International Journal of Mathematical Analysis
Vol. 19, 2025, no. 1, 1 - 21
HIKARI Ltd, www.m-hikari.com
https://doi.org/10.12988/ijma.2025.912555

Comparison Criterion for Second Order Riccati Equations

G. A. Grigorian

0019 Armenia c. Yerevan, str. M. Bagramian 24/5
Institute of Mathematics NAS of Armenia

Abstract

Three comparison criteria are obtained for second order Riccati equations. On the basis of these criteria some global existence theorems are proved for mentioned equations. The results obtained are used to derive a non oscillation criterion for three dimensional linear systems of ordinary differential equations.

Mathematics Subject Classification: 34D20

Keywords: comparison criteria, second order Riccati equations, global solvability, three dimensional linear systems, non oscillation

1. Introduction

Let $a(t)$, $b(t)$, $c(t)$, $d(t)$ and $e(t)$ be real-valued continuous functions on $[t_0, +\infty)$. Consider the second order Riccati equation

$$y'' + 3a(t)yy' + b(t)y' + a^2(t)y^3 + c(t)y^2 + d(t)y + e(t) = 0, \qquad t \geq t_0. \tag{1.1}$$

Throughout we will assume that $a(t)$, $b(t) \in C^1([t_0, \infty))$. Equation (1.1) serves as an important tool for the study of properties of solutions of first order three dimensional linear systems of ordinary differential equations equations, in particular, for the study of properties of solutions of third order linear ordinary differential equations. One of ways to start to apply this equation to the mentioned above systems is to obtain comparison criteria for it. It should be noticed that this approach has been used to first order Riccati equations (see [1,2]) allowing to obtain several results for qualitative study of solutions of some types of equations (see e. g. [3–13]).

In this paper we prove three comparison criteria for second order Riccati equations. We use these criteria to obtain some global solvability theorems for Eq. (1.1) and some nonoscillation criteria for first order three dimensional linear systems of ordinary differential equations.

2. Auxiliary propositions.

Let $a_1(t)$, $b_1(t)$, $c_1(t)$, $d_1(t)$ and $e_1(t)$ be real-valued continuous functions on $[t_0,\infty)$. Along with (1.1) consider the equation

$$y'' + 3a_1(t)yy' + b_1(t)y' + a_1^2(t)y^3 + c_1(t)y^2 + d_1(t)y + e_1(t) = 0, \qquad t \ge t_0. \quad (2.1)$$

We set $\nu(t,u,v,u_1,v_1) \equiv u_1 - v_1 + \frac{3}{2}a(t)(u^2 - v^2) + b(t)(u-v)$, $\Gamma(t,u,v) \equiv a^2(t)[u^2 + uv + v^2] + (c(t) - \frac{3}{2}a'(t))(u+v) - b'(t) + d(t)$, $J(t,u,v) \equiv (u-v)\Gamma(t,u,v)$, $L(t,u,v) \equiv 3(a_1(t) - a(t)))uv + (b_1(t) - b(t))v + (a_1^2(t) - a^2(t))u^3 + (c_1(t) - c(t))u^2 + (d_1(t) - d(t))u + e_1(t) - e(t)$, $t \ge t_0$, $u,v,u_1,v_1 \in \mathbb{R}$. Let $y_0(t)$ and $y_1(t)$ be solutions of Eq. (1.1) and Eq. (2.1) respectively on $[t_1,t_2) \subset [t_0,\infty)$. Then it is not difficult to verify that $[y_0(t) - y_1(t)]'' + \frac{3}{2}a(t)[(y_0(t) - y_1(t))(y_0(t) + y_1(t))]' + b(t)[y_0(t) - y_1(t)]' + a^2(t)[y_0(t) - y_1(t)][y_0^2(t) + y_0(t)y_1(t) + y_1^2(t)] + c(t)(y_0(t) - y_1(t))(y_0(t) + y_1(t)) + d(t)(y_0(t) - y_1(t)) - L(t, y_1(t), y_1'(t)) = 0$, $t \in [t_1,t_2)$. Let us integrate this equality from t_1 to t. After using the rule of integration by parts and making some simplifications we obtain

$$[y_0(t) - y_1(t)]' + \Big\{\frac{3}{2}a(t)(y_0(t) + y_1(t)) + b(t)\Big\}(y_0(t) - y_1(t)) - \nu(t, y_0(t), y_1(t), y_0'(t), y_1'(t)) -$$

$$- \int\limits_{t_1}^{t} J(\tau, y_0(\tau), y_1(\tau))d\tau - \int\limits_{t_1}^{t} L(\tau, y_1(\tau), y_1'(\tau))d\tau = 0, \ \ t \in [t_1,t_2). \quad (2.2)$$

It is clear from here that $y_0(t) - y_1(t)$ is a solution of the linear equation

$$x' + \Big\{\frac{3}{2}a(t)(y_0(t) + y_1(t)) + b(t)\Big\}x = \nu(t, y_0(t), y_1(t), y_0'(t), y_1'(t)) +$$

$$+ \int\limits_{t_1}^{t} J(\tau, y_0(\tau), y_1(\tau))d\tau + \int\limits_{t_1}^{t} L(\tau, y_1(\tau), y_1'(\tau))d\tau = 0, \ \ t \in [t_1,t_2).$$

Then by the Cauchy formula we have

$$y_0(t) - y_1(t) = \exp\Big\{-\int\limits_{t_1}^{t}\Big\{\frac{3}{2}a(\tau)(y_0(\tau) + y_1(\tau)) + b(\tau)\Big\}d\tau\Big\}\times$$

$$\times\Bigg[y_0(t_1) - y_1(t_1) + \int\limits_{t_1}^{t} \exp\Bigg\{\int\limits_{t_1}^{\tau}\Big[\frac{3}{2}a(s)(y_0(s)+y_1(s)) + b(s)\Big]ds\Bigg\}\times$$

$$\times\Bigg(\nu(t_1, y_0(t_1), y_1(t_1), y_0'(t_1), y_1'(t_1))(\tau - t_1) + \int\limits_{t_1}^{\tau}[J(s, y_0(s), y_1(s)) +$$

$$+L(s, y_1(s), y_1'(s))]ds\Bigg)d\tau\Bigg], \qquad t \in [t_1, t_2). \tag{2.3}$$

We set $D(t) \equiv 2(2c(t) - 3a'(t))^2 + a^2(t)(d(t) - b'(t)).\ t \geq t_0$.

Lemma 2.1. *If* $a(t) \neq 0,\ \ D(t) \geq 0,\ \ t \geq t_0$, *then* $\Gamma(t,u,v) \geq 0,\ \ t \geq t_0,\ \ u, v \in \mathbb{R}$.

Proof. We have

$$\begin{cases} \frac{\partial \Gamma(t,u,v)}{\partial u} = a^2(t)(2u+v) + c(t) - \frac{3}{2}a'(t), \\ \\ \frac{\partial \Gamma(t,u,v)}{\partial u} = a^2(t)(u+2v) + c(t) - \frac{3}{2}a'(t),\ \ t \in [t_1, t_2),\ \ u, v \in \mathbb{R}. \end{cases} \tag{2.4}$$

Since $a(t) \neq 0,\ \ t \in [t_1, t_2)$ one can easily verify that the functions $u_0(t) = v_0(t) = \frac{3a'(t)2c(t)}{2a^2(t)},\ \ t \in [t_1, t_2)$ form the unique solution $(u_0(t), v_0(t))$ of the system

$$\begin{cases} a^2(t)(2u+v) = \frac{3}{2}a'(t) - c(t), \\ \\ a^2(t)(u+2v) = \frac{3}{2}a'(t) - c(t),\ \ t \in [t_1, t_2),\ \ u, v \in \mathbb{R}. \end{cases}$$

Then by virtue of (2.4) it follows from the conditions of the lemma that

$$\min_{u,v\in\mathbb{R}} \Gamma(t,u,v) = \Gamma(t, u_0(t), v_0(t)) = \frac{D(t)}{a^2(t)} \geq 0,\ \ t \in [t_1, t_2).$$

Therefore, $\Gamma(t,u,v) = \frac{D(t)}{a^2(t)} \geq 0,\ \ t \in [t_1, t_2)$. The lemma is proved.

Let $X(t),\ \ Y(t),\ \ Z(t)$ and $W(t)$ be real-valued continuous functions on $[t_0, \infty)$ and let $X(t) \neq 0,\ \ t \geq t_0,\ \ X(t) \in C^1([t_0, \infty))$. Consider the linear system

$$\begin{cases} \phi' = \qquad a(t)\psi, \\ \\ \psi' = \qquad\qquad X(t)\chi, \\ \\ \chi' = Y(t)\phi + Z(t)\psi + W(t)\chi,\ \ t \geq t_0. \end{cases} \tag{2.5}$$

The substitution

$$\psi = y\phi \tag{2.6}$$

reduces this system into

$$\begin{cases} \phi' = \ \ a(t)y\phi, \\ [y' + a(t)y^2]\phi = \ \ X(t)\chi, \\ \chi' = [Y(t)\phi + Z(t)y]\phi + W(t)\chi, \ \ t \ge t_0. \end{cases} \tag{2.7}$$

Since $X(t) \ne 0, \ \ t \ge t_0$, from here we get

$$\chi = \frac{y' + a(t)y^2}{X(t)}\phi, \qquad t \ge t_0. \tag{2.8}$$

This together with the last equation of the system (2.7) implies

$$\Big(\frac{1}{X(t)}\Big)'(y'+a(t)y^2)\phi + \frac{1}{X(t)}[y''+a'(t)y'+2a(t)yy']\phi = [Y(t)+Z(t)+W(t)\frac{y'+a(t)y^2}{X(t)}]\phi,$$

$t \ge t_0$. After some simplifications from here we derive

$$y''+3a(t)yy'-\Big[\frac{X'(t)}{X(t)}+W(t)\Big]y'+a^2(t)y^3+\Big[W(t)a(t)-\frac{X'(t)}{X(t)}a(t)+a'(t)\Big]y^2+X(t)Z(t)y+$$

$$+X(t)Y(t) = 0, \qquad t \ge t_0. \tag{2.9}$$

This equation coincides with Eq. (1.1) provided

$$\begin{cases} \frac{X'(t)}{X(t)} + W(t) = -b(t), \\ W(t)a(t) - \frac{X'(t)}{X(t)}a(t) + a'(t) = c(t), \\ X(t)Z(t) = d(t), \ \ X(t)Y(t) = e(t), \ \ t \ge t_0. \end{cases}$$

Assume $a(t) \ne 0, \ \ t \ge t_0$. Then the last system is equivalent to the following one.

$$\begin{cases} X'(t) = \Big[\frac{a'(t)}{a(t)} - \frac{c(t)}{2a(t)} - \frac{b(t)}{2}\Big]X(t), \\ W(t) = -\frac{X'(t)}{X(t)} - b(t), \\ Y(t) = \frac{e(t)}{X(t)}, \ \ Z(t) = \frac{d(t)}{X(t)}, \ \ t \ge t_0. \end{cases}$$

Hence, if in particular, $X(t) = \exp\Big\{\int\limits_{t_0}^{t}\Big[\frac{a'(\tau)}{2a(\tau)} - \frac{c(\tau)}{2a(\tau)} - \frac{b(\tau)}{2}\Big]d\tau\Big\}, \ \ Y(t) = \frac{e(t)}{X(t)}, \ \ Z(t) = \frac{d(t)}{X(t)}, \ \ W(t) = \frac{b(t)}{2} + \frac{c(t)}{2a(t)} - \frac{a'(t)}{a(t)}, \ \ t \ge t_0$, then Eq. (2.9) coincides with Eq. (1.1). It follows from here and (2.6) - (2.8) that all solutions $y(t)$ of Eq. (1.1), existing on any interval $[t_1, t_2) \subset [t_0, \infty)$, are connected with solutions $(\phi(t), \psi(t), \chi(t))$ of the system (2.5) by the relations

$$\begin{cases} \phi(t) = \phi(t_1) \exp\left\{\int\limits_{t_1}^{t} a(\tau) y(\tau) d\tau\right\}, \ \ \phi(t_1) \neq 0, \\ \\ \psi(t) = y(t)\phi(t), \ \ \chi(t) = \frac{y'(t)+a(t)y^2(t)}{X(t)}, \ \ t \in [t_1, t_2). \end{cases} \tag{2.10}$$

Definition 2.1. *An interval $[t_1, t_2) \subset [t_0, \infty)$ is called the maximum existence interval for a solution $y(t)$ of Eq. (1.1), if $y(t)$ exists on $[t_1, t_2)$ and cannot be continued to the right from t_2 as a solution of Eq. (1.1).*

Lemma 2.2. *Let $a(t) \neq 0, \ \ t \in [t_1, t_2) \subset [t_0, \infty)$ and $y(t)$ be a solution of Eq. (1.1) on $[t_1, t_2)$. If the function $F(t) \equiv \int\limits_{t_0}^{t} a(\tau) y(\tau) d\tau, \ \ t \in [t_1, t_2)$ is bounded from below on $[t_1, t_2)$, then $[t_1, t_2)$ is not the maximum existence interval for $y(t)$.*

Proof. Since $a(t) \neq 0, \ \ t \in [t_1, t_2)$ by (2.10) the functions $\phi(t) \equiv \exp\left\{\int\limits_{t_1}^{t} a(\tau) y(\tau) d\tau\right\}$, $\psi(t) \equiv y(t)\phi(t)$ and $\chi(t) \equiv \frac{y'(t)+a(t)y^2(t)}{X(t)}, \ \ t \in [t_1, t_2)$ form a solution $(\phi(t), \psi(t), \chi(t))$ of the system (2,5) on $[t_1, t_2)$, which is continuable on $[t_1, \infty)$ as a solution of the system (2.5) on $[t_1, \infty)$. It follows from here and the boundedness from below of $F(t)$ that $\phi(t) \neq 0$, $t \in [t_1, t_3)$ for some $t_3 > t_2$. Then by (2.6) $\widetilde{y}(t) \equiv \frac{\psi(t)}{\phi(t)}, \ \ t \in [t_1, t_3)$ is a solution of Eq. (1.1) on $[t_1, t_3)$. Obviously $\widetilde{y}(t) = y(t), \ \ t \in [t_1, t_2)$. Hence, $\widetilde{y}'(t) = y'(t), \ \ t \in [t_1, t_2)$. It follows from here that $[t_1, t_2)$ is not the maximum existence interval for $y(t)$. The lemma is proved.

Consider the differential inequality

$$\eta' + 3a(t)\eta\eta' + b(t)\eta' + a^2(t)\eta^3 + c(t)\eta^2 + d(t)\eta + e(t) \geq 0, \ \ t \geq t_0. \tag{2.11}$$

Remark 2.1. *It is not difficult to verify that if $a(t) \neq 0, \ \ t \in [t_0, T]$ for some $T > t_0$, then $\eta_\lambda \equiv \lambda + \max\limits_{t \in [t_0, T]} \frac{|c(t)|+|d(t)|+|e(t)|}{a^2(t)}, \ \ t \in [t_0, T]$, where λ is a constant and ≥ 0, is a solution of the inequality (2.11), and if $a(t) \neq 0, \ \ t \geq t_0, \ \ M \equiv \sup\limits_{t \geq t_0} \frac{c(t)|+|d(t)|+|e(t)|}{a^2(t)} < \infty$, then $\eta_\lambda(t) \equiv \lambda + M, \ \ t \geq t_0$, is a solution of the inequality (2.11) on $[t_0, \infty)$.*

Lemma 2.3. *Let $y(t)$ be a solution of Eq. (1.1) on $[t_1, t_2)$ and $\eta(t)$ be a solution of the inequality (2.11) on $[t_1, t_2)$ such that $\eta(t_1) \geq y(t_1), \ \ \nu(t_1, \eta(t_1), y(t_1)\eta'(t_1), y(t_1)) \geq 0$. If $a(t) \neq 0$ and $D(t) \geq 0, \ \ t \in [t_1, t_2)$, then*

$$\eta(t) \geq y(t), \ \ t \in [t_1, t_2), \tag{2.12}$$

$$\nu(t, \eta(t), y(t), \eta'(t), y'(t)) \geq 0, \ \ t \in [t_1, t_2), \tag{2.13}$$

Proof. We have

$$[\eta(t)-y(t)]''+\frac{3}{2}a(t)[(\eta(t)+y(t))(\eta(t)-y(t))]'+b(t)(\eta(t)-y(t))'+$$
$$+a^2(t)(\eta(t)-y(t))[\eta^2(t)+\eta(t)y(t)+y^2(t)]+c(t)(\eta(t)+y(t))(\eta(t)-y(t))+$$
$$+d(t)(\eta(t)-y(t))\ge 0,\ \ t\in[t_1,t_2).$$

Integrating this inequality from t_1 to t and making some simplifications we obtain

$$\mu(t)\equiv[\eta(t)-y(t)]'+\Big\{\frac{3}{2}a(t)(\eta(t)+y(t))+b(t)\Big\}(\eta(t)-y(t))-$$
$$-\nu(t_1,\eta(t_1),y(t_1)\eta'(t_1),y(t_1))-\int\limits_{t_1}^{t}J(\tau,\eta(\tau),y(\tau))d\tau\ge 0,\ \ t\in[t_1,t_2). \tag{2.14}$$

It is clear from here that $\eta(t)-y(t)$ is a solution of the linear equation

$$x'+\frac{3}{2}a(t)[(\eta(t)+y(t))x-\nu(t_1,\eta(t_1),y(t_1)\eta'(t_1),y(t_1))-\int\limits_{t_1}^{t}J(\tau,\eta(\tau),y(\tau))d\tau-\mu(t)=0,$$

$t\in[t_1,t_2)$. Then by the Cauchy formula we have

$$\eta(t)-y(t)=\exp\Big\{-\int\limits_{t_1}^{t}\Big[\frac{3}{2}a(\tau)(\eta(\tau)+y(\tau))+b(\tau)\Big]d\tau\Big\}\times$$
$$\times\Big[\eta(t_1)-y(t_1)+\int\limits_{t_1}^{t}\exp\Big\{\int\limits_{t_1}^{\tau}\Big[\frac{3}{2}a(s)(\eta(s)+y(s))+b(s)\Big]ds\Big\}\times$$
$$\times\Big((\tau-t)\nu(t_1,\eta(t_1),y(t_1),\eta'(t_1),y'(t_1))+\int\limits_{t_1}^{\tau}[J(s,\eta(s),y(s))+\mu(s)]ds\Big)d\tau\Big], \tag{2.15}$$

$t\in[t_1,t_2)$. By Lemma 2.1 it follows from the conditions $a(t)\ne 0$ and $D(t)\ge 0$, $t\in[t_1,t_2)$ that $J(t,\eta(t),y(t))\ge 0,\ \ t\in[t_1,t_2)$. Moreover, it follows from (2.14) that $\mu(t)\ge\ 0,\ \ t\in[t_1,t_2)$. Hence, under the initial conditions of the lemma it follows the inequality (2.12) from (2.15). The inequality (2.13) follows immediately from (2.14), the initial condition $\nu(t_1,\eta(t_1),y(t_1),\eta'(t_1),y'(t_1))\ge 0$ and the inequality $J(t,\eta(t),y(t))\ge\ 0,\ t\in[t_1,t_2)$. The lemma is proved.

Consider the differential inequality

$$\zeta'+3a(t)\zeta\zeta'+b(t)\zeta'+a^2(t)\zeta^3+c(t)\zeta^2+d(t)\zeta+e(t)\le 0,\qquad t\ge t_0. \tag{2.16}$$

Remark 2.2. *It is not difficult to verify that if* $a(t) \ne 0, \ \ t \in [t_0, T]$ *for some* $T > t_0$, *then* $\zeta_\lambda \equiv -\lambda - \max\limits_{t\in[t_0,T]} \frac{|c(t)|+|d(t)|+|e(t)|}{a^2(t)}, \ \ t \in [t_0, T]$, *where* $\lambda = const \ge 0$, *is a solution of the inequality (2.16), and if* $a(t) \ne 0, \ \ t \ge t_0, \ \ M \equiv \sup\limits_{t\ge t_0} \frac{c(t)|+|d(t)|+|e(t)|}{a^2(t)} < \infty$, *then* $\zeta_\lambda(t) \equiv -\lambda - M, \ \ t \ge t_0$, *is a solution of the inequality (2.16) on* $[t_0, \infty)$.

By analogy with the proof of Lemma 2.3 one can prove the following lemma.

Lemma 2.4. *Let* $y(t)$ *be a solution of Eq. (1.1) on* $[t_1, t_2)$ *and* $\zeta(t)$ *be a solution of the inequality (2.16) on* $[t_1, t_2)$ *such that*

$$\zeta(t_1) \le y(t_1), \ \ \nu(t_1, \zeta(t_1), y(t_1), \zeta'(t_1), y'(t_1)) \le 0.$$

If $a(t) \ne 0$ *and* $D(t) \ge 0, \ \ t \in [t_1, t_2)$, *then*

$$\zeta(t) \le y(t), \qquad \nu(t, \zeta(t), y(t), \zeta'(t), y'(t)) \le 0, \qquad t \in [t_1, t_2).$$

■

Consider the nonlinear system

$$Y' = F(t, Y), \qquad t \ge t_0. \tag{2.17}$$

Every solution $Y(t) = Y(t, t_0, Y_0)$ of this system exists either only on a finite interval $[t_0, T)$ or is continuable on $[t_0, \infty)$

Lemma 2.5([14, p. 204, Lemma]). *If a solution* $Y(t)$ *of the system (2.17) exists only on a finite interval* $[t_0, T)$, *then*

$$||Y(t)|| \to \infty \ \ as \ \ t \to T - 0,$$

where $||Y(t)||$ *is any euclidian norm of* $Y(t)$ *for every fixed* $t \in [t_0, T)$.

■

3. Comparison criteria

In this section we use the results of the previous section to derive three comparison criteria for second order Riccati equations. These criteria we use in the next section to obtain some global solvability criteria for Eq. (1.1).

Theorem 3.1. *Let the following conditions be satisfied*
(1) $a(t) > 0, \ \ t \in [t_1, t_2)$,
(2) $D(t) \ge 0, \ \ t \in [t_1, t_2)$,
(3) Eq. (2.1) has a solution $y_1(t)$ *on* $[t_1, t_2)$ *such that*

$$\gamma - y_1(t_1) + \int\limits_{t_1}^{t} \exp\Big\{\int\limits_{t_1}^{\tau}\Big[\frac{3}{2}a(s)(\eta(s)+y_1(s))+b(s)\Big]ds\Big\}\Big(\int\limits_{t_1}^{\tau} L(s, y_1(s), y_1'(s))ds\Big)d\tau \ge 0,$$

$t\in[t_1,t_2)$ for some $\gamma\ge y(t_1)$, where $\eta(t)$ is a solution of the inequality (2.11) on $[t_1,t_2)$ with $\eta(t_1)\ge\gamma$.
Then every solution $y(t)$ of Eq. (1.1) with the initial conditions $y(t_1)\in[\gamma,\eta(t_1)]$, $\nu(t_1,\eta(t_1),y(t_1),\eta'(t_1),y'(t_1)) > 0$, $\nu(t_1,y(t_1),y_1(t_1),y'(t_1),y_1'(t_1))\ge 0$, exists on $[t_1,t_2)$ and

$$y_1(t)\le y(t)\le\eta(t),\ \ t\in[t_1,t_2), \tag{3.1}$$

$$\nu(t,\eta(t),y(t),\eta'(t),y'(t))\ge 0,\ \ t\in[t_1,t_2). \tag{3.2}$$

In addition if $\int\limits_{t_1}^{t} L(s,y_1(s),y_1'(s))ds\ge 0,\ \ t\in[t_1,t_2)$, then

$$\nu(t,y(t),y_1(t),y'(t),y_1'(t))\ge 0,\ \ t\in[t_1,t_2). \tag{3.3}$$

Proof. Let $\eta(t)$ be a solution of the inequality (2.11) on $[t_1,t_2)$ with $\eta(t_1)\ge\gamma$ and $y(t)$ be a solution of Eq. (1.1) with $y(t_1)\in(\gamma,\eta(t_1)]$, $\nu(t_1,\eta(t_1),y(t_1)\eta'(t_1),y'(t_1)) > 0$, $\nu(t_1,y(t_1),y_1(t_1),y'(t_1),y_1'(t_1))\ge 0$. Let $[t_1,t_3)$ be the maximum existence interval for $y(t)$. We must show that

$$t_3\ge t_2. \tag{3.4}$$

Suppose $t_3<t_2$. We claim that

$$y(t)>y_1(t),\ \ t\in[t_1,t_3). \tag{3.5}$$

Assume this is not true. Then there exists $t_4\in(t_1,t_3)$ such that (since $y(t_1)>\gamma\ge y_1(t_1)$)

$$y(t_4)=y_1(t_4). \tag{3.6}$$

By (2.3) we have

$$y(t)-y_1(t)=\exp\Big\{-\int\limits_{t_1}^{t}\Big\{\frac{3}{2}a(\tau)(y(\tau)+y_1(\tau))+b(\tau)\Big\}d\tau\Big\}\times$$
$$\times\Bigg[y(t_1)-y_1(t_1)+\int\limits_{t_1}^{t}\exp\Big\{\int\limits_{t_1}^{\tau}\Big[\frac{3}{2}a(s)(y(s)+y_1(s))+b(s)\Big]ds\Big\}\times$$
$$\times\Bigg(\nu(t_1,y(t_1),y_1(t_1),y'(t_1),y_1'(t_1))(\tau-t_1)+\int\limits_{t_1}^{\tau}[J(s,y(s),y_1(s))+$$
$$+L(s,y_1(s),y_1'(s))]ds\Bigg)d\tau\Bigg],\qquad t\in[t_1,t_3). \tag{3.7}$$

By Lemma 2.3 it follows from the conditions (1) and (2) of the theorem that

$$y(t)\le\eta(t),\qquad t\in[t_1,t_3), \tag{3.8}$$

$$\nu(t,\eta(t),y(t),\eta'(t),y'(t)) \ge 0, \qquad t\in[t_1,t_3). \tag{3.9}$$

By the mean value theorems for integrals (see [15, p. 869]) it follows from (3.8) that

$$\int\limits_{t_1}^{t} \exp\Big\{\int\limits_{t_1}^{\tau}\Big[\frac{3}{2}a(s)(y(s)+y_1(s))+b(s)\Big]ds\Big\}\times$$

$$\times\Big(\nu(t_1,y(t_1),y_1(t_1),y'(t_1),y_1'(t_1))(\tau-t_1)+\int\limits_{t_1}^{\tau}[J(s,y(s),y_1(s))+$$

$$+L(s,y_1(s),y_1'(s))]ds\Big)d\tau =$$

$$=\int\limits_{t_1}^{\alpha(t)} \exp\Big\{\int\limits_{t_1}^{\tau}\Big[\frac{3}{2}a(s)(\eta(s)+y_1(s))+b(s)\Big]ds\Big\}\times$$

$$\times\Big(\nu(t_1,y(t_1),y_1(t_1),y'(t_1),y_1'(t_1))(\tau-t_1)+\int\limits_{t_1}^{\tau}[J(s,y(s),y_1(s))+$$

$$+L(s,y_1(s),y_1'(s))]ds\Big)d\tau, \qquad t\in[t_1,t_4],$$

for some $\alpha(t)\in[t_1,t]$, $t\in[t_1,t_4]$ By Lemma 2.1 it follows from the conditions (1) and (2) of the theorem that

$$J(t,y(t),y_1(t)) \ge 0, \qquad t\in[t_1,t_4]. \tag{3.10}$$

This together with (3.7) and the condition (3) of the theorem implies that (since $y(t_1)>\gamma$) $y(t_4)>y_1(t_4)$, which contradicts (3.5). It follows from (3.5) and the condition (1) of the theorem that the function $F(t)\equiv\int\limits_{t_1}^{t} a(\tau)y(\tau)d\tau$, $t\in[t_1.t_3)$ is bounded from below on $[t_1,t_3)$. In virtue of Lemma 2.2 it follows from here that $[t_1,t_3)$ is not the maximum existence interval for $y(t)$. We obtain a contradiction, proving (3.4). From (3.4), (3.5) and (3.9) it follows (3.1) and (3.2). If $\int\limits_{t_1}^{t} L(s,y_1(s),y_1'(s))ds \ge 0$, $t\in[t_1,t_2)$, then by (2,2) from the initial condition $\nu(t_1,y(t_1),y_1(t_1),y'(t_1),y_1'(t_1)) \ge 0$ and (3.10) it follows (3.3). Thus the theorem is proved in the case $y(t_1)>\gamma$. It remains to prove the theorem for the case $y_1(t_1)=\gamma$. For any $\delta>0$ let the function $\widetilde{y}_\delta(t)$ be a solution of Eq. (1.1) with $\widetilde{y}_\delta(t_1)=\gamma+\delta$, $|\widetilde{y}_\delta'(t_1)-y'(t_1)|<\delta$, $\nu(t_1,\widetilde{y}_\delta(t_1),y_1(t_1),\widetilde{y}_\delta'(t_1),y'(t_1))>0$ (the last two inequalities are always satisfiable for all enough small $\delta>0$ due to initial conditions) and $\eta_\delta(t)$ be a solution of the inequality (2.11) on $[t_1,t_2)$ with $\eta_\delta(t_1)\ge\widetilde{y}_\delta(t_1)$,

$\nu(t_1, \eta_\delta(t_1), \widetilde{y}_\delta(t_1), \eta'_\delta(t_1), \widetilde{y}'_\delta(t_1)) \geq 0$. Then by already proven (by Remark 2.1 if $t_2 < \infty$, then $\eta_\delta(t)$ always exists and the proof of the case $t_2 = \infty$ is reducible to the proof of the case $t_2 < \infty$) $\widetilde{y}_\delta(t)$ exists on $[t_1, t_2)$ and

$$\widetilde{y}_\delta(t) > y_1(t), \qquad t \in [t_1, t_2). \tag{3.11}$$

Let $[t_1, t_3)$ be the maximum existence interval for $y(t)$. Show that

$$t_3 \geq t_2. \tag{3.12}$$

Suppose $t_3 < t_2$. We claim that

$$y(t) \geq y_1(t), \qquad t \in [t_1, t_3). \tag{3.13}$$

Suppose this is not true. Then there exists $t_4 \in (t_1, t_3)$ such that

$$y(t_4) < y_1(t_4). \tag{3.14}$$

Since the solutions of Eq. (1.1) are continuously dependent on their initial values we chose $\delta > 0$ enough small such that

$$|\widetilde{y}_\delta(t_1) - y(t_1)| < \frac{y_1(t_4) - y(t_4)}{2}.$$

Then $y(t_4) - y_1(t_4) = y(t_4) - \widetilde{y}_\delta(t_1) + \widetilde{y}_\delta(t_1) - y_1(t_4) \geq -|y(t_1) - \widetilde{y}_\delta(t_1)| > \frac{y(t_4) - y_1(t_4)}{2}$. We obtain a contradiction with (3.14), which proves (3.13). It follows from (3.13) and the condition (1) of the theorem that the function $F(t) \equiv \int\limits_{t_1}^{t} a(\tau) d\tau, \ \ t \in [t_1, t_3)$ is bounded from below on $[t_1, t_3)$. Then by virtue of Lemma 2.2 $[t_1, t_3)$ is not the maximum existence interval for $y(t)$, which contradicts our supposition. The obtained contradiction proves (3.12). From(3.12) and (3.13) it follows

$$y(t) \geq y_1(t), \qquad t \in [t_1, t_2).$$

Then by Lemma 2.3 we obtain (3.1) and (3.2). The inequality (3.3) can be proved by analogy with the proof of the already proven case $y(t_1) > \gamma$. The theorem is proved.

Remark 3.1. *It is clear from the proof of Theorem 3.1 that the conditions* $y(t_1) \in [\gamma, \eta(t_1)], \ \nu(t_1, \eta(t_1), y(t_1), \eta'(t_1), y'(t_1)) > 0$ *of Theorem 3.1 can be replaced by the following ones* $y(t_1) \in (\gamma, \eta(t_1)], \ \nu(t_1, \eta(t_1), y(t_1), \eta'(t_1), y'(t_1)) \geq 0$.

Using Lemma 2.4 instead of Lemma 2.3 by analogy with the proof of Theorem 3.1 one can prove the following theorem

Theorem 3.2. *Let the following conditions be satisfied*
$a(t) > 0, \ \ D(t) \geq 0, \ \ t \in [t_1, t_2),$

Eq. (2.1) has a solution $y_1(t)$ *on* $[t_1, t_2)$ *such that*

$$\gamma - y_1(t_1) + \int\limits_{t_1}^{t} \exp\Big\{\int\limits_{t_1}^{\tau}\Big[\frac{3}{2}a(s)(\zeta(s)+y_1(s))+b(s)\Big]ds\Big\}\Big(\int\limits_{t_1}^{\tau} L(s, y_1(s), y_1'(s))ds\Big)d\tau \le 0,$$

$t \in [t_1, t_2)$ *for some* $\gamma \le y(t_1)$, *where* $\zeta(t)$ *is a solution of the inequality (2.16) on* $[t_1, t_2)$ *with* $\zeta(t_1) \le \gamma$.
Then every solution $y(t)$ *of Eq. (1.1) with* $y(t_1) \in [\zeta(t_1), \gamma]$, $\nu(t_1, \zeta(t_1), y(t_1), \zeta'(t_1), y'(t_1)) < 0$, $\nu(t_1, y(t_1), y_1(t_1), y'(t_1), y_1'(t_1)) \le 0$, *exists on* $[t_1, t_2)$ *and*

$$\zeta(t) \le y(t) \le y_1(t), \ \ t \in [t_1, t_2),$$

$$\nu(t, \zeta(t), y(t), \zeta'(t), y'(t)) \le 0, \ \ t \in [t_1, t_2).$$

In addition if $\int\limits_{t_1}^{t} L(s, y_1(s), y_1'(s))ds \le 0, \ \ t \in [t_1, t_2)$, *then*

$$\nu(t, y(t), y_1(t), y'(t), y_1'(t)) \le 0, \ \ t \in [t_1, t_2).$$

■

Remark 3.2. *It is clear from the Remark 3.1 and the similarity of the proofs of Theorem 3.1 and Theorem 3.2 that the initial conditions* $y(t_1) \in [\zeta(t_1), \gamma]$, $\nu(t_1, \zeta(t_1), y(t_1), \zeta'(t_1), y'(t_1)) < 0$ *of Theorem 3.2 can be replaced by the following ones* $y(t_1) \in [\zeta(t_1), \gamma)$, $\nu(t_1, \eta(t_1), y(t_1), \eta'(t_1), y'(t_1)) \le 0$.

Let $a_2(t)$, $b_2(t)$, $c_2(t)$, $d_2(t)$ and $e_2(t)$ be real-valued continuous functions on $[t_0, \infty)$. We set $L_1(t, u, v) \equiv 3(a_2(t) - a(t))uv + (b_2(t) - b(t))v + (a_2^2(t) - a^2(t))u^3 + (c_2(t) - c(t))u^2 + (d_2(t) - d(t))u + e_2(t) - e(t)$, $t \ge t_0$. Consider the equation

$$y'' + 3a_2(t)yy' + b_2(t)y' + a_2^2(t)y^3 + c_2(t)y^2 + d_2(t)y + e_2(t) = 0, \qquad t \in [t_1, 2). \quad (3.15)$$

Theorem 3.3. *Let the following conditions be satisfied*
(4) If $a(t) = 0$, *then* $c(t) = \frac{3}{2}a'(t)$, $d(t) \ge b'(t)$, *otherwise* $D(t) \ge 0$, $t \in [t_1, t_2)$,
(5) Eq. (2.1) has a solution $y_1(t)$ *on* $[t_1, t_2)$ *such that* $\int\limits_{t_1}^{t} L(\tau, y_1(\tau), y_1'(\tau))d\tau \ge 0$, $t \in [t_1, t_2)$.
(6) Eq. (3.15) has a solution $y_2(t)$ *on* $[t_1, t_2)$ *such that* $y_1(t_1) \le y_2(t_1)$ *and* $\int\limits_{t_1}^{t} L_1(s, y_2(s), y_2'(s))ds \ge 0$, $t \in [t_1, t_2)$.
Then every solution $y(t)$ *of Eq. (1.1) with*
(7) $y_1(t_1) \le y(t_1) \le y_2(t_1)$,
(8) $\nu(t_1, y(t_1), y_1(t_1), y'(t_1), y_1'(t_1)) \ge 0$,
(9) $\nu(t_1, y(t_1), y_2(t_1), y'(t_1), y_2'(t_1)) \le 0$,

exists on $[t_1,t_2)$ *and*

$$y_1(t) \le y(t) \le y_2(t), \ \ [t_1,t_2), \tag{3.16}$$

$$\nu(t,y(t),y_1(t),y'(t),y_1'(t)) \ge 0, \qquad \nu(t,y(t),y_2(t),y'(t),y_2'(t)) \le 0, \qquad t\in[t_1,t_2). \tag{3.17}$$

Proof. Let $y(t)$ be a solution of Eq. (1.1) with $y_1(t_1) \le y(t_1) \le y_2(t_1)$, satisfying the initial conditions (8) and (9) and let $[t_1,t_3)$ be its maximum existence interval. We must show that

$$t_3 \ge t_2. \tag{3.18}$$

By (2.3) we have

$$y(t)-y_1(t) = \exp\Big\{-\int\limits_{t_1}^{t}\Big\{\frac{3}{2}a(\tau)(y(\tau)+y_1(\tau))+b(\tau)\Big\}d\tau\Big\}\times$$

$$\times\Bigg[y(t_1)-y_1(t_1)+\int\limits_{t_1}^{t}\exp\Big\{\int\limits_{t_1}^{\tau}\Big[\frac{3}{2}a(s)(y(s)+y_1(s))+b(s)\Big]ds\Big\}\times$$

$$\times\Bigg(\nu(t_1,y(t_1),y_1(t_1),y'(t_1),y_1'(t_1))(\tau-t_1)+\int\limits_{t_1}^{\tau}[J(s,y(s),y_1(s))+$$

$$+L(s,y_1(s),y_1'(s))]ds\Bigg)d\tau\Bigg], \ \ t\in[t_1,t_3),$$

$$y_2(t)-y(t) = \exp\Big\{-\int\limits_{t_1}^{t}\Big\{\frac{3}{2}a(\tau)(y_2(\tau)+y(\tau))+b(\tau)\Big\}d\tau\Big\}\times$$

$$\times\Bigg[y_2(t_1)-y(t_1)+\int\limits_{t_1}^{t}\exp\Big\{\int\limits_{t_1}^{\tau}\Big[\frac{3}{2}a(s)(y_2(s)+y(s))+b(s)\Big]ds\Big\}\times$$

$$\times\Bigg(\nu(t_1,y_2(t_1),y(t_1),y_2'(t_1),y'(t_1))(\tau-t_1)+\int\limits_{t_1}^{\tau}[J(s,y_2(s),y(s))-$$

$$-L_1(s,y_2(s),y_2'(s))]ds\Bigg)d\tau\Bigg], \ \ t\in[t_1,t_3).$$

Multiplying both sides of the obtained equalities by $\exp\Big\{\int\limits_{t_1}^{t}\Big[\frac{3}{2}a(s)(y(s)+y_1(s))\Big]ds\Big\}$ and $\exp\Big\{\int\limits_{t_1}^{t}\Big[\frac{3}{2}a(s)(y_2(s)+y(s))\Big]ds\Big\}$ respectively and changing order of integration of the right parts of the obtained relations we get

$$U_1(t) = F_1(t) + \int\limits_{t_1}^{t} V_1(t,\tau)U_1(\tau)d\tau, \ \ t \in [t_1, t_3), \tag{3.19}$$

$$U_2(t) = F_2(t) + \int\limits_{t_1}^{t} V_2(t,\tau)U_1(\tau)d\tau, \ \ t \in [t_1, t_3), \tag{3.20}$$

where

$$U_1(t) \equiv [y(t) - y_1(t)] \exp\biggl\{\int\limits_{t_1}^{t}\Bigl[\frac{3}{2}a(s)(y(s)+y_1(s))\Bigr]ds\biggr\},$$

$$U_2(t) \equiv [y_2(t) - y(t)] \exp\biggl\{\int\limits_{t_1}^{t}\Bigl[\frac{3}{2}a(s)(y_2(s)+y(s))\Bigr]ds\biggr\},$$

$$F_1(t) \equiv y(t_1) - y_1(t_1) + \int\limits_{t_1}^{t} \exp\biggl\{\int\limits_{t_1}^{\tau}\Bigl[\frac{3}{2}a(s)(y(s)+y_1(s)) + b(s)\Bigr]ds\biggr\}\times$$

$$\times\Bigl(\nu(t_1, y(t_1), y_1(t_1), y'(t_1), y_1'(t_1)(\tau - t_1) + \int\limits_{t_1}^{\tau} L(s, y_1(s), y_1'(s))ds\Bigr)$$

$$F_2(t) \equiv y_2(t_1) - y(t_1) + \int\limits_{t_1}^{t} \exp\biggl\{\int\limits_{t_1}^{\tau}\Bigl[\frac{3}{2}a(s)(y_2(s)+y(s)) + b(s)\Bigr]ds\biggr\}\times$$

$$\times\Bigl(\nu(t_1, y_2(t_1), y(t_1), y_2'(t_1), y'(t_1)(\tau - t_1) + \int\limits_{t_1}^{\tau} L(s, y_2(s), y_2'(s))ds\Bigr), \ \ t \in [t_1, t_3),$$

$$V_k(t,s) \equiv \int\limits_{s}^{t} \exp\biggl\{\int\limits_{s}^{\tau}\Bigl[\frac{3}{2}a(\xi)(y(\xi)+y_k(\xi)+b(\xi))\Bigr]d\xi\biggr\}\Gamma(\tau, y(\tau), y_k(\tau))d\tau, \ \ k = 1,2,$$

$t_1 \le s \le t \le t_3$. It follows from the condition (4) of the theorem that $\Gamma(t, y(t), y_k(t)) \ge 0$, $t \in [t_1, t_2), \ \ k = 1,2$. Hence,

$$V_k(s,t) \ge 0, \ \ k = 1,2, \ \ t_1 \le s \le t \le t_3. \tag{3.21}$$

It follows from the conditions (5) - (9) of the theorem that

$$F_k(t) \ge 0, \ \ t \in [t_1.t_3), \ \ k = 1,2. \tag{3.22}$$

Obviously, $U_k(t)$ is a solution of the Volterra integral equation

$$U(t) = F_k(t) + \int\limits_{t_1}^{t} V_k(t,\tau)U(\tau)d\tau, \ \ t \in [t_1, t_3)$$

on $[t_1, t_2)$, $k = 1, 2$. Then it follows from (3.21) and (3.22) that $U_k(t) \ge 0, \ t \in [t_1, t_3)$,
$k = 1, 2$. Therefore,

$$y_1(t) \le y(t) \le y_2(t), \ \ t \in [t_1, t_3). \tag{3.23}$$

Suppose $t_3 < t_2$. It follows from (3.23) that the function $y(t), \ t \in [t_1, t_3)$ is bounded on $[t_1, t_3)$. By virtue of Lemma 2.5 it follows from here that $[t_1, t_3)$ is not the maximum existence interval for $y(t)$, which contradicts our supposition. The obtained contradiction proves (3.18). From (3.18) and (3,23) it follows (3.6). By (2.2) we have

$$\nu(t, y(t), y_1(t), y'(t), y_1'(t)) = \nu(t_1, y(t_1), y_1(t_1), y'(t_1), y_1'(t_1)) +$$

$$+ \int\limits_{t_1}^{t} [J(\tau, y(\tau), y_1(\tau)) + L(\tau, y_1(\tau), y_1'(\tau))]d\tau, \qquad t \in [t_1, t_2), \tag{3.24}$$

$$\nu(t, y_2(t), y(t), y_2'(t), y'(t)) = \nu(t_1, y_2(t_1), y(t_1), y_2'(t_1), y'(t_1)) +$$

$$+ \int\limits_{t_1}^{t} [J(\tau, y_2(\tau), y(\tau)) + L_1(\tau, y_2(\tau), y_2'(\tau))]d\tau, \qquad t \in [t_1, t_2), \tag{3.25}$$

It follows from (4) and (3.16) that

$$J(t, y(t), y_1(t)) \ge 0, \qquad J(t, y_2(t), y(t)) \le 0, \qquad t \in [t_1, t_2).$$

These inequalities with (3.14), (3.25) and the conditions (5), (6) imply (3.17). The theorem is proved.

4. Global solvability criteria

Theorem 4.1. *Let the following conditions be satisfied*

(α) $a(t) > 0, \ D(t) \ge 0, \ t \ge t_0$,

(β) $M \equiv \sup\limits_{t \ge t_0} \frac{|c(t)| + |d(t)| + |e(t)|}{a^2(t)} < +\infty$,

(γ) $\int\limits_{t_0}^{t} \exp\Big\{\int\limits_{t_0}^{\tau}\Big[\frac{3}{2}Ma(s) + b(s)\Big]d\tau\Big\}\Bigg(\int\limits_{t_0}^{\tau} e(s)ds\Bigg)d\tau \le 0, \ \ t \ge t_0.$

Then every solution $y(t)$ *of Eq. (1.1) with* $y(t_0) \in [0, M]$, $\nu(t_0, M, y(t_0), 0, y'(t_0)) \ge 0$, $\nu(t_0, 0, y(t_0), 0, y'(t_0)) \le 0$ *exists on* $[t_0, \infty)$ *and*

$$0 \le y(t) \le M, \qquad t \ge t_0, \tag{4.1}$$

$$\nu(t, M, y(t), 0, y'(t)) \ge 0, \;\; t \ge t_0. \tag{4.2}$$

$$\nu(t, 0, y(t), 0, y'(t)) \le 0, \;\; t \ge t_0. \tag{4.3}$$

Proof. By Remark 2.1 it follows from the conditions (α) and (β) that $\eta(t) \equiv M$, $t \ge t_0$ is a solutions of the inequality (2.11) on $[t_0, \infty)$. We put $a_1(t) = a(t)$, $b_1(t) = b(t)$, $c_1(t) = c(t)$, $d_1(t) = d(t)$, $e_1(t) \equiv 0$, $t \ge t_0$. in Eq. (2.1). Then, obviously, $y_1(t) \equiv 0$ is a solution of Eq. (2.1) on $[t_0, \infty)$. Then by Theorem 3.1

$$\int\limits_{t_0}^{t} \exp\biggl\{\int\limits_{t_0}^{\tau}\Bigl[\frac{3}{2}Ma(s) + b(s)\Bigr]ds\biggr\}\biggl(\int\limits_{t_0}^{\tau} L(s,0,0)ds\biggr)d\tau \ge 0, \qquad t \ge t_0 \tag{4.11}$$

and the condition (α) holds, then every solution $y(t)$ of Eq. (1.1) with $y(t_0) \in [0, M]$, $\nu(t, M, y(t), 0, y'(t)) \ge 0$, $\nu(t, 0, y(t), 0, y'(t)) \le 0$ exists on $[t_0, \infty)$ and the inequalities (4.1)-(4.3) hold. Since $L(t, 0, 0) = -e(t)$, $t \ge t_0$ it follows from the condition (γ) the inequality (4.1). The theorem is proved.

Using Theorem 3.2 instead of Theorem 3.1 by analogy with the proof of Theorem 3.1 one can prove the following theorem

Theorem 4.2. *Let the following conditions be satisfied*
$a(t) > 0$, $D(t) \ge 0$, $t \ge t_0$, $M \equiv \sup\limits_{t \ge t_0} \frac{|c(t)|+|d(t)|+|e(t)|}{a^2(t)} < +\infty$,
$\int\limits_{t_0}^{t} \exp\biggl\{\int\limits_{t_0}^{\tau}\Bigl[\frac{3}{2}Ma(s) + b(s)\Bigr]d\tau\biggr\}\Biggl(\int\limits_{t_0}^{\tau} e(s)ds\Biggr)d\tau \le 0, \;\; t \ge t_0.$
Then every solution $y(t)$ *of Eq. (1.1) with* $y(t_0) \in [-M, 0]$, $\nu(t_0, M, y(t_0), 0, y'(t_0)) \le 0$, $\nu(t_0, 0, y(t_0), 0, y'(t_0)) \ge 0$ *exists on* $[t_0, \infty)$ *and*

$$-M \le y(t) \le 0, \;\; \nu(t, M, y(t), 0, y'(t)) \le 0, \;\; \nu(t, 0, y(t), 0, y'(t)) \ge 0, \;\; t \ge t_0.$$

■

Let $t_0 < t_1 \ldots < t_n < \ldots$ be an infinitely large sequence. We set

$$M_n \equiv \max_{t \in [t_0, t_n]} \frac{|c(t)| + |d(t)| = |e(t)|}{a^2(t)|}, \qquad n = 1, 2, \ldots.$$

Theorem 4.3 *Let the following conditions be satisfied.*
(α) $a(t) > 0$, $D(t) \ge 0$, $t \ge t_0$,

(δ) $\int\limits_{t_n}^{t} \exp\biggl\{\int\limits_{t_n}^{\tau}\Bigl[\frac{3}{2}a(s)M_{n+1}+b(s)\Bigr]ds\biggr\}\biggl(\int\limits_{t_n}^{\tau} e(s)ds\biggr)d\tau \le 0, \ \ t\in[t_n,t_{n+1}), \ \ n=0,1,2,\dots.$

Then every solution $y(t)$ *of Eq. (1.1) with* $y(t_0)\in[0,M_1]$, $\nu(t_0,y(t_0),M_1,y'(t_0),0)\le 0$, $\nu(t_0,y(t_0),0,y'(t_0),0)\ge 0$ *exists on* $[t_0,\infty)$ *and*

$$0\le y(t)\le M_n, \qquad t\in[t_{n-1},t_n], \tag{4.5}$$

$$\nu(t,y(t),M_n,y'(t),0)\le 0, \ \ \nu(t,M_n,y(t),0,y'(t))\ge 0, \ \ t\in[t_{n-1},t_n], \ \ n=1,2,\dots. \tag{4.6}$$

Proof. According to Remark 2.1 $\eta_n(t)\equiv M_n$, $t\in[t_{n-1},t_n]$ is a solution of the inequality (2.11) on $[t_{n-1},t_n]$, $n=1,2,\dots$. We put $a_1(t)=a(t)$, $b_1(t)=b(t)$, $c_1(t)=c(t)$, $d_1(t)=d(t)$, $e_1(t)\equiv 0$, $t\ge t_0$ in Eq. (2.1). In this case, obviously, $y_1(t)\equiv 0$ is a solution of Eq. (2.1) on $[t_0,\infty)$. Then by Theorem 3.1 if

$$\int\limits_{t_n}^{t}\exp\biggl\{\int\limits_{t_n}^{\tau}\Bigl[\frac{3}{2}a(s)M_{n+1}=b(s)\Bigr]ds\biggr\}\biggl(\int\limits_{t_n}^{\tau}L(s,0,0)ds\biggr)d\tau\ge 0, \ \ t\in[t_n,t_{n+1}], \tag{4.7}$$

$n=0,1,2,\dots$ every solution $y_n(t)$ of Eq. (1.1) with $y_n(t_{n-1})\in[0,M_n]$, $\nu(t_{n-1},y_n(t_{n-1}),0,y_n'(t_{n-1}),0)\ge 0$, $\nu(t_{n-1},M_n,y_n(t_{n-1}),0,y_n'(t_{n-1}),)\ge 0$ exists on $[t_{n-1},t_n]$ and

$$0\le y_n(t)\le M_n, \qquad t\in[t_{n-1},t_n], \ \ n=1,2,\dots. \tag{4.8}$$

$$\nu(t,y_n(t),0,y_n'(t),0)\ge 0, \ \ \nu(t,M_n,y_n(t),0)\ge 0, \ \ t\in[t_{n-1},t_n], \ \ n=1,2,\dots. \tag{4.9}$$

Since $M_1\le M_2\le\dots$ according to (4.8) we can take that $y_{n+1}(t_n)=y_n(t_n)$, $y_{n+1}'(t_n)=y_n'(t_n)$, $n=1,2,\dots$. Note that according to (4.9) in this particular case of chose of initial conditions the relations (initial conditions) $\nu(t_{n-1},M_n,y_{n-1}(t_{n-1}),0,y_n'(t_{n-1}))\ge 0$,
$\nu(t_{n-1},y_{n-1}(t_{n-1}),0,y_n'(t_{n-1}),0)\ge 0$, $n=2,3\dots$ remain valid. Hence the function

$$y(t)=y_n(t), t\in[t_{n-1},t_n], \qquad n=1,2,\dots$$

is a solution of Eq. (1.1) on $[t_0,\infty)$, satisfying the initial conditions $y(t_0)\in[0,M_1]$, $\nu(t_0,M_n,y(t_0),0,y'(t_{n-1}))\ge 0$, $\nu(t_0,y(t_0),M,y'(t_0),0)\le 0$, for which according to (4.8) and (4.9) the inequalities hold. Then since $L(t,y_1(t),y_1'(t))=L(t,0,0)=-e(t)$, $t\ge t_0$ the inequality (4.7) follows from the condition (δ). The theorem is proved.

Using Theorem 3.2 instead of Theorem 3.1 by analogy with the proof of Theorem 4.3 can be proved the following theorem.

Theorem 4.4. *Let the following conditions be satisfied*
$a(t)<0$, $D(t)\ge 0$, $t\ge t_0$,
$\int\limits_{t_n}^{t}\exp\biggl\{-\int\limits_{t_n}^{\tau}\Bigl[\frac{3}{2}a(s)M_{n+1}+b(s)\Bigr]ds\biggr\}\biggl(\int\limits_{t_n}^{\tau}e(s)ds\biggr)d\tau\ge 0, \ \ t\in[t_n,t_{n-1}), \ \ n=0,1,2,\dots.$

Then every solution $y(t)$ *of Eq. (1.1) with* $y(t)\in[-M_1,0]$, $\nu(t_0,y(t_0),0,y'(t_0))\le 0$, $\nu(t_0,-M_1,y(t_0),0,y'(t_0))\le 0$ *exists on* $[t_0,\infty)$ *and*

$$-M_n\le y(t)\le 0,\qquad \nu(t,y(t),0,y'(t),0)\le 0,\qquad \nu(t,-M_n,y(t),0y'(t))\le 0,\ \ t\in[t_{n-1},t_n],$$

$n=1,2,\ldots.$

■

We set

$$\rho_\pm(t)\equiv\frac{-[sgn\, c(t)]d(t)\pm\sqrt{d^2(t)-4c(t)e(t)}}{2|c(t)|},\qquad t\ge t_0.$$

Theorem 4.5. *Let the conditions (4) of Theorem 3.3 and the conditions*
(10) $c(t)\ne 0$, $d^2(t)-4c(t)e(t)\ge 0$, $\rho_\pm(t)\in C^2([t_0,\infty))$, $t\ge t_0$,
(11) $\int\limits_{t_0}^{t}[\rho''_-(\tau)+3a(\tau)\rho_-(\tau)\rho'_-(\tau)+a^2(\tau)\rho^3_-(\tau)]d\tau\le 0$, $t\ge t_0$,
(12) $\int\limits_{t_0}^{t}[\rho''_+(\tau)+3a(\tau)\rho_+(\tau)\rho'_+(\tau)+a^2(\tau)\rho^3_+(\tau)]d\tau\ge 0$, $t\ge t_0$
be satisfied.
Then every solution $y(t)$ *of Eq. (1.1) with the initial conditions* $y(t_0)\in[\rho_-(t_0),\rho_+(t_0)]$, $\nu(t_0,y(t_0),\rho_-(t_0),y'(t_0),\rho'_-(t_0))\ge 0$, $\nu(t_0,y(t_0),\rho_+(t_0),y'(t_0),\rho'_+(t_0))\le 0$ *exists on* $[t_0,\infty)$ *and*

$$\rho_-(t)\le y(t)\le\rho_+(t),\ \ t\ge t_0, \tag{4.10}$$

$$\nu(t,y(t),\rho_-(t),y'(t),\rho'_-(t))\ge 0,\qquad \nu(t,y(t),\rho_+(t),y'(t),\rho'_+(t))\le 0,\ \ t\ge t_0. \tag{4.11}$$

Proof. We set $a_1(t)=a_2(t)=a(t)$, $b_1(t)=b_2(t)=b(t)$, $c_1(t)=c_2(t)=c(t)$, $d_1(t)=d_2(t)=d(t)$, $e_1(t)=e(t)-\rho''_-(t)-3a(t)\rho_-(t)\rho'_-(t)-a^2(t)\rho^3_-(t)$, $e_2(t)=e(t)-\rho''_+(t)-3a(t)\rho_+(t)\rho'_+(t)-a^2(t)\rho^3_+(t)$, $t\ge t_0$. Then it is not difficult to verify that $\rho_-(t)$ is a solution of Eq. (2.1) on $[t_0,\infty)$ and $\rho_+(t)$ is a solution of Eq. (3.15) on $[t_0,\infty)$ (obviously $\rho_-(t)\le\rho_+(t)$, $t\ge t_0$). Then the conditions (4), (10)-(12) provide the satisfiability all of the conditions of Theorem 3.3. Hence, every solution $y(t)$ of Eq. (1.1) with the initial conditions $y(t_0)\in[\rho_-(t_0),\rho_+(t_0)]$, $\nu(t_0,y(t_0),\rho_-(t_0),y'(t_0),\rho'_-(t_0))\ge 0,\nu(t_0,y(t_0),\rho_+(t_0),y'(t_0),\rho'_+(t_0))\le 0$ exists on $[t_0,\infty)$ and the relations (4.10) and (4.11) are fulfilled. The theorem is proved.

5. An application to three dimensional linear systems of ordinary differential equations

Let $a_{jk}(t)$, $j,k=\overline{1,3}$ be real-valued continuous functions on $[t_0,\infty)$. Consider the linear system of ordinary differential equations

$$\begin{cases} \phi' = a_{11}(t)\phi + a_{12}(t)\psi + a_{13}(t)\chi, \\ \psi' = a_{21}(t)\phi + a_{22}(t)\psi + a_{23}(t)\chi, \\ \chi' = a_{31}(t)\phi + a_{32}(t)\psi + a_{33}(t)\chi, \ \ t \ge t_0. \end{cases} \tag{5.1}$$

Assume

$$a_{13}(t) \equiv 0, \qquad t \ge t_0. \tag{5.2}$$

Let us substitute

$$\psi = y\phi \tag{5.3}$$

in the system (5.1). We obtain

$$\begin{cases} \phi' = [a_{11}(t) + a_{12}(t)y], \\ y\phi' = [a_{21}(t) - y' + a_{22}(t)y]\phi + a_{23}(t)\chi, \\ \chi' = [a_{31}(t) + a_{32}(t)y]\phi + a_{33}(t)\chi, \ \ t \ge t_0. \end{cases} \tag{5.4}$$

From the first and second equations of the last system we obtain

$$[y' + a_{12}(t)y^2 + B(t)y - a_{21}(t)]\phi = a_{23}(t)\chi, \qquad t \ge t_0, \tag{5.5}$$

where $B(t) \equiv a_{11}(t) - a_{22}(t), \ \ t \ge t_0$. Assume

$$a_{23}(t) \ne 0, \qquad t \ge t_0. \tag{5.6}$$

Then from (5.5) we get

$$\chi = \frac{y' + a_{12}(t)y^2 + B(t)y - a_{21}(t)}{a_{23}(t)}\phi, \qquad t \ge t_0. \tag{5.7}$$

This together with the first and the third equations of the system (5.4) implies

$$\left\{\frac{y' + a_{12}(t)y^2 + B(t)y - a_{21}(t)}{a_{23}(t)}\right\}'\phi +$$

$$+\left\{\frac{y' + a_{12}(t)y^2 + B(t)y - a_{21}(t)}{a_{23}(t)}\right\}[a_{11}(t) + a_{12}(t)y]\phi =$$

$$= \left[a_{31}(t) + a_{32}(t)y + a_{33}(t)\frac{y' + a_{12}(t)y^2 + B(t)y - a_{21}(t)}{a_{23}(t)}\right]\phi, \ \ t \ge t_0.$$

By (5.3) and (5.7) it follows from here that under the restrictions (5.2), (5.6) all solutions $y(t)$ of the equation

$$\left\{\frac{y' + a_{12}(t)y^2 + B(t)y - a_{21}(t)}{a_{23}(t)}\right\}' +$$

$$+[a_{11}(t) - a_{33}(t) + a_{12}(t)y]\left\{\frac{y' + a_{12}(t)y^2 + B(t)y - a_{21}(t)}{a_{23}(t)}\right\} - a_{32}(t)y - a_{31}(t) = 0, \tag{5.8}$$

$t \geq t_0$, existing on any interval $[t_1.t_2) \subset [t_0, \infty)$, are connected with solutions $(\phi(t), \psi(t), \chi(t))$ of the system (5.1) by relations

$$\phi(t) = \phi(t_1)\exp\left\{\int\limits_{t_1}^{t}[a_{11}(\tau) + a_{12}(\tau)y(\tau)]d\tau\right\}, \ \ \phi(t_1) \neq 0, \ \ t \in [t_1.t_2), \tag{5.9}$$

$$\psi(t) = y(t)\phi(t), \ \ \chi(t) = \frac{y'(t) + a_{12}(t)y^2(t) + B(t)y(t) - a_{21}(t)}{a_{23}(t)}\phi(t), \ \ t \in [t_1.t_2). \tag{5.10}$$

Assume

$$B(t), \ \ a_{12}(t), \ \ a_{23}(t) \in C^1([t_0, \infty)). \tag{5.11}$$

Then making some differentiations and arithmetic simplifications in (5.8) we obtain

$$y'' + 3a_{12}(t)yy' + A(t)y' + a_{12}^2(t)y^3 + C(t)y^2 + F(t)y + E(t) = 0, \qquad t \geq t_0, \tag{5.12}$$

where

$$A(t) \equiv 2a_{11}(t) - a_{22}(t) - a_{33}(t) - \frac{a_{23}'(t)}{a_{23}(t)},$$

$$C(t) \equiv a_{12}(t)[2a_{11}(t) - a_{22}(t) - a_{33}(t)] + a_{23}(t)\left(\frac{a_{12}(t)}{a_{23}(t)}\right)',$$

$$F(t) \equiv [a_{11}(t) - a_{22}(t)][a_{11}(t) - a_{33}(t)] - a_{23}(t)a_{32}(t) + a_{23}(t)\left(\frac{a_{11}(t) - a_{22}(t)}{a_{23}(t)}\right)',$$

$$E(t) \equiv [a_{33}(t) - a_{11}(t)]a_{21}(t) - a_{23}(t)a_{31}(t) - a_{23}(t)\left(\frac{a_{21}(t)}{a_{23}(t)}\right)', \qquad t \geq t_0.$$

Remark 5.1. *If $\frac{a_{13}(t)}{a_{12}(t)}$ is well defined on $[t_0, \infty)$ and $\in C^1([t_0, \infty))$ (that is $a_{13}(t) = a_{12}(t)\lambda(t), \ \ \lambda(t) \in C^1([t_0, \infty))$), then the linear transformation*

$$\psi = \eta - \lambda(t)\chi$$

reduces the general system (5.1) to its particular case of (5.2) (of $a_{13}(t) \equiv 0, \ \ t \geq t_0$.)

Definition 5.1. *The system (5.1) is called non oscillatory, if it has a solution* $(\phi(t), \psi(t), \chi(t))$ *such that* $\phi(t) \neq 0, \ \ t \geq t_0$.

By (5.9) from the results of the section 4 we obtain

Theorem 5.1. *Let the conditions (5.2), (5.6) and (5.11) be satisfied and let for* $a(t) = \ a_{12}(t), \ b(t) = A(t), \ c(t) = C(t), \ d(t) = D(t), \ e(t) = E(t), \ t \geq t_0$ *the conditions of one of Theorems 4.1 - 4.5 be satisfied. Then the system (5.1) is non oscillatory.*

References

[1] 1. G. A. Grigorian, On two comparison tests for second-order linear ordinary differential equations, *Diff. Urav.*, **47** (2011), 1225–1240. (in Russian), Diff. Eq., vol. 47, (2011), 1237–1252 (in English).
https://doi.org/10.1134/s0012266111090023

[2] G. A. Grigorian, Two comparison criteria for scalar Riccati equations and some of their applications Izv. Vyssh. Uchebn. Zaved. Mat.(2012), no. 11, 20-35. *Russian Math.*, (Iz. VUZ) **56** (2012), no. 11, 17-30.
https://doi.org/10.3103/s1066369x12110023

[3] G. A. Grigorian, Interval oscillation criteria for linear matrix Hamiltonian systems, *Rocky Mountain J. Math.*, **50** (2020), no. 6, 2047-2057.
https://doi.org/10.1216/rmj.2020.50.2047

[4] G. A. Grigorian, New reducibility criteria for systems of two linear first-order ordinary differential equations, *Monatsh. Math.*, **198** (2022), no. 2, 311-322.
https://doi.org/10.1007/s00605-022-01671-8

[5] G. A. Grigorian, Oscillation and non-oscillation criteria for linear nonhomogeneous systems of two first-order ordinary differential equations, *J. Math. Anal. Appl.*, **507** (2022), no. 1, Paper No. 125734.
https://doi.org/10.1016/j.jmaa.2021.125734

[6] G. A. Grigorian, Oscillatory and non-oscillatory criteria for linear four-dimensional Hamiltonian systems, *Math. Bohem.*, **146** (2021), no. 3, 289-304.
https://doi.org/10.21136/mb.2020.0149-19

[7] G. A. Grigorian, On the reducibility of systems of two linear first-order ordinary differential equations, *Monatsh. Math.*, **195** (2021), no. 1, 107-117.
https://doi.org/10.1007/s00605-020-01503-7

[8] G. A. Grigorian, Oscillation criteria for linear matrix Hamiltonian systems, *Proc. Amer. Math. Soc.*, **148** (2020), no. 8, 3407-3415.
https://doi.org/10.1090/proc/14973

[9] G. A. Grigorian, Oscillatory and non oscillatory criteria for the systems of two linear first order two by two dimensional matrix ordinary differential equations, *Arch. Math. (Brno)*, **54** (2018), no. 4, 189-203. https://doi.org/10.5817/am2018-4-189

[10] G. A. Grigorian, Stability criterion for systems of two first-order linear ordinary differential equations, *Mat. Zametki*, **103** (2018), no. 6, 831-840. Math. Notes 103 (2018), no. 5-6, 892-900. https://doi.org/10.1134/s0001434618050255

[11] G. A. Grigorian, Oscillatory criteria for the systems of two first-order linear differential equations, *Rocky Mountain J. Math.*, **47** (2017), no. 5, 1497-1524. https://doi.org/10.1216/rmj-2017-47-5-1497

[12] G. A. Grigorian, Some properties of the solutions of third order linear ordinary differential equations, *Rocky Mountain J. Math.*, **46** (2016), no. 1, 147-168. https://doi.org/10.1216/rmj-2016-46-1-147

[13] G. A. Grigorian, Stability criteria for systems of two first-order linear ordinary differential equations, *Math. Slovaca*, **72** (2022), no. 4, 935-944. https://doi.org/10.1515/ms-2022-0064

[14] B. P. Demidovich, *Lectures on the Mathematical Theory of Stability*, Moskow, "Nauka", 1967.

[15] R. E. Edvards, *A Formal Background to Mathematics*, Springer-Verlag, New York, Heidelberg, Berlin, 1980. MR0636095

Received: January 17, 2025; Published: January 30, 2025