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Iterative Algorithm for Zeros of Maximal Monotone Mappings in Uniformly Smooth and Uniformly Convex Banach Spaces

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Abstract

Let E be a uniformly smooth and uniformly convex real Banach space and E^* be its dual space. Let $A : E \to E^*$ be a bounded maximal monotone mappping such that $A^{-1}(0) \neq 0$. Define the algorithm $\{x_n\}$ as follows: for given $x_1 \in E$, $x_{n+1} = J^{-1}\big(Jx_n - \lambda_n Ax_n - \lambda_n\theta_n(Jx_n - Jx_1)\big)$ where J is the normalized duality mapping from E into E^* and $\{\lambda_n\}$ and θ_n are positive real numbers in $(0, 1)$ satisfying suitable conditions. It is proved that x_n converges strongly to some $x^* \in A^{-1}(0)$. The results extend our recent works [18] to larger class of Banach spaces with numerical simulations.

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1 Introduction

Let E be a real normed vector space and E^* be its dual. Let $A : E \to E^*$ be a map with a nonempty domain $D(A)$ and Gr(A) be as its graph. One says that A is monotone if and only if for any $x, y \in D(A)$ the following holds:

$$\langle Ax - Ay, x - y\rangle \geq 0,$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between E and E^*. More generaly, considering a multi-valued map $A : E \to 2^{E^*}$, it is monotone if and only if

$$\langle y_1 - y_2, x_1 - x_2\rangle \geq 0,\ \forall\ (x_i, y_i) \in \mathrm{Gr(A)},\ i = 1, 2.$$

The map A is said to be maximal monotone if its graph is not properly contained in the graph of any other monotone mapping.

Nonlinear monotone and maximal monotone operators in Hilbert spaces was introduced in the 60's by Minty [15] and has been of great interest in the theory of nonlinear partial differential equations. In order to prove existence results for nonlinear operator problems in Banach spaces, Browder [5, 6, 7, 8, 9] and Minty [16] extended the study of nonlinear monotone and maximal monotone operator in Banach spaces. Typical examples of monotone mappings and maximal monotone mappings are gradients and subgradients of convex functions. Moreover it is known that for a lower semicontinuous convex function $f : E \to \mathbb{R}\cup\{+\infty\}$, its subdifferential ∂f is maximal monotone and for $x_0 \in E$, one has x_0 is a minimizer of f if and only if $0 \in \partial f(x_0)$, where $\partial f : E \to 2^{E^*}$ defined by

$$\partial f(x) := \{x^* \in E^* : \langle x^*, x - y\rangle + f(x) \leq f(y),\ \forall\ y \in E\}.$$

The wide class of monotone and maximal monotone mappings and their connection in optimization theory, optimal control, variational analysis etc..has motivated intensive research (see, e.g., Bruck Jr [12], Martinet [10], Reich [26, 27], Rockafellar [13, 14], Pascali and Sburlan [17], Chidume [19]) and references therein.

Several convergence results to approximate solutions of nonlinear operators equations has been established. Among the iterative method to approximate the zeros of maximal monotone mapping is the popular *Proximal Point Algorithm* introduced by Martinet [10] in Hilbert space H. It is defined as follows: given $x_0 \in H$,

$$x_{n+1} = J_{r_n}x_n,\ \ n \geq 1$$

where $J_r := (I + rA)^{-1}$ is the resolvent of A.

Assuming that $\liminf r_n > 0$ and $A^{-1}(0) \neq \emptyset$, Rockaffelar [14] proved that the proximal point algorithm converges weakly to some $x^* \in A^{-1}(0)$. Then, the strong convergence of this algorithm has been of interest for some authors. For instance, Kamimura and Takahashi [22] proved the strong convergence of the following modified proximal point algorithm: $u \in H$, $x_0 \in H$

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)J_{r_n}x_n,\ \ n \geq 1$$

with $\{\alpha_n\} \subset (0, 1)$ and $\{r_n\} \subset (0, +\infty)$ satisfying appropriate conditions. This results was extended in Banach spaces by Koshada and Takahashi [21] as follows: $u \in H$, $x_0 \in H$:

$$x_{n+1} = J^{-1}(\alpha_n Ju + (1 - \alpha_n)JJ_{r_n}x_n),\ \ n \geq 1.$$

More strong convergence results for the solution of $Au = 0$ were obtained in Banach spaces (see, e.g. Kamimura and Takahashi [21], Zegueye [4], Moudafi [24], Chidume and Djitte [20] etc..). Recently, Mendy, Sene and Djitte [18] introduced a new iterative scheme to approximate the solution of $Au = 0$ for a bounded maximal monotone operator as follows: $x_1 \in E$

$$x_{n+1} = J^{-1}\Big(Jx_n - \lambda_n A x_n - \lambda_n \theta_n (Jx_n - Jx_1)\Big), \quad n \geq 1. \tag{1.1}$$

where $\{\lambda_n\}$ and $\{\theta_n\}$ are sequences in $(0,1)$ satisfying the following conditions:

(i) $\lim_{n\to\infty} \theta_n = 0$; (ii) $\sum_{n=1}^{\infty} \lambda_n \theta_n = \infty$ and $\lambda_n = o(\theta_n)$;

(iii) $\limsup_{n\to\infty} \dfrac{\left(\frac{\theta_{n-1}}{\theta_n} - 1\right)}{\lambda_n \theta_n} \leq 0$ and $\sum_{n=1}^{\infty} \lambda_n^2 < \infty$.

In fact, they prove the following theorem.

Theorem 1.1 (Mendy, Sene and Djitte [18]) *For $q > 1$, let E be a 2-uniformly convex and q-uniformly smooth real Banach space and $A : E \to E^*$ be a bounded maximal monotone mapping such that $A^{-1}(0) \neq \emptyset$. Then, there exists $\gamma_0 > 0$ such that if $\lambda_n < \gamma_0 \theta_n$ for all $n \geq 1$, the sequence $\{x_n\}$ given by* (1.1) *converges strongly to some $x^* \in A^{-1}(0)$.*

It is our purpose to prove the strong convergence of the sequence given by (1.1) in the larger class of uniformly convex and uniformly smooth Banach spaces. Our technique of proofs consist to define appropriate new conditions on the sequences λ_n and θ_n and by using some results (see e.g. [1] p. 45 and p. 46) in uniformly convex and uniformly smooth Banach spaces. Finally we give an application on convex minimization problem and numerical results.

2 Preliminaries

Let E be a real normed space and let $S := \{x \in E : \|x\| = 1\}$. One says that E is *smooth* if the limit

$$\lim_{t\to 0^+} \frac{\|x + ty\| - \|x\|}{t} \tag{2.1}$$

exists for each $x, y \in S$. If E is smooth and the limit in (2.1) is attained uniformly for $y \in S_E$ then E is said to be Fréchet differentiable. Finally E is uniformly smooth if it is smooth and the limit in (2.1) is attained uniformly for each $x, y \in S_E$. If E is a normed linear space of dimension ≥ 2, then, the *modulus of smoothness* of E , ρ_E, is defined by

$$\rho_E(\tau) := \sup\left\{\frac{\|x + y\| + \|x - y\|}{2} - 1 : \|x\| = 1, \|y\| = \tau\right\}; \quad \tau > 0.$$

The normed linear space E is called *uniformly smooth* if

$$\lim_{\tau\to 0}\frac{\rho_E(\tau)}{\tau}=0.$$

If there exists a constant $c>0$ and a real number $q>1$ such that $\rho_E(\tau)\le c\tau^q$, then E is said to be *q-uniformly smooth*.
The modulus of convexity of the normed linear space E is defined by the function $\delta_E:(0,2]\to[0,1]$ such that for any $\varepsilon\in(0,2]$,

$$\delta_E(\varepsilon):=\inf\{1-\frac{1}{2}\|x+y\|:\|x\|=\|y\|=1,\ \|x-y\|\ge\varepsilon\}.$$

Then, E is said to be uniformly convex if and only if

$$\lim_{\varepsilon\to 0}\frac{\delta_E(\varepsilon)}{\varepsilon}=0.$$

For any real $p>1$, E is said to be p-uniformly convex if there exists some constant $c>0$ such that $\delta_E(\varepsilon)\ge c\varepsilon^p,\ \forall\,\varepsilon\in(0,2]$. Note that q-uniformly smooth real Banach spaces are uniformly smooth and p-uniformly convex real Banach spaces are uniformly convex.
Typical examples of such spaces are the L_p, ℓ_p and W_p^m- spaces for $1<p<\infty$ where,

$$L_p\ (or\ \ell_p)\ or\ W_p^m \text{ is } \begin{cases} 2-\text{uniformly smooth and } p-\text{uniformly convex} & \text{if } \ 2\le p<\infty;\\ \\ 2-\text{uniformly convex and } p-\text{uniformly smooth} & \text{if } \ 1<p<2.\end{cases}$$

Let J_q denote the *generalized duality mapping* from E to 2^{E^*} defined by

$$J_q(x):=\left\{f\in E^*:\langle x,f\rangle=\|x\|^q \text{ and } \|f\|=\|x\|^{q-1}\right\}.$$

J_2 is called the *normalized duality mapping* and is denoted simply by J.

It is well known that E is smooth if and only if J is single valued. Moreover, if E is a reflexive, smooth and strictly convex real Banach space, then J^{-1} is single valued, one-to-one, surjective and it is the duality mapping from E^* into E.

Remark 1 *It is known that the duality mapping is well defined in any Banach space. Moreover this mapping is known explicitly in $\ell_p, L_P, W^{m,p}$-spaces, $1<p<\infty$ (see e.g. [21]):*

(i) $\ell_p:\ Jx=\|x\|_{l_p}^{2-p}\tilde{x}\in\ell_q,\ \ x=(x_1,x_2,\cdots,x_n,\cdots),\ \ \tilde{x}=(x_1|x_1|^{p-2},\cdots,x_n|x_n|^{p-2},\cdots)$

(ii) $L_p:\ Jx=\|x\|_{L_p}^{2-p}|x|^{p-2}x\in L_q$ *and for all* $x\neq 0, J_p(x)=\|x\|_{L_p}^{p-2}Jx$

(iii) $W^{m,p}: \ Jx = \|x\|^{2-p}_{W^{m,p}} \sum_{|\alpha|\le m}(-1)^{|\alpha|}D^{\alpha}\Big(|D^{\alpha}x|^{p-2}D^{\alpha}x\Big) \in W^{-m,q}$,

where $1 < q < \infty$ *is such that* $1/p + 1/q = 1$.

In the sequel, we shall need the following definitions and results.

Theorem 2.1 *(Takahashi [23]) Let* E *be a uniformly convex real Banach space with Fréchet differentiable norm. Let* $A : E^* \to 2^E$ *be a multivalued maximal monotone mapping with* $A^{-1}(0) \neq \emptyset$. *Then, for* $u \in E$,

$$\lim_{\lambda_n\to\infty}(I+\lambda_n AJ)^{-1}u \quad \text{exists and belongs to} \quad (AJ)^{-1}(0), \tag{2.2}$$

where J *is the normalized duality mapping from* E *into* E^*. *Moreover, if* $Ru := y^* = \lim_{\lambda_n\to\infty}(I+\lambda_n AJ)^{-1}u$, *then* R *is a sunny generalized nonexpansive retraction of* E *into* $(AJ)^{-1}(0)$.

Lemma 2.2 *(Alber and Ryazantseva [1], p.45) Let* E *be a uniformly convex Banach space. Then, for any* $R > 0$ *and any* $x, y \in E$ *such that* $\|x\| \le R$, $\|y\| \le R$, *the following inequality holds:*

$$\langle Jx - Jy, x - y\rangle \ge (2L)^{-1}\delta_E\big(c_2^{-1}\|x-y\|\big),$$

where $c_2 = 2\,max\{1, R\}$, $1 < L < 1.7$.

Lemma 2.3 *(Alber and Ryazantseva [1], p.46) Let* E *be a uniformly smooth and strictly convex Banach space. Then, for any* $R > 0$ *and any* $x, y \in E$ *such that* $\|x\| \le R$, $\|y\| \le R$, *the following inequality holds:*

$$\langle Jx - Jy, x - y\rangle \ge (2L)^{-1}\delta_{E^*}\big(c_2^{-1}\|Jx-Jy\|\big),$$

where $c_2 = 2\,max\{1, R\}$, $1 < L < 1.7$.

Let E be a smooth real Banach space with dual E^*. Let $\phi : E \times E \to \mathbb{R}$ be defined as follows

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \quad \forall\ x, y \in E. \tag{2.3}$$

The function ϕ was introduced by Alber [3] and has been studied by Guerre-Delabriere [2], Kamimura and Takahashi [21], Reich [27] and a host of other authors. If $E = H$, a real Hilbert space, then the relation (2.3) reduces to $\phi(x, y) = \|x - y\|^2$ for all $x, y \in H$. It follows from the definition of ϕ that

$$(\|x\| - \|y\|)^2 \le \phi(x,y) \le (\|x\| + \|y\|)^2. \tag{2.4}$$

Define the functional $V : E \times E^* \to \mathbb{R}$ by

$$V(x,x^*) = \|x\|^2 - 2\langle x, x^* \rangle + \|x^*\|^2.$$

Then it is easy to see that

$$V(x,x^*) = \phi(x, J^{-1}x^*), \quad \forall\, x \in E,\ x^* \in E^*.$$

Lemma 2.4 *(Alber and Ryazantseva [1]) Let E be a reflexive strictly convex and smooth Banach space with E^* as its dual. Then*

$$V(x,x^*) + 2\langle J^{-1}x^* - x, y^* \rangle \le V(x, x^* + y^*) \tag{2.5}$$

for all $x \in E$ and $x^*, y^* \in E^*$.

Lemma 2.5 *(Kamimura and Takahashi [21]) Let E be a smooth real Banach space. Then for all $x,y,z \in E$ one has the following*

$$\phi(x,z) = \phi(x,y) + \phi(y,z) + 2\langle x - y, Jy - Jz \rangle.$$

From inequality (2.4) and the above lemma, we observe that for all $x, y \in E$,

$$\phi(y,x) \ge 0 \quad \text{and} \quad 2\langle x - y, Jx - Jy \rangle - \phi(x,y) = \phi(y,x).$$

The following lemma is an immediate consequence of the above relation.

Lemma 2.6 *Soit E be a smooth real Banach space. Then, for all $x, y \in E$ the following holds*

$$\phi(x,y) \le 2\langle Jy - Jx, y - x \rangle.$$

We complete this paragrah with the next important lemmas.

Lemma 2.7 *(Kamimura and Takahashi [21]) Let E be a real smooth and uniformly convex Banach space, and let $\{x_n\}$ and $\{y_n\}$ be two sequences of E. If either $\{x_n\}$ or $\{y_n\}$ is bounded and $\phi(x_n, y_n) \to 0$ as $n \to \infty$, then $\|x_n - y_n\| \to 0$ as $n \to \infty$.*

Lemma 2.8 *(See e.g.,[25]) Let* $\{a_n\}$ *be a sequence of nonnegative numbers and* $\{\alpha_n\} \subseteq (0,1)$ *a sequence such that* $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$. *Let the recursive inequality*

$$a_{n+1} \leq (1-\alpha_n)a_n + \gamma_n, n = 1, 2, ..., \tag{2.6}$$

be given. If $\gamma_n = o(\alpha_n)$ *Then* $a_n \to 0$, *as* $n \to \infty$.

Lemma 2.9 *[18] Let E be a uniformly convex and uniformly smooth real Banach space,* $x_1 \in E$ *and* $A : E \to E^*$ *be a maximal monotone mapping such that* $A^{-1}(0) \neq 0$. *Let* $\theta_n \in (0,1)$ *be a sequence such that* θ_n *converges to* 0. *Then, there exists a sequence* $\{y_n\} \in E$ *such that:*

$$\theta_n(Jy_n - Jx_1) + Ay_n = 0, \ \forall\, n \geq 1 \tag{2.7}$$

$$y_n \to y^* \quad . \tag{2.8}$$

where $y^* \in A^{-1}(0)$ *and* J *is the normalized duality mapping from* E *into* E^*.

3 Main results

Let E be a uniformly convex and uniformly smooth real Banach space with norm $\|\cdot\|$ and E^* be as its dual. Let A: $E \to E^*$ be a bounded maximal monotone mapping. Define the sequence $\{x_n\}$ as follows: $x_1 \in E$ is chosen arbitrarily and

$$x_{n+1} = J^{-1}\Big(Jx_n - \lambda_n Ax_n - \lambda_n\theta_n(Jx_n - Jx_1)\Big), \quad n \geq 1. \tag{3.1}$$

where $\{\lambda_n\}_{n=1}^{\infty}$ and $\{\theta_n\}_{n=1}^{\infty}$ are real sequences in $(0,1)$.
Let the conditions $(i)-(iii)$ be defined as follows:

(i) : $\displaystyle\sum_{n=1}^{\infty} \lambda_n\theta_n = \infty$ and $\theta_n \to 0$ as $n \to +\infty$,

(ii) : $\delta_E^{-1}(\lambda_n M_0) = o(\theta_n)$ for some constant $M_0 > 0$,

(iii) : $\displaystyle\left(\frac{\theta_{n-1}-\theta_n}{\theta_n}K\right) \in (0,1)$ and $\displaystyle\delta_E^{-1}\left(\frac{\theta_{n-1}-\theta_n}{\theta_n}K\right) = o(\lambda_n\theta_n)$,

for some constant $K := 4R_0L$ where $\sqrt{R_0} := \|x_1\| + \sup_{n\geq 1}\|y_n\|$ and $\{y_n\}$ is the sequence from the above Lemma 2.9 and L the constant given in Lemma 2.2.

Theorem 3.1 *Let* E *be a uniformly smooth and uniformly convex real Banach space and* E^* *be as its dual. Let* $A : E \to E^*$ *be a bounded maximal monotone mapping such that* $A^{-1}(0) \neq 0$. *Assume that the conditions* (i) *and* (ii) *are satisfied. Then, there exists some constants* M_0 *and* γ_0 *such that if* $\delta_E^{-1}(\lambda_n M_0) \leq \gamma_0\theta_n$, $\forall\, n \geq 1$, *the sequence* $\{x_n\}$ *given by* (3.1) *is bounded.*

Proof. We have, let $x^* \in A^{-1}(0)$, there exists a real number $r > 0$ such that:

$$\sqrt{r} > \max\Big\{ \|x^*\|, 12\|Jx^* - Jx_1\|, \sqrt{\phi(x^*, x_1)}\Big\}. \tag{3.2}$$

Since A is bounded the following constants are well defined and finite:

$$M := \sup\{ \|Ax + \theta(Jx - Jx_1)\| : \phi(x, x^*) \leq r,\ 0 \leq \theta \leq 1\}. \tag{3.3}$$

$$M_1 := \sup\{ \|J^{-1}(Jx - \lambda(Ax - \theta(Jx - Jx_1)) - x\| : \phi(x, x^*) \leq r,\ \lambda, \theta \in (0,1)\}. \tag{3.4}$$

Define

$$\gamma_0 := \frac{r}{4Mc_2} \quad \text{and} \quad M_0 = 2LMM_1, \tag{3.5}$$

where c_2 and L are the constants appearing in Lemma 2.2.
We show by induction that $\phi(x^*, x_n) \leq r$ for all $n \geq 1$. Assume that $\phi(x^*, x_n) \leq r$ for some $n \geq 1$ and let us show that $\phi(x^*, x_{n+1}) \leq r$.
From the relation (3.1) we make the following observations

$$\|x_{n+1} - x_n\| = \|J^{-1}(Jx_n - \lambda_n Ax_n - \lambda_n\theta_n(Jx_n - Jx_1)) - J^{-1}Jx_n\| \leq M_1 \tag{3.6}$$

and

$$\|Jx_{n+1} - Jx_n\| = \lambda_n\|Ax_n - \theta_n(Jx_n - Jx_1)\| \leq \lambda_n M. \tag{3.7}$$

Therefore, applying Lemma 2.2 and Schwarz inequality, we have

$$\begin{aligned} (2L)^{-1}\delta_E(c_2^{-1}\|x_{n+1} - x_n\|) &\leq \langle Jx_{n+1} - Jx_n, x_{n+1} - x_n\rangle \\ &\leq \|Jx_{n+1} - Jx_n\|\,\|x_{n+1} - x_n\| \\ &\leq \lambda_n MM_1. \end{aligned}$$

This implies that

$$\|x_{n+1} - x_n\| \leq c_2\delta_E^{-1}(\lambda_n M_0). \tag{3.8}$$

Using Lemma 2.4 with $y^* = \lambda_n Ax_n + \lambda_n\theta_n(Jx_n - Jx_1)$, we have

$$\begin{aligned} \phi(x^*, x_{n+1}) &= \phi(x^*, J^{-1}(Jx_n - \lambda_n(Ax_n) - \lambda_n\theta_n(Jx_n - Jx_1))) \\ &= V(x^*, Jx_n - \lambda_n(Ax_n) - \lambda_n\theta_n(Jx_n - Jx_1))) \\ &\leq V(x^*, Jx_n) \\ &\quad -2\lambda_n\langle J^{-1}(Jx_n - \lambda_n(Ax_n + \theta_n(Jx_n - Jx_1)) - x^*, Ax_n + \theta_n(Jx_n - Jx_1)\rangle \\ &\leq V(x^*, Jx_n) - 2\lambda_n\langle x_{n+1} - x^*, Ax_n + \theta_n(Jx_n - Jx_1)\rangle \\ &= \phi(x^*, x_n) - 2\lambda_n\theta_n\langle x_n - x^*, Jx_n - Jx^*\rangle - 2\lambda_n\langle x_n - x^*, Ax_n - Ax^*\rangle \\ &\quad -2\lambda_n\theta_n\langle x_n - x^*, Jx^* - Jx_1\rangle - 2\lambda_n\langle x_{n+1} - x_n, Ax_n + \theta_n(Jx_n - Jx_1))\rangle. \end{aligned}$$

From Lemma 2.6, Swarchz Inequality and the relation (3.3) it follows that

$$\phi(x^*, x_{n+1}) \leq (1 - \lambda_n\theta_n)\phi(x^*, x_n) + 2\lambda_n\theta_n\|x_n - x^*\| \cdot \|Jx^* - Jx_1\| + 2\lambda_n M\|x_{n+1} - x_n\|.$$

Therefore, by (3.8) we have

$$\phi(x^*, x_{n+1}) \leq (1-\lambda_n\theta_n)\phi(x^*, x_n) + 2\lambda_n\theta_n\|x_n - x^*\| \cdot \|Jx^* - Jx_1\| + 2\lambda_n M c_2 \delta_E^{-1}(\lambda_n M_0).$$

Relation (2.4), the induction assumption and inequality (3.2) imply that

$$\|x_n - x^*\| \leq 3\sqrt{r} \quad \text{and} \quad 2\|x_n - x^*\| \cdot \|Jx^* - Jx_1\| \leq \frac{r}{2}.$$

Combining this with the fact that $\delta_E^{-1}(\lambda_n M_0) \leq \gamma_0\theta_n$, for all $n \geq 1$, and using the definition (3.5) of γ_0, it follows that

$$\begin{aligned}\phi(x^*, x_{n+1}) &\leq \Big(1-\lambda_n\theta_n\Big)r + \lambda_n\theta_n\frac{r}{2} + 2\lambda_n\theta_n M c_2\gamma_0 \\ &= (1-\lambda_n\theta_n)r + \lambda_n\theta_n\frac{r}{2} + \lambda_n\theta_n\frac{r}{2} = r.\end{aligned}$$

Therefore $\phi(x^*, x_{n+1}) \leq r$. Hence, by induction $\phi(x^*, x_n) \leq r$ for all $n \geq 1$. That is $\{x_n\}$ is bounded. This completes the proof. □ Let us now prove the next convergence theorem.

Theorem 3.2 *Let E a uniformly convex and uniformly smooth real Banach space with E^* be its dual. And let $A : E \to E^*$ be a bounded maximal monotone mapping such that $A^{-1}(0) \neq 0$. Assume that the conditions (i) - (iii) are satisfied. Then, there exist some constants M_0 and $\gamma_0 > 0$ such that if $\delta_E^{-1}(\lambda_n M_0) \leq \gamma_0\theta_n$, $\forall\ n \geq 1$, the sequence $\{x_n\}$ given by (3.1) converges strongly to some $x^* \in A^{-1}(0)$.*

Proof. Based on Lemma 2.9, we need to prove that $\|x_n - y_n\| \to 0$ as $n \to \infty$. We estimate as follows:

$$\begin{aligned}\phi(y_n, x_{n+1}) &= V(y_n, Jx_{n+1}) \\ &= V(y_n, Jx_n - \lambda_n Ax_n - \lambda_n\theta_n(Jx_n - Jx_1)).\end{aligned}$$

Applying Lemma 2.4 with $y^* = \lambda_n Ax_n + \lambda_n\theta_n(Jx_n - Jx_1)$, we have

$$\begin{aligned}\phi(y_n, x_{n+1}) &= V(y_n, Jx_n - \lambda_n Ax_n - \lambda_n\theta_n(Jx_n - Jx_1)) \\ &\leq V(y_n, Jx_n) - 2\lambda_n\langle x_{n+1} - y_n, Ax_n + \theta_n(Jx_n - Jx_1)\rangle \\ &= \phi(y_n, x_n) - 2\lambda_n\langle x_n - y_n, Ax_n + \theta_n(Jx_n - Jx_1)\rangle \\ &\quad -2\lambda_n\langle x_{n+1} - x_n, Ax_n + \theta_n(Jx_n - Jx_1)\rangle \\ &= \phi(y_n, x_n) - 2\lambda_n\langle x_n - y_n, Ax_n - Ay_n\rangle - 2\lambda_n\langle x_n - y_n, Ay_n + \theta_n(Jx_n - Jx_1)\rangle \\ &\quad -2\lambda_n\langle x_{n+1} - x_n, Ax_n + \theta_n(Jx_n - Jx_1)\rangle.\end{aligned}$$

From the relation (2.7) in Lemma 2.9 and the fact that A is monotone we obtain

$$\begin{aligned}\phi(y_n, x_{n+1}) &\leq \phi(y_n, x_n) - 2\lambda_n\langle x_n - y_n, \theta_n(Jx_n - Jy_n)\rangle - 2\lambda_n\langle x_{n+1} - x_n, Ax_n + \theta_n(Jx_n - Jx_1)\rangle \\ &\leq \phi(y_n, x_n) - 2\lambda_n\theta_n\langle x_n - y_n, Jx_n - Jy_n\rangle + 2\lambda_n M\|x_{n+1} - x_n\|.\end{aligned}$$

Using Lemma 2.6 and inequality (3.8), we have

$$\phi(y_n, x_{n+1}) \leq (1-\lambda_n\theta_n)\phi(y_n, x_n) + 2\lambda_n M\delta_E^{-1}(\lambda_n M_0). \tag{3.9}$$

Now, by Lemma 2.5 we have

$$\begin{aligned}\phi(y_n, x_{n+1}) &= \phi(y_n, y_{n+1}) + \phi(y_{n+1}, x_{n+1}) + 2\langle y_n - y_{n+1}, Jy_{n+1} - Jx_{n+1}\rangle \\ &\geq \phi(y_{n+1}, x_{n+1}) + 2\langle y_n - y_{n+1}, Jy_{n+1} - Jx_{n+1}\rangle.\end{aligned}$$

Therefore

$$\phi(y_{n+1}, x_{n+1}) \leq \phi(y_n, x_{n+1}) + 2\langle y_{n+1} - y_n, Jy_{n+1} - Jx_{n+1}\rangle. \tag{3.10}$$

Using Schwartz inequality, the boundedness of $\{x_n\}$ and $\{y_n\}$ we have,

$$2\langle y_{n+1} - y_n, Jy_{n+1} - Jx_{n+1}\rangle \leq C\|y_{n+1} - y_n\|, \tag{3.11}$$

for some positive constant C.

Combining (3.11) with (3.9) and (3.10), we obtain

$$\phi(y_{n+1}, x_{n+1}) \leq (1-\lambda_n\theta_n)\phi(y_n, x_n) + 2\lambda_n M\delta_E^{-1}(\lambda_n M_0) + C\|y_{n+1} - y_n\|. \tag{3.12}$$

Now computing from Lemma 2.9 we have

$$Jy_n - Jy_{n+1} + \frac{1}{\theta_{n+1}}(Ay_n - Ay_{n+1}) = \frac{\theta_n - \theta_{n+1}}{\theta_{n+1}}(Jx_1 - Jy_n).$$

This implies the following

$$\langle Jy_n - Jy_{n+1}, y_n - y_{n+1}\rangle \leq \frac{\theta_n - \theta_{n+1}}{\theta_{n+1}}\|Jx_1 - Jy_n\| \cdot \|y_{n+1} - y_n\|.$$

Let $K := 4R_0L$ where $\sqrt{R_0} := \|x_1\| + \sup_{n\geq 1}\|y_n\|$, we have from Lemma 2.2,

$$\|y_{n+1} - y_n\| \leq c_2\delta_E^{-1}\Big(\frac{\theta_n - \theta_{n+1}}{\theta_{n+1}}K\Big). \tag{3.13}$$

Therefore (3.13) and (3.12) together give the following

$$\phi(y_{n+1}, x_{n+1}) \leq (1-\lambda_n\theta_n)\phi(y_n, x_n) + 2\lambda_n M\delta_E^{-1}(\lambda_n M_0) + C\delta_E^{-1}\Big(\frac{\theta_{n-1} - \theta_n}{\theta_n}K\Big)$$

Applying Lemma 2.8 with $a_n = \phi(y_n, x_n)$, $\alpha_n = \lambda_n\theta_n$ and

$$\gamma_n = 2\lambda_n M\delta_E^{-1}(\lambda_n M_0) + C\delta_E^{-1}\Big(\frac{\theta_{n-1} - \theta_n}{\theta_n}K\Big),$$

it follows from the conditions $(i)-(iv)$ that $\phi(y_n, x_n) \to 0$ as $n \to \infty$. Finally from Lemma 2.7, we deduce that $x_n \to y^* \in A^{-1}(0)$. This completes the proof. □

In the following corollary we deduce the result in Mendy et al. [18].

Corollary 3.3 *For $q > 1$ and $p > 1$, let E be a 2-uniformly convex and q-uniformly smooth or p-uniformly convex and 2-uniformly smooth real Banach space and E^* be its dual. Let $A : E \to E^*$ be a bounded maximal monotone mapping such that $A^{-1}(0) \neq \emptyset$. Assume that the conditions (i) - (iii) are satisfied. Then, there exists a constant $\gamma_0 > 0$ such that if $\delta_E^{-1}(\lambda_n M_0) \leq \gamma_0\theta_n$, $\forall\, n \geq 1$, the sequence $\{x_n\}$ defined by (3.1) converges strongly to some $x^* \in A^{-1}(0)$.*

Proof. It's clear that E is uniformly smooth and uniformly convex. So the result follows immediately from Theorem 3.2. □

4 Application and numerical simulations

Let E be a Banach space and E^* be its dual. Let $f : E \to \mathbb{R}$ be a real valued convex and differentiable function, we consider the optimization problem of minimizing f.

Let $df : E \to E^*$ denotes the differential map associated to f and $a \in E$. It is well known that the point a is a minimizer of f on E if and only if $df(a) = 0$.

Lemma 4.1 *[18] Let E be normed linear space and $f : E \to \mathbb{R}$ be a real-valued differentiable convex function. Assume that f is bounded. Then the differential map $df : E \to E^*$ is bounded.*

Theorem 4.2 *Let E be a uniformly convex and uniformly smooth real Banach space. Let $f : E \to \mathbb{R}$ be a differentiable, bounded and convex real-valued function which satisfies the growth condition: $f(x) \to +\infty$ as $\|x\| \to +\infty$. Assume that the conditions (i) - (iii) are satisfied. Then, there exists a constant $\gamma_0 > 0$ such that if $\delta_E^{-1}(\lambda_n M_0) \leq \gamma_0\theta_n$, $\forall\, n \geq 1$, the sequence $\{x_n\}$ given (3.1) with $A = df$ converges strongly to x^*, a minimizer of f.*

Example 4.3 *Let $E = L_p, 1 < p < \infty$ and the following real sequences:*

$$\lambda_n = (n+1)^{-a}, \quad \theta_n = (n+1)^{-b},\ n \geq 1 \quad \text{for} \quad 0 < b < \frac{1}{p}a \quad \text{and} \quad a + b < \frac{1}{p}.$$

For instance one can choose $a := \frac{1}{p+1}$ and $b := \frac{1}{2p(p+1)}$.

For $p \geq 2$, we check that the conditions $(i)-(ii)$ are satisfied.

It is clear that $\theta_n \to 0$ as $n \to +\infty$ and

$$\sum_{n=1}^{\infty} \lambda_n\theta_n = \sum_{n=1}^{\infty} \frac{1}{(n+1)^{a+b}} = +\infty.$$

So i) is satisfied. Let us check for ii).
It is known that (see. [11])

$$\delta_E(\tau) = 1 - \left(1 - \left(\frac{\tau}{2}\right)^p\right)^{\frac{1}{p}}, \ \forall\, \tau \in [0,2].$$

Therefore

$$\delta_E^{-1}(\varepsilon) = 2\left[1-(1-\varepsilon)^p\right]^{\frac{1}{p}}, \ \forall\, \varepsilon \in [0,1]. \tag{4.1}$$

Observe that $f(\varepsilon) := (1-\varepsilon)^p - 1 + p\varepsilon$ *is increasing and* $f(0) = 0$*. Hence*

$$(1-\varepsilon)^p > 1 - p\varepsilon. \tag{4.2}$$

Combining (4.2) *with the relation* (4.1) *gives the following inequality*

$$\delta_E^{-1}(\varepsilon) < 2p^{\frac{1}{p}}\varepsilon^{\frac{1}{p}}.$$

Now, for some $M_0 > 0$ *we have*

$$\begin{aligned}
\delta_E^{-1}(\lambda_n M_0) &< 2(qM_0)^{\frac{1}{p}}\lambda_n^{\frac{1}{p}} \\
&< 2(pM_0)^{\frac{1}{p}}(n+1)^{-\frac{a}{p}} \\
&= 2(pM_0)^{\frac{1}{p}}(n+1)^{b-\frac{a}{p}}(n+1)^{-b} \\
&= 2(pM_0)^{\frac{1}{p}}\theta_n(n+1)^{b-\frac{a}{p}}.
\end{aligned}$$

Therefore $\delta_E^{-1}(\lambda_n M_0) = o(\theta_n)$*. So ii) is satisfied.*

Let us check for condition (iii)*. We have*

$$\begin{aligned}
\frac{\theta_{n-1}-\theta_n}{\theta_n} &= \left(\frac{n}{n+1}\right)^{-b} \\
&= \left(1+\frac{1}{n}\right)^b - 1 \to 0 \quad as \quad n \to +\infty.
\end{aligned}$$

Thefore, we may assume without loss of generality that

$$\left(\frac{\theta_{n-1}-\theta_n}{\theta_n}K\right) \in (0,1), \ \forall\, n \geq 1.$$

Using Bernoulli's inequality $(1+x)^s \leq 1 + sx$ *for* $x > -1$ *and* $0 < s < 1$*, we have*

$$\begin{aligned}
\frac{\delta_E^{-1}\left(\frac{\theta_{n-1}-\theta_n}{\theta_n}K\right)}{\lambda_n\theta_n} &< \frac{2K^{\frac{1}{p}}p^{\frac{1}{p}}(\frac{\theta_{n-1}}{\theta_n}-1)^{\frac{1}{p}}}{\lambda_n\theta_n} \\
&< \frac{2K^{\frac{1}{p}}p^{\frac{1}{p}}(\frac{\theta_{n-1}}{\theta_n}-1)^{\frac{1}{p}}}{\lambda_n\theta_n} = 2K^{\frac{1}{p}}p^{\frac{1}{p}}\left[\left(1+\frac{1}{n}\right)^b - 1\right]^{\frac{1}{p}}(n+1)^{a+b} \\
&\leq 2K^{\frac{1}{p}}p^{\frac{1}{p}}b^{\frac{1}{p}}\frac{(n+1)^{a+b}}{n^{\frac{1}{p}}} \leq 2^{a+b}C_0 n^{a+b-\frac{1}{p}} \to 0 \quad as\ n \to +\infty
\end{aligned}$$

for some constant $C_0 > 0$*. Therefore iii) is satisfied.*
In conlusion, for $E = L_p([0,1])$, $p \geq 2$*, the above sequences* λ_n *and* θ_n *satisfy the conditions of Theorem 3.2.*
Now, define the map $f : E \to \mathbb{R}$ *by* $f(u) = \int_0^1 u^2(t)\,dt$*. It is clear that* f *is differentiable and for each* $u \in E$ *one has* $df(u)(\cdot) = 2\langle u, \cdot\rangle_{E,E^*} \in E^*$*. Moreover, since* f *is convex and continuous,* $df(u)$ *is maximal monotone.*
Consider the following algorithm: $x_1 \in E$,

$$x_{n+1} = J^{-1}\Big(Jx_n - \lambda_n df(x_n) - \lambda_n\theta_n(Jx_n - Jx_1)\Big), \quad n \geq 1. \tag{4.3}$$

By Theorem 4.2, the sequence $\{x_n\}$ *converges strongly to some* x^* *satisfying*

$$f(x^*) = \min_E f.$$

Now, for the numerical simulation, let us take $p = 3$, $q = \frac{3}{2}$ *and* $x_1(t) = t^2$*.*
This give us $\lambda_n = (n+1)^{-\frac{1}{4}}$ *and* $\theta_n = (n+1)^{-\frac{1}{24}}$*.*
Now, we rewrite the sequence (4.3) as follows: $y_1 = Jx_1$,

$$x_n = \|y_n\|^{2-q}|y_n|^{q-2}y_n \quad \text{and} \quad y_{n+1} = y_n - 2\lambda_n x_n - \lambda_n\theta_n(y_n - y_1) \ \ \forall\, n \geq 1.$$

The simulation for the algorithm gives the following results:

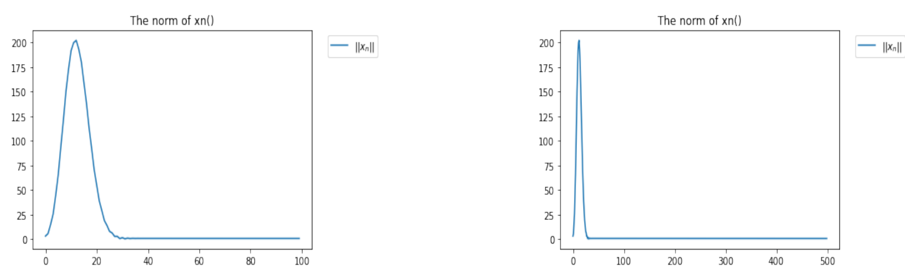

Figure 1: The convergence of $\|x_n\|$ to 0

The above results show that the sequence x_n *converges strongly to* 0 *which is a minimizer of* f*.*

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