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Moment and Sample-Path Behavior of Ito Population Models under Time-Varying Random Catastrophes

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Abstract

This paper develops a time-inhomogeneous Ito diffusion population model subject to multiplicative random catastrophes. Closed-form transient moments of arbitrary integer order are derived. For a homogeneous benchmark, the expected-population exponent depends on E[Y], whereas the almost-sure sample-path exponent depends on E[log Y] and the diffusion correction. This creates a parameter region in which the expected population grows while typical sample paths decline exponentially. A nonnegative mean-path gap is obtained from log y <= y-1. Random catastrophe severity and seasonal environments are also considered. Monte Carlo experiments illustrate that a rising ensemble mean can coexist with a sharply declining median population.

Mathematics Subject Classification: 60H10, 60G55, 92D25

Keywords: Ito diffusion; population process; random catastrophes; jump diffusion; moments; almost-sure growth; seasonal intensity

1. Introduction

Population systems may experience continuous environmental fluctuations together with sudden losses caused by disasters, epidemics, food shortages, severe weather, or other external shocks.

The earlier study from which the present work is developed considered an Ito diffusion population model with a general catastrophe process and derived the mean and variance, with a further specialization to uniformly distributed catastrophe sizes.

The broader literature has developed several complementary approaches to population dynamics with abrupt losses. [5] studied stochastic Lotka-Volterra population systems with jumps, including moment boundedness, sample Lyapunov exponents, and extinction. [11] analyzed logistic growth with multiplicative Poisson catastrophes and obtained exact stationary moments, a diffusion limit, and extinction-time comparisons. More recently, [9] investigated population growth with random proportional collapses governed by state-dependent catastrophe rates, focusing on recurrence, transience, and extinction.

Recent work also emphasizes non-autonomous stochastic environments. [13] developed moment and almost-sure exponential-stability criteria for non-autonomous stochastic differential equations with Markovian switching. [8] studied time-inhomogeneous diffusion models for population growth with time-varying growth intensity, transition densities, first-passage quantities, and periodic environments. In a closely related applied direction, [4] used an Ito diffusion model with Gamma catastrophe magnitudes to derive the mean, variance, and sample path for resilience analysis. These studies motivate a careful positioning of the present paper: the contribution sought here is the combined treatment of transient all-order moments, sample-path growth, mean-path mismatch, and time-varying multiplicative catastrophe environments rather than the introduction of jumps or time dependence alone.

The present extension formulates catastrophes as multiplicative marked jumps in continuous time, allows a time-varying environment, derives arbitrary-order transient moments, and distinguishes moment behavior from almost-sure sample-path behavior.

A central issue is that expected population growth need not describe a typical trajectory. Rare large paths may lift the ensemble mean even when most paths decline.

2. Model and Assumptions

$$dX_t = b(t)X_{t-}\,dt + \sigma(t)X_{t-}\,dB_t + X_{t-}\int_0^1 (y-1)\,N(dt,dy)$$
$$E[N(dt,dy)] = \lambda(t)F_t(dy)\,dt$$

Here B_t is a standard Brownian motion and $N(dt,dy)$ is a Poisson random measure, independent of B_t, with deterministic compensator $\lambda(t)F_t(dy)\,dt$. The mark $Y_t \in (0,1]$ is the remaining population fraction after a catastrophe; thus, at a catastrophe time, $X_t = Y_t X_{t^-}$.

A1. $X_0 = x_0 > 0$.

A2. $b(t)$ and $\sigma(t)$ are deterministic measurable functions such that $\int_0^T |b(s)|\,ds < \infty$ and $\int_0^T \sigma^2(s)\,ds < \infty$ for every finite T.

A3. $\lambda(t) \geq 0$ is deterministic and measurable, with $\int_0^T \lambda(s)\, ds < \infty$ for every finite T; F_t is a measurable probability kernel on (0,1].
A4. For integer $n \geq 1$, $\mu_n(t) = \int_{(0,1]} y^n\, F_t(dy)$ exists. Because $0 < Y_t \leq 1$, these positive moments are automatically finite.
A5. For the homogeneous sample-path result, catastrophe marks are iid, independent of B_t and of the Poisson arrival process, with $E|\log Y| < \infty$. These conditions are sufficient for the strong laws used below.

3. General Transient Moment Structure

Theorem 1. General integer-order moments

$$dM_n(t)/dt = [n\, b(t) + \tfrac{1}{2}n(n-1)\sigma^2(t) + \lambda(t)(\mu_n(t) - 1)]\, M_n(t)$$
$$M_n(t) = x_0^n \exp\{\textstyle\int_0^t [n\, b(s) + \tfrac{1}{2}n(n-1)\sigma^2(s) + \lambda(s)(\mu_n(s) - 1)]\, ds\}$$

Proof.
Apply Ito's formula for semimartingales with jumps to $f(x) = x^n$. The continuous part contributes $nb(t)X_{t^-}^n\, dt + \frac{n(n-1)}{2}\sigma^2(t)X_{t^-}^n\, dt$. At a jump with mark y, the complete jump increment in X^n is $X_{t^-}^n(y^n - 1)$. Using the compensator $\lambda(t)F_t(dy)\, dt$ and taking expectations gives $M_n{}'(t) = \left[nb(t) + \frac{n(n-1)}{2}\sigma^2(t) + \lambda(t)(\mu_n(t) - 1)\right] M_n(t)$. Solving this scalar linear ODE yields the stated formula.

Corollaries

$$E[X_t] = x_0 \exp\{\textstyle\int_0^t [b(s) - \lambda(s)(1 - \mu_1(s))]\, ds\}$$
$$E[X_t^2] = x_0^2 \exp\{\textstyle\int_0^t [2b(s) + \sigma^2(s) - \lambda(s)(1 - \mu_2(s))]\, ds\}$$
$$\mathrm{Var}(X_t) = E[X_t^2] - (E[X_t])^2$$

4. Moment Growth and Decay

$$\Gamma_n(t) = (1/t) \textstyle\int_0^t [n\, b(s) + \tfrac{1}{2}n(n-1)\sigma^2(s) - \lambda(s)(1 - \mu_n(s))]\, ds$$
Since $M_n(t) = x_0^n \exp\{t\Gamma_n(t)\}$, $\limsup \Gamma_n(t) < 0$ implies moment decay, while $\liminf \Gamma_n(t) > 0$ implies exponential moment growth.

5. Almost-Sure Behavior and Mean-Path Mismatch

Theorem 2. Homogeneous sample-path exponent

$$\Lambda\_AS = b - \tfrac{1}{2}\sigma^2 + \lambda\, E[\log Y]$$
Under A1-A5 with constant b, σ and λ, the Brownian strong law gives $B_t/t \to 0$ almost surely, the Poisson strong law gives $N_t/t \to \lambda$, and the iid mark strong law gives the average log-mark $\to E[\log Y]$. Consequently $t^{-1}\log(X_t/x_0)$ converges almost surely to Λ_{AS}. The same limit holds for $t^{-1}\log X_t$ because $\log x_0/t \to 0$.

Theorem 3. Mean-path gap

$$\Lambda\_M = b + \lambda(E[Y] - 1)$$

$$D = \Lambda\_M - \Lambda\_AS = ½\sigma^2 + \lambda\{E[Y] - 1 - E[\log Y]\} \geq 0$$

Hence Λ_AS ≤ Λ_M. The mismatch region Λ_M > 0 > Λ_AS describes positive expected growth together with almost-sure exponential decline.

$$\text{For deterministic } Y = y\text{:} \quad \lambda(1 - y) < b < ½\sigma^2 - \lambda \log y$$

6. Random Severity and Seasonal Extensions

For fixed E[Y], Jensen's inequality E[log Y]<=log E[Y] shows that non-degenerate severity variability reduces the almost-sure exponent without changing the mean exponent.

$$\lambda(t) = \lambda_0[1 + \rho \sin(2\pi t/T)], \quad 0 \leq \rho < 1$$

For constant b, σ and F, a zero-mean sinusoidal modulation changes finite-horizon moments but preserves their long-run exponential rates because $t^{-1} \int_0^t \lambda(s)\, ds \to \lambda_0$. Thus the long-run mean exponent is $b + \lambda_0(E[Y] - 1)$, and the corresponding sample-path exponent is $b - \sigma^2/2 + \lambda_0 E[\log Y]$, under the analogous strong-law conditions. If both catastrophe frequency and severity vary periodically, their phase relationship can affect the period-average catastrophe contribution.

7. Numerical Study

Reproducible Monte Carlo experiment: seed=20260820, paths=500,000, $x_0 = 1000$, $b = 0.04$, $\sigma = 0.20$, $\lambda = 0.10$, $y = 0.70$.

$$\Lambda\_M = 0.010000; \quad \Lambda\_AS = -0.015667; \quad D = 0.025667$$

t	Theo. mean	Sim. mean	Median	P10	P90	Below x0 (%)
25	1284.03	1279.37	685.77	153.55	2897.09	62.99
50	1648.72	1655.18	464.13	56.62	3636.65	68.22
100	2718.28	2703.59	212.78	10.89	3925.38	75.12

The theoretical and simulated means agree closely, while the median declines strongly. At t=100, most simulated paths are below the initial population even though the ensemble mean is substantially above it.

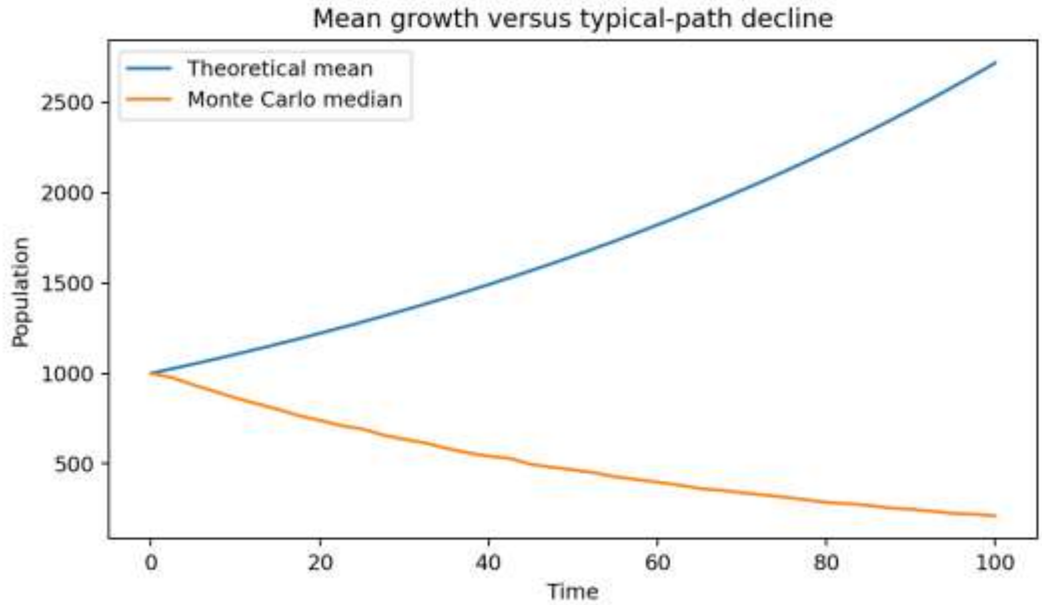

Figure 1. Theoretical mean versus Monte Carlo median.

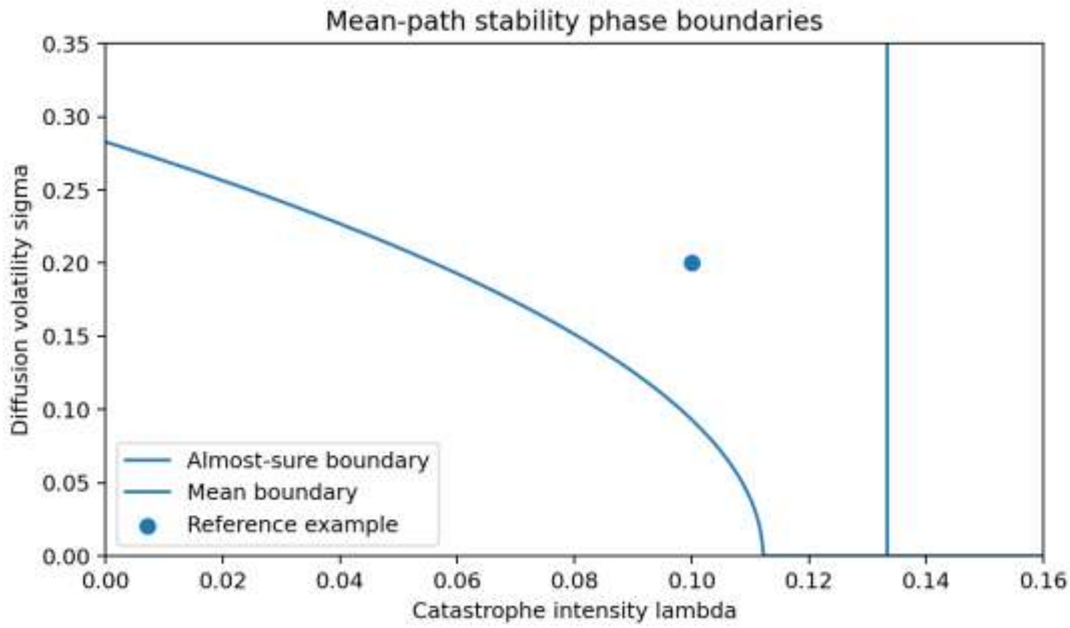

Figure 2. Mean and almost-sure stability boundaries.

8. Discussion and Conclusions

The analysis shows that the ensemble mean can be an incomplete measure of persistence in catastrophe-driven populations. Diffusion volatility and variability in catastrophe severity widen the gap between expected behavior and typical sample-path behavior. The seasonal extension separates finite-horizon seasonal effects from long-run average effects.

The present results should also be interpreted against established jump-population and stability theory. In particular, sample Lyapunov exponents and extinction criteria for population SDEs with jumps are well established in related models [5], while [11] demonstrates that multiplicative Poisson catastrophes can generate extinction behavior distinct from Gaussian environmental noise. The time-inhomogeneous formulation is likewise related to recent non-autonomous stability and population-diffusion studies [13, 8]. Accordingly, the contribution of the present paper lies in the combined treatment, within the specified multiplicative jump-diffusion framework, of transient all-order moments, almost-sure sample-path growth, the mean-path mismatch, and time-varying catastrophe environments.

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