac{2ab}{(2-\lambda_k-\lambda_{k+1})b+(\lambda_k+\lambda_{k+1})a}\Big)\\
\le\ &\sum_{k=0}^{n}\frac{ab}{b-a}\int_{\frac{ab}{(1-\lambda_k)b+\lambda_k a}}^{\frac{ab}{(1-\lambda_{k+1})b+\lambda_{k+1}a}}\frac{f(x)}{x^2}dx\\
\le\ &\sum_{k=0}^{n}(\lambda_{k+1}-\lambda_k)\frac{f\Big(\frac{ab}{(1-\lambda_k)b+\lambda_k a}\Big)+f\Big(\frac{ab}{(1-\lambda_{k+1})b+\lambda_{k+1}a}\Big)}{2},
\end{aligned}$$

that is

$$l(\lambda_1\cdots,\lambda_n)\le\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}dx\le L(\lambda_1,\cdots,\lambda_n),$$

where $l(\lambda_1\cdots,\lambda_n)$ and $L(\lambda_1,\cdots,\lambda_n)$ are defined as in Theorem 3.1.

To prove the remaining two inequalities:

$$f\Big(\frac{2ab}{a+b}\Big)\le l(\lambda_1,\cdots,\lambda_n)\le L(\lambda_1,\cdots,\lambda_n)\le\frac{f(a)+f(b)}{2},$$

we use the fact that f is harmonically convex and observe that $\sum\limits_{k=0}^{n}(\lambda_{k+1}-\lambda_k)=1$, $\sum\limits_{k=0}^{n}(\lambda_{k+1}^2-\lambda_k^2)=1$, $\sum\limits_{k=0}^{n}\Big[(1-\lambda_k)^2-(1-\lambda_{k+1})^2\Big]=1$, respectively, then we obtain

$$\begin{aligned}
f\Big(\frac{2ab}{a+b}\Big)\ &=\ f\left(\frac{1}{\sum_{k=0}^{n}\frac{(\lambda_{k+1}-\lambda_k)(2-\lambda_k-\lambda_{k+1})}{2}\frac{1}{a}+\sum_{k=0}^{n}\frac{(\lambda_{k+1}-\lambda_k)(\lambda_k+\lambda_{k+1})}{2}\frac{1}{b}}\right)\\
&=\ f\left(\frac{1}{\sum_{k=0}^{n}(\lambda_{k+1}-\lambda_k)\Big(\frac{(2-\lambda_k-\lambda_{k+1})}{2}\frac{1}{a}+\frac{(\lambda_k+\lambda_{k+1})}{2}\frac{1}{b}\Big)}\right)\\
&\le\ \sum_{k=0}^{n}(\lambda_{k+1}-\lambda_k)f\left(\frac{1}{\frac{(2-\lambda_k-\lambda_{k+1})}{2}\frac{1}{a}+\frac{(\lambda_k+\lambda_{k+1})}{2}\frac{1}{b}}\right)\\
&=\ \sum_{k=0}^{n}(\lambda_{k+1}-\lambda_k)f\left(\frac{ab}{\frac{(2-\lambda_k-\lambda_{k+1})}{2}b+\frac{(\lambda_k+\lambda_{k+1})}{2}a}\right)\\
&\le\ \frac{1}{2}\sum_{k=0}^{n}[(1-\lambda_k)-(1-\lambda_{k+1})][(1-\lambda_k)+(1-\lambda_{k+1})]f(a)\\
&\quad+\frac{1}{2}\sum_{k=0}^{n}(\lambda_{k+1}-\lambda_k)(\lambda_{k+1}+\lambda_k)f(b)\\
&=\ \frac{1}{2}\sum_{k=0}^{n}\Big((1-\lambda_k)^2-(1-\lambda_{k+1})^2\Big)f(a)+\frac{1}{2}\sum_{k=0}^{n}(\lambda_{k+1}^2-\lambda_k^2)f(b)\\
&=\ \frac{f(a)+f(b)}{2},
\end{aligned}$$

where the first inequality follows from the Jensen's inequality for f. The proof is completed.

Remark 2. With notations in Theorem 3.1, then

$$\begin{aligned} f\Big(\frac{2ab}{a+b}\Big) &\le \sup_{0\le\lambda_1\le\cdots\le\lambda_n\le 1} l(\lambda_1,\cdots,\lambda_n) \\ &\le \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \\ &\le \inf_{0\le\lambda_1\le\cdots\le\lambda_n\le 1} L(\lambda_1,\cdots,\lambda_n) \\ &\le \frac{f(a)+f(b)}{2}. \end{aligned}$$

Remark 3. Choosing $n = 1$ in Theorem 3.1 gives

$$\begin{aligned} f\Big(\frac{2ab}{a+b}\Big) &\le l(\lambda) \\ &\le \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \\ &\le L(\lambda) \\ &\le \frac{f(a)+f(b)}{2}, \end{aligned}$$

where

$$l(\lambda) = \lambda f\Big(\frac{2ab}{(2-\lambda)a+\lambda b}\Big) + (1-\lambda) f\Big(\frac{2ab}{(1-\lambda)a+(1+\lambda)b}\Big)$$

and

$$L(\lambda) = \frac{f\Big(\frac{ab}{(1-\lambda)b+\lambda a}\Big) + \lambda f(a) + (1-\lambda)f(b)}{2}.$$

Remark 4. Putting $n = 1$ and $\lambda = \frac{1}{2}$ in Theorem 3.1, then

$$\begin{aligned} f\Big(\frac{2ab}{a+b}\Big) &\le \frac{f\Big(\frac{4ab}{3a+b}\Big) + f\Big(\frac{4ab}{a+3b}\Big)}{2} \\ &\le \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \\ &\le \frac{1}{2}\Big(f\Big(\frac{2ab}{a+b}\Big) + \frac{f(a)+f(b)}{2}\Big) \\ &\le \frac{f(a)+f(b)}{2}. \end{aligned}$$

Conflict of Interests. The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgements. We are grateful to the referee for the suggested improvements. We would like to acknowledge the support from the National Natural Science Foundation of China (11671249) and the Natural Science Foundation of Chongqing Municipal Education Commission (No.KJ1501009).

References

[1] S. Abramovich, G. Farid, and J. Pečarić, More about Hermite-Hadamard inequalities, Cauchy's Means, and superquadracity, *J. Inequal. Appl.,* **2010**, Article ID 102467, 14 pages. https://doi.org/10.1155/2010/102467

[2] M. W. Alomari, M. Darus and U. S. Kirmaci, Some inequalities of Hermite-Hadamard type for s-convex functions, *Acta Math. Sci.,* **4** (31) (2011), 1643-1652. https://doi.org/10.1016/s0252-9602(11)60350-0

[3] M. Avci, H. Kavurmaci, M. E. Özdemir, New inequalities of Hermite-Hadamard type via s-convex functions in the second sense with applications, *Appl. Math. Comput.,* **217** (12) (2011), 5171-5176. https://doi.org/10.1016/j.amc.2010.11.047

[4] M. Bessenyei and Z. Páles, Hadamard-type inequalities for generalized convex functions, *Math. Inequal. Appl.,* **6** (3) (2003), 379-392. https://doi.org/10.7153/mia-06-35

[5] F. X. Chen and X. F. Liu, Refinements on the Hermite-Hadamard inequalities for r-convex functions, *J. Appl. Math.,* **2013**, Article ID 978493, 5 pages. https://doi.org/10.1155/2013/978493

[6] F. X. Chen, A note on Hermite-Hadamard inequalities for products of convex functions, *J. Appl. Math.,* **2013**, Article ID 935020, 5 pages. https://doi.org/10.1155/2013/935020

[7] F. X. Chen, On Hermite-Hadamard type inequalities for s-convex functions on the coordinates via Riemann-Liouville fractional integrals, *J. Appl. Math.,* **2014**, Article ID 248710, 8 pages. https://doi.org/10.1155/2014/248710

[8] Y. M. Chu, G. D. Wang and X. H. Zhang, Schur convexity and Hadamard's inequality. *Math. Inequal. Appl.,* **13** (4) (2010), 725-731. https://doi.org/10.7153/mia-13-51

[9] S. S. Dragomir, C. E. M. Pearce, *Selected Topics on Hermite-Hadamard Inequalities and Applications*, RGMIA Monographs, Victoria University, 2000.

[10] S. S. Dragomir, On the Hadamard's inequality for convex functions on the co-ordinates in a rectangle from the plane, *Taiwanese J. Math.,* **5** (4) (2001), 775-788. https://doi.org/10.11650/twjm/1500574995

[11] A. E. Farissi, Simple proof and refinement of Hermite-Hadamard inequality. *J. Math. Inequal.,* **4** (3) (2010), 365-369. https://doi.org/10.7153/jmi-04-33

[12] X. Gao, A note on the Hermite-Hadamard inequality, *J. Math. Inequal.,* **4** (4), (2010), 587-591. https://doi.org/10.7153/jmi-04-52

[13] İ. İşcan and S. H. Wu, Hermite-Hadamard type inequalities for harmonically convex functions via fractional integrals, *Appl. Math. Comput.,* **238** (2014), 237-244. https://doi.org/10.1016/j.amc.2014.04.020

[14] İ. İşcan, Hermite-Hadamard type inequalities for harmonically convex functions, *Hacet. J. Math. Stat.,* **43** (6) (2014), 935-942.

[15] D. S. Mitrinović and I. B. Lacković, Hermite and convexity, *Aequationes Math.,* **28** (1) (1985), 229-232. https://doi.org/10.1007/bf02189414

[16] C. P. Niculescu and L. E. Persson, Old and new on the Hermite-Hadamard inequality, *Real Anal. Exchange.,* **29** (2) (2003), 663-685. https://doi.org/10.14321/realanalexch.29.2.0663

Received: December 26, 2025; Published: September 15, 2026