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A Generalized $\mathbf{L_1}$-Analytic Feynman Integral Over Paths in Abstract Wiener Space

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Abstract

Let $C^{\mathbb{B}}[a,b]$ denote an analogue of Wiener space over paths in abstract Wiener space $\mathbb{B}$, the space of $\mathbb{B}$-valued continuous functions on $[a,b]$. In this paper, we introduce a positive finite measure on $C^{\mathbb{B}}[a,b]$ with a scale and arbitrary variance function. We then introduce a generalized analytic Feynman integral for the functions on $C^{\mathbb{B}}[a,b]$. With regards to the definition, we establish the generalized analytic Feynman integral of the product of the cylinder functions and the functions in a Banach algebra corresponding to the Fresnel class. We note that the established analytic Feynman integrals are of interest in quantum mechanics, especially in Feynman integration theory.

Mathematics Subject Classification: 28C20, 60G15

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1 Introduction

The Feynman integral was introduced in 1948 [7] and this integral has long had a major influence in several areas in mathematics and mathematical physics. One approach to the Feynman integral is via analytic continuation from the Wiener integral on $C_0[0,T]$ which is the classical Wiener space, that is, the space of continuous real-valued functions x on $[0,T]$ with $x(0) = 0$ [1, 2, 3, 4, 14]. An important mathematical problem of evaluating this Feynman integral arises quite naturally in elementary-particle physics, especially, in the Brownian motion.

As another approach, Kuelbs and Lepage [10] mentioned an existence of a non-zero, stationary increment Gaussian measure $w_{a,b}^{\mathbb{B}}$ over paths in abstract Wiener space $C_0^{\mathbb{B}}[a,b]$, the space of $\mathbb{B}$-valued continuous functions on $[a,b]$ that vanish at a, where $\mathbb{B}$ is an abstract Wiener space [11]. Ryu [12] established the concrete form of $w_{a,b}^{\mathbb{B}}$ and gave an integration formula for it, as an extension of the Wiener integration formula [14]. Moreover, in [13], he introduced an analogue of Wiener measure $w_{a,b;\varphi_a}^{\mathbb{B}}$ with an initial distribution φ_a over paths in abstract Wiener space $C^{\mathbb{B}}[a,b]$ which is the space of $\mathbb{B}$-valued continuous functions on $[a,b]$, where φ_a is a positive finite measure on the Borel class $\mathcal{B}(\mathbb{B})$ of $\mathbb{B}$. Taking $\varphi_a = \delta_0$ which is the Dirac measure at $0 \in \mathbb{B}$, he also proved that $w_{a,b;\varphi_a}^{\mathbb{B}}$ can be reduced to $w_{a,b}^{\mathbb{B}}$. In other words, $w_{a,b}^{\mathbb{B}}$ on $C_0^{\mathbb{B}}[a,b]$ can be extended to $w_{a,b;\varphi_a}^{\mathbb{B}}$ on $C^{\mathbb{B}}[a,b]$.

In this paper, we introduce a positive finite measure on $C^{\mathbb{B}}[a,b]$ with a scale and arbitrary variance function, which generalizes the measures in [12, 13]. We then introduce a generalized L_1-analytic Feynman integral for the functions on $C^{\mathbb{B}}[a,b]$. With regards to the definition, we establish the generalized L_1-analytic Feynman integral of the product of the cylinder functions and the functions in a Banach algebra corresponding to the Fresnel class in [9]. We note that the established analytic Feynman integrals are of interest in quantum mechanics, especially in Feynman integration theory. We also note that while every path in $C_0^{\mathbb{B}}[a,b]$ starts at $0 (\in \mathbb{B})$, the paths in $C^{\mathbb{B}}[a,b]$ need not start at 0. Furthermore, the measure used in this paper has a scale and need not be a probability measure so that this paper serves to generalize the related results on $\mathbb{B}$, $C_0[0,T]$, $C_0^{\mathbb{B}}[a,b]$ and $C^{\mathbb{B}}[a,b]$ [1, 2, 3, 4, 9, 12, 13].

2 An analogue of Wiener measure

In this section, we introduce a generalized analogue of Wiener measure over paths in abstract Wiener space as our underlying measure of this paper.

Let $(\mathbb{B}, \mathcal{B}(\mathbb{B}), \mu)$ be an abstract Wiener space with the norm $\|\cdot\|_{\mathbb{B}}$ (see [11]). It is well-known that $\mathbb{B}$ is an infinite dimensional real separable Banach space and there exists an infinite dimensional real separable Hilbert space $\mathcal{H}$ satisfying $\mathbb{B}^* \subseteq \mathcal{H}^* \cong \mathcal{H} \subseteq \mathbb{B}$, where $\mathbb{B}^*$ and $\mathcal{H}^*$ denote the dual spaces of $\mathbb{B}$ and $\mathcal{H}$, respectively. The symbol $(\cdot,\cdot)_1$ will denote the inner product on $\mathcal{H}$ with the norm $|\cdot|_1$ and $(\cdot,\cdot)_2$ will be the dual paring between $\mathbb{B}$ and $\mathbb{B}^*$ with the norm $|\cdot|_2$. We note that for $h \in \mathbb{B}^*$, $|h|_1 = |h|_2$ by the Hahn-Banach theorem, but $|\cdot|_1$ need not be a restriction of the norm $\|\cdot\|_{\mathbb{B}}$ on $\mathbb{B}$, that is, $|\cdot|_1$ need not be a measurable (semi-) norm on $\mathcal{H}$. Let φ_a be a finite positive measure on $\mathcal{B}(\mathbb{B})$ which is the Borel class of $\mathbb{B}$. For a positive real ρ, let φ_a^ρ and μ_ρ be the measures defined by

$$\mu_\rho(B) = \mu(\rho^{-1}B) \text{ and } \varphi_a^\rho(B) = \varphi_a(\rho^{-1}B)$$

for $B \in \mathcal{B}(\mathbb{B})$. Let $C^{\mathbb{B}}[a,b]$ denote the space of $\mathbb{B}$-valued continuous functions on $[a,b]$. Then $C^{\mathbb{B}}[a,b]$ is a real separable Banach space with the supremum norm $\|x\|_C = \sup_{a\le t\le b} \|x(t)\|_{\mathbb{B}}$ for $x \in C^{\mathbb{B}}[a,b]$. Let $\beta : [a,b] \to \mathbb{R}$ be a strictly increasing function. For any $\vec{t_k} = (t_0, t_1, \ldots, t_k)$ with $a = t_0 < t_1 < \cdots < t_k \le b$, let $T_{\vec{t_k},\beta} : \mathbb{B}^{k+1} \to \mathbb{B}^{k+1}$ and $J_{\vec{t_k}} : C^{\mathbb{B}}[a,b] \to \mathbb{B}^{k+1}$ be the functions given by

$$\begin{aligned} & T_{\vec{t_k},\beta}(x_0, x_1, \ldots, x_k) \\ = & \left(x_0, x_0 + \sqrt{\beta(t_1) - \beta(t_0)}x_1, \ldots, x_0 + \sum_{j=1}^{k} \sqrt{\beta(t_j) - \beta(t_{j-1})}x_j \right) \end{aligned}$$

for $(x_0, x_1, \ldots, x_k) \in \mathbb{B}^{k+1}$ and

$$J_{\vec{t_k}}(x) = (x(t_0), x(t_1), \ldots, x(t_k))$$

for $x \in C^{\mathbb{B}}[a,b]$. We note that $J_{\vec{t_k}}$ is continuous on $C^{\mathbb{B}}[a,b]$ since each component function $x(t_j)$, $j = 0, 1, \ldots, k$, is continuous on $C^{\mathbb{B}}[a,b]$. For $B_j \in \mathcal{B}(\mathbb{B})$, $j = 0, 1, \ldots, k$, the subset $J_{\vec{t_k}}^{-1}(\prod_{j=0}^k B_j)$ of $C^{\mathbb{B}}[a,b]$ is called an interval I^* and let $\mathcal{I}$ be the set of all intervals I^*. We note that $\mathcal{I}$ is an algebra. Define a premeasure $m_{a,b;\varphi_a}^{\rho,\beta,\mathbb{B}}$ on $\mathcal{I}$ by

$$\begin{aligned} & m_{a,b;\varphi_a}^{\rho,\beta,\mathbb{B}} \left[J_{\vec{t_k}}^{-1} \left(\prod_{j=0}^{k} B_j \right) \right] \\ = & \int_{\mathbb{B}^{k+1}} \chi_{\prod_{j=0}^k B_j}(T_{\vec{t_k},\beta}(x_0, x_1, \ldots, x_k)) d\mu_\rho^k(x_1, \ldots, x_k) d\varphi_a^\rho(x_0). \qquad (1) \end{aligned}$$

Using arguments used in the proof of [13, Theorem 2.1], we can show that $m_{a,b;\varphi_a}^{\rho,\beta,\mathbb{B}}$ is well-defined on $\mathcal{I}$, but its proof is long and tedious. So, we omit the proof.

Lemma 2.1 *The premeasure $m_{a,b;\varphi_a}^{\rho,\beta,\mathbb{B}}$ is countably additive on the algebra $\mathcal{I}$.*

Proof. Take intervals I^* and I^{**} in $\mathcal{I}$. For $B_j \in \mathcal{B}(\mathbb{B})$, $j = 0, 1, \ldots, k$ and $B'_p \in \mathcal{B}(\mathbb{B})$, $p = 0, 1, \ldots, q$, suppose that

$$I^* = J_{\vec{t_k}}^{-1}\left(\prod_{j=0}^{k} B_j \right) \text{ and } I^{**} = J_{\vec{s_q}}^{-1}\left(\prod_{p=0}^{q} B'_p \right),$$

where $\vec{t_k} = (t_0, t_1, \ldots, t_k)$ with $a = t_0 < t_1 < \cdots < t_k \le b$ and $\vec{s_q} = (s_0, s_1, \ldots, s_q)$ with $a = s_0 < s_1 < \cdots < s_q \le b$. Suppose $t_j \notin \{s_0, s_1, \ldots, s_q\}$

for some $j \in \{0, 1, \ldots, k\}$. Let $s' = t_j$ and $B_{s'} = \mathbb{B}$. Then $I^{**} = \{x \in C^{\mathbb{B}}[a,b] : x(s') \in B_{s'}\} \cap J_{\vec{s}_q}^{-1}(\prod_{p=0}^{q} B'_p)$. From this fact, we can assume that $k = q$ and $t_j = s_j$ for all $j \in \{0, 1, \ldots, k\}$. We note that $I^* = J_{\vec{t}_k}^{-1}(\prod_{j=0}^{k} B_j)$ and $I^{**} = J_{\vec{s}_q}^{-1}(\prod_{p=0}^{q} B'_p)$ need not be expressed by minimal numbers of the B_js and B'_ps, respectively. Since the premeasure $m_{a,b;\varphi_a}^{\rho,\beta,\mathbb{B}}$ is defined by (1) which is an integral, it suffices to prove that if $I^* \cap I^{**} = \emptyset$, then $\prod_{j=0}^{k} B_j \cap \prod_{j=0}^{k} B'_j = \emptyset$, or equivalently, if $\prod_{j=0}^{k} B_j \cap \prod_{j=0}^{k} B'_j \neq \emptyset$, then $I^* \cap I^{**} \neq \emptyset$. Now, suppose that $\prod_{j=0}^{k} B_j \cap \prod_{j=0}^{k} B'_j \neq \emptyset$, and take $(x_0, x_1, \ldots, x_k) \in \prod_{j=0}^{k} B_j \cap \prod_{j=0}^{k} B'_j$. Define $K : \mathbb{B}^{k+1} \to C^{\mathbb{B}}[a,b]$ by

$$\begin{aligned} K(x_0, x_1, \ldots, x_k)(t) &= \chi_{\{a\}}(t)x_0 + \sum_{j=1}^{k} \chi_{(t_{j-1},t_j]}(t)\Big[x_{j-1} \\ &\quad + \frac{t - t_{j-1}}{t_j - t_{j-1}}(x_j - x_{j-1})\Big] + \chi_{(t_k,b]}(t)x_k \qquad (2) \end{aligned}$$

for $t \in [a,b]$. We note that K is well-defined, that is, K is continuous on $[a,b]$ since it is the polygonal function connecting the points $(t_0, x(t_0)), (t_1, x(t_1)), \ldots, (t_k, x(t_k))$. Then we have

$$(J_{\vec{t}_k} \circ K)(x_0, x_1, \ldots, x_k) = (x_0, x_1, \ldots, x_k) \qquad (3)$$

so that $K(x_0, x_1, \ldots, x_k) \in I^* \cap I^{**}$ which implies $I^* \cap I^{**} \neq \emptyset$. Now, the proof is complete. $\square$

Using Lemma 2.1 and similar arguments in the proof of [11, Theorem 3.3, p.47], we can show that the Borel σ-algebra $\mathcal{B}(C^{\mathbb{B}}[a,b])$ of $C^{\mathbb{B}}[a,b]$ with the norm $\|\cdot\|_C$, coincides with the σ-algebra generated by $\mathcal{I}$. By [8, Theorem 1.14], there exists a unique positive finite measure $w_{a,b;\varphi_a}^{\rho,\beta,\mathbb{B}}$ on $\mathcal{B}(C^{\mathbb{B}}[a,b])$ with $w_{a,b;\varphi_a}^{\rho,\beta,\mathbb{B}}(I^*) = m_{a,b;\varphi_a}^{\rho,\beta,\mathbb{B}}(I^*)$ for all $I^* \in \mathcal{I}$. This measure $w_{a,b;\varphi_a}^{\rho,\beta,\mathbb{B}}$ is called a generalized analogue of Wiener measure over paths in abstract Wiener space according to β, φ_a and ρ.

We now have the following theorem.

Theorem 2.2 *Let $f : \mathbb{B}^{k+1} \to \mathbb{C}$ be a function. Then f is Borel measurable on $\mathbb{B}^{k+1}$ if and only if $f \circ J_{\vec{t}_k}$ is Borel measurable on $C^{\mathbb{B}}[a,b]$. If either of the equivalent conditions just stated hold, we have*

$$\begin{aligned} &\int_{C^{\mathbb{B}}[a,b]} f(J_{\vec{t}_k}(x))\,dw_{a,b;\varphi_a}^{\rho,\beta,\mathbb{B}}(x) \\ \overset{*}{=} &\int_{\mathbb{B}^{k+1}} (f \circ T_{\vec{t}_k,\beta})(x_0, x_1, \ldots, x_k)\,d\mu_\rho^k(x_1, \ldots, x_k)\,d\varphi_a^\rho(x_0), \qquad (4) \end{aligned}$$

where $\overset{*}{=}$ means that if either side exists, then both sides exist and they are equal.

Proof. Let $K:\mathbb{B}^{k+1}\to C^{\mathbb{B}}[a,b]$ be the function given by (2). From (3), we have $K^{-1}[J_{\vec{t}_k}^{-1}(B)]=B$ for any subset B of $\mathbb{B}^{k+1}$. Suppose that $f\circ J_{\vec{t}_k}$ is Borel measurable on $C^{\mathbb{B}}[a,b]$. Then for a Borel measurable subset B_1 of $\mathbb{C}$, we have $J_{\vec{t}_k}^{-1}[f^{-1}(B_1)]\in\mathcal{B}(C^{\mathbb{B}}[a,b])$. Replacing B with $f^{-1}(B_1)$, we have $f^{-1}(B_1)=K^{-1}[J_{\vec{t}_k}^{-1}[f^{-1}(B_1)]]\in\mathcal{B}(\mathbb{B}^{k+1})$ since K is continuous on $\mathbb{B}^{k+1}$. The converse follows immediately from the continuity of $J_{\vec{t}_k}$. To prove (4), suppose that f is Borel measurable on $\mathbb{B}^{k+1}$. We note that $\mathcal{B}(\mathbb{B}^{k+1})=\prod_{j=0}^{k}\mathcal{B}(\mathbb{B})$ since $\mathbb{B}$ is a separable Banach space. Hence if $f=\chi_B$ for any $B\in\mathcal{B}(\mathbb{B}^{k+1})$, then we have $f=\chi_{\prod_{j=0}^{k}B_j}$ for some $B_j\in\mathcal{B}(\mathbb{B})$, $j=0,1,\ldots,k$. Now, (4) follows from (1) if f is a characteristic function since $w_{a,b;\varphi_a}^{\rho,\beta,\mathbb{B}}=m_{a,b;\varphi_a}^{\rho,\beta,\mathbb{B}}$ on $\mathcal{I}$. Hence (4) follows immediately if f is a simple function since f is a finite linear combination of characteristic functions. Let $\nu_{\vec{t}_k}=w_{a,b;\varphi_a}^{\rho,\beta,\mathbb{B}}\circ J_{\vec{t}_k}^{-1}$ be the image measure on $\mathcal{B}(\mathbb{B}^{k+1})$ induced by $J_{\vec{t}_k}$. If $f:\mathbb{B}^{k+1}\to\mathbb{R}$ is nonnegative $\nu_{\vec{t}_k}$-a.e., then we can take a nondecreasing sequence $\{\phi_j\}_{j=1}^{\infty}$ of simple functions on $\mathbb{B}^{k+1}$ such that $\lim_{j\to\infty}\phi_j(\vec{x}_0)=f(\vec{x}_0)$ pointwise for $\nu_{\vec{t}_k}$-a.e $\vec{x}_0\in\mathbb{B}^{k+1}$. Then we have by the change of variables theorem and the monotone convergence theorem

$$
\begin{aligned}
&\int_{C^{\mathbb{B}}[a,b]} f(J_{\vec{t}_k}(x))\,dw_{a,b;\varphi_a}^{\rho,\beta,\mathbb{B}}(x)\\
\overset{*}{=}\ &\lim_{j\to\infty}\int_{C^{\mathbb{B}}[a,b]}\phi_j(J_{\vec{t}_k}(x))\,dw_{a,b;\varphi_a}^{\rho,\beta,\mathbb{B}}(x)\\
\overset{*}{=}\ &\lim_{j\to\infty}\int_{\mathbb{B}^{k+1}}(\phi_j\circ T_{\vec{t}_k,\beta})(x_0,x_1,\ldots,x_k)\,d\mu_\rho^k(x_1,\ldots,x_k)\,d\varphi_a^\rho(x_0)\\
\overset{*}{=}\ &\lim_{j\to\infty}\int_{\mathbb{B}^{k+1}}\phi_j(\vec{x}_0)\,d((\mu_\rho^k\times\varphi_a^\rho)\circ T_{\vec{t}_k,\beta}^{-1})(\vec{x}_0)\\
\overset{*}{=}\ &\int_{\mathbb{B}^{k+1}} f(\vec{x}_0)\,d((\mu_\rho^k\times\varphi_a^\rho)\circ T_{\vec{t}_k,\beta}^{-1})(\vec{x}_0)\\
\overset{*}{=}\ &\int_{\mathbb{B}^{k+1}}(f\circ T_{\vec{t}_k,\beta})(x_0,x_1,\ldots,x_k)\,d\mu_\rho^k(x_1,\ldots,x_k)\,d\varphi_a^\rho(x_0)
\end{aligned}
$$

since $w_{a,b;\varphi_a}^{\rho,\beta,\mathbb{B}}=m_{a,b;\varphi_a}^{\rho,\beta,\mathbb{B}}$ on $\mathcal{I}$ and $\nu_{\vec{t}_k}=(\varphi_a^\rho\times\mu_\rho^k)\circ T_{\vec{t}_k,\beta}^{-1}$ on $\mathcal{B}(\mathbb{B}^{k+1})$ by (1). If f is a complex-valued function, then f can be expressed by $f=f_1^+-f_1^-+i(f_2^+-f_2^-)$ for some real-valued Borel measurable functions f_1 and f_2 on $\mathbb{B}^{k+1}$, so that (4) follows immediately. Now, the proof is complete. □

Remark 2.3 We have by (4)

$$
\begin{aligned}
w_{a,b;\varphi_a}^{\rho,\beta,\mathbb{B}}(C^{\mathbb{B}}[a,b]) &= \int_{C^{\mathbb{B}}[a,b]}\chi_{\mathbb{B}}(x(a))\,dw_{a,b;\varphi_a}^{\rho,\beta,\mathbb{B}}(x)\\
&= \int_{\mathbb{B}}\chi_{\mathbb{B}}(x_0)\,d\varphi_a^\rho(x_0)=\varphi_a(\mathbb{B}) \qquad (5)
\end{aligned}
$$

since $\varphi_a^\rho(\mathbb{B}) = \varphi_a(\rho^{-1}\mathbb{B}) = \varphi_a(\mathbb{B})$, so that $w_{a,b;\varphi_a}^{\rho,\beta,\mathbb{B}}$ is a probability on $\mathcal{B}(C^{\mathbb{B}}[a,b])$ if and only if $\varphi_a(\mathbb{B}) = 1$.

Throughout the remaining part of this paper, let $W_t(x) = x(t)$ for $(t,x) \in [a,b] \times C^{\mathbb{B}}[a,b]$. We note that if $\varphi_a(\mathbb{B}) = 1$, then $w_{a,b;\varphi_a}^{\rho,\beta,\mathbb{B}}$ is a probability measure by (5), so that $\{W_t : a \le t \le b\}$ is a vector-valued stochastic process.

Lemma 2.4 *For $t \in (a,b]$, let $X_t : C^{\mathbb{B}}[a,b] \to \mathbb{B}$ be the function defined by*

$$X_t(x) = W_t(x) - W_a(x)$$

for $x \in C^{\mathbb{B}}[a,b]$ and let m_{X_t} be the image measure on $(\mathbb{B}, \mathcal{B}(\mathbb{B}))$ induced by X_t. Then we have

$$m_{X_t} = \varphi_a(\mathbb{B})\mu_{\rho\sqrt{\beta(t)-\beta(a)}}.$$

Proof. Let B_1 be in $\mathcal{B}(\mathbb{B})$. Then we have by (4) and the change of variables theorem

$$\begin{aligned} m_{X_t}(B_1) &= \int_{\mathbb{B}} \chi_{B_1}(x_1) dm_{X_t}(x_1) \\ &= \int_{C^{\mathbb{B}}[a,b]} \chi_{B_1}(x(t) - x(a)) dw_{a,b;\varphi_a}^{\rho,\beta,\mathbb{B}}(x) \\ &= \int_{\mathbb{B}^2} \chi_{B_1}(\sqrt{\beta(t)-\beta(a)}x_1) d\mu_\rho(x_1) d\varphi_a^\rho(x_0) \\ &= \varphi_a(\mathbb{B})\mu_{\rho\sqrt{\beta(t)-\beta(a)}}(B_1) \end{aligned}$$

since $\varphi_a^\rho(\mathbb{B}) = \varphi_a(\rho^{-1}\mathbb{B}) = \varphi_a(\mathbb{B})$. This completes the proof. □

A set N in $\mathcal{B}(\mathbb{B})$ is called a scale-invariant null set if $\mu_\rho(N) = 0$ for any $\rho > 0$ [5]. A property is said to hold s-a.e. if it holds except for a scale-invariant null set in $\mathcal{B}(\mathbb{B})$. Similar definitions are understood on $C^{\mathbb{B}}[a,b]$ with $w_{a,b;\varphi_a}^{\rho,\beta,\mathbb{B}}$. Define a relation $\approx$ on the set of Borel measurable functions on $C^{\mathbb{B}}[a,b]$ by

$$F \approx G \text{ if and only if } F = G \ s\text{-a.e.}$$

for Borel measurable functions $F, G : C^{\mathbb{B}}[a,b] \to \mathbb{C}$. Then $\approx$ is an equivalence relation on $C^{\mathbb{B}}[a,b]$ and the equivalence classes by $\approx$ are called s-a.e. equivalence classes of the functions on $C^{\mathbb{B}}[a,b]$.

Let $\mathbb{C}_+$ and $\mathbb{C}_+^\sim$ denote the sets of complex numbers with positive real parts and nonnegative real parts, respectively. For $\lambda > 0$, suppose that $F(\lambda^{-\frac{1}{2}}\cdot)$ is integrable over $C^{\mathbb{B}}[a,b]$. Let

$$I_F^\lambda = \int_{C^{\mathbb{B}}[a,b]} F(\lambda^{-\frac{1}{2}}x) dw_{a,b;\varphi_a}^{\rho,\beta,\mathbb{B}}(x). \tag{6}$$

If I_F^λ has the analytic extension J_F^λ on $\mathbb{C}_+$, then we write $J_F^\lambda = GE^{anw_\lambda}[F]$ for $\lambda \in \mathbb{C}_+$ and call it the generalized L_1-analytic Wiener integral of F with parameter λ. For a non-zero real q, if the limit $\lim_{\lambda\to -iq} GE^{anw_\lambda}[F]$ exists, where λ approaches $-iq$ through $\mathbb{C}_+$, then we write $\lim_{\lambda\to -iq} GE^{anw_\lambda}[F] = GE^{anf_q}[F]$. We call $GE^{anf_q}[F]$ the generalized L_1-analytic Feynman integral of F with parameter q. For $\lambda \in \mathbb{C}$, $\lambda^{\frac{1}{2}}$ will be understood as the square root of λ with nonnegative real part.

3 Evaluation formulas and examples

In this section, we introduce a Banach algebra corresponding to the Fresnel class in [9] and evaluate the generalized L_1-analytic Feynman integrals of the cylinder functions and the functions in the established Banach algebra.

Let $\{e_j : j \in \mathbb{N}\}$ be a complete orthonormal subset of $\mathcal{H}$ such that the e_js are in $\mathbb{B}^*$. For each $h \in \mathcal{H}$ and $y \in \mathbb{B}$, define the stochastic inner product $(h, y)^\sim$ by

$$(h,y)^\sim = \begin{cases} \lim_{n\to\infty} \sum_{j=1}^n (h,e_j)_1 (y,e_j)_2 & \text{if the limit exists;} \\ 0, & \text{otherwise.} \end{cases}$$

Suppose that $(h, y)_1^\sim$ is defined by the right-hand side of the above equality using another complete orthonormal subset of $\mathcal{H}$. By [9, Lemma 2.1 and Remark 1], we have $(h, y)^\sim = (h, y)_1^\sim$ for μ-a.e. $y \in \mathbb{B}$. Hence $(h, y)^\sim$ is essentially independent of the choice of the complete orthonormal set used in its definition. Further, we note that for each h in $\mathcal{H}$, $(h, \cdot)^\sim$ is a (degenerate) Gaussian random variable on $(\mathbb{B}, \mathcal{B}(\mathbb{B}), \mu)$ with mean zero and variance $|h|_1^2$, and $(h, \lambda y)^\sim = (\lambda h, y)^\sim = \lambda(h, y)^\sim$ for all $\lambda \in \mathbb{R}$ (see [9, Lemma 2.1]). Moreover, if $\{h_j : j = 1, \ldots, r\}$ is an orthogonal subset $\mathcal{H}$, then the random variables $(h_j, \cdot)^\sim$s are independent.

3.1 A Banach algebra

Let $a < t \le b$ be fixed, but arbitrary, and let $\mathcal{M}(\mathcal{H})$ be the class of all $\mathbb{C}$-valued Borel measures on $\mathcal{H}$ with bounded variation. Let $\mathcal{F}(C^{\mathbb{B}}[a, b]; t)$ be the space of all s-a.e. equivalence classes $[F_\sigma]$ of functions F_σ on $C^{\mathbb{B}}[a, b]$ which for $\sigma \in \mathcal{M}(\mathcal{H})$ have the form

$$F_\sigma(x) = \int_{\mathcal{H}} \exp\{i(h, W_t(x) - W_a(x))^\sim\} d\sigma(h) \tag{7}$$

for $x \in C^{\mathbb{B}}[a, b]$.

Theorem 3.1 *The space $\mathcal{F}(C^{\mathbb{B}}[a,b];t)$ is an algebra over $\mathbb{C}$ under pointwise addition, pointwise multiplication and complex scalar multiplication. Moreover, $\mathcal{F}(C^{\mathbb{B}}[a,b];t)$ is a Banach algebra with the norm $\|\cdot\|$ defined by*

$$\|F_\sigma\| = \|\sigma\|,$$

where F_σ and σ are related by (7) *and $\|\sigma\|$ denotes the total variation of the complex measure σ.*

Proof. It is easily verified that the operations of pointwise addition, pointwise multiplication, and scalar multiplication can be regarded as operations on the s-a.e. equivalence classes of $\mathcal{F}(C^{\mathbb{B}}[a,b];t)$. For σ_1, σ_2 in $\mathcal{M}(\mathcal{H})$, let F_{σ_1} and F_{σ_2} be the functions defined by (7), respectively. Then it is obvious that $F_{\sigma_1} + F_{\sigma_2} = F_{\sigma_1+\sigma_2}$ and $\lambda F_{\sigma_1} = F_{\lambda\sigma_1}$ for $\lambda \in \mathbb{C}$. Since the convolution $\sigma_1 * \sigma_2$ of σ_1 and σ_2 is defined by

$$(\sigma_1 * \sigma_2)(G) = (\sigma_1 \times \sigma_2)(\{(h_1,h_2) \in \mathcal{H}^2 : h_1 + h_2 \in G\})$$

for every Borel subset G of $\mathcal{H}$ (see (8.47) in [8, p.270]), we have by (7) and the Fubini theorem

$$\begin{aligned} F_{\sigma_1}(x)F_{\sigma_2}(x) &= \int_{\mathcal{H}^2} \exp\{i(h_1+h_2, W_t(x) - W_a(x))^\sim\} d\sigma_1(h_1) d\sigma_2(h_2) \\ &= \int_{\mathcal{H}} \exp\{i(h, W_t(x) - W_a(x))^\sim\} d(\sigma_1 * \sigma_2)(h) = F_{\sigma_1 * \sigma_2}(x) \end{aligned}$$

for $x \in C^{\mathbb{B}}[a,b]$, so that we have $F_{\sigma_1}F_{\sigma_2} = F_{\sigma_1 * \sigma_2}$. Hence

$$\sigma \longmapsto [F_\sigma] \tag{8}$$

defines an onto map from $\mathcal{M}(\mathcal{H})$ to $\mathcal{F}(C^{\mathbb{B}}[a,b];t)$ and it is an algebra homomorphism. To complete the proof, it remains only to show that (8) is one-to-one since $\mathcal{M}(\mathcal{H})$ is a Banach algebra. Now, suppose that $F_\sigma = 0$ s-a.e. with respect to $w^{\rho,\beta,\mathbb{B}}_{a,b;\varphi_a}$. We will show that $\sigma = 0$. Let $F_\sigma(x) = F_1(W_t(x) - W_a(x))$ for x in $C^{\mathbb{B}}[a,b]$, where

$$F_1(x_1) = \int_{\mathcal{H}} \exp\{i(h, x_1)^\sim\} d\sigma(h)$$

for $x_1 \in \mathbb{B}$. We note that if $F_1(x_1) = 0$ for s-a.e. $x_1 \in \mathbb{B}$ with respect to the measure μ, then we have $\sigma = 0$ by Proposition 2.1 of [9]. Thus, we must show that $F_1(x_1) = 0$ for s-a.e. x_1 in $\mathbb{B}$, which will be complete the proof. Now, let $B_1 = \{x_1 \in \mathbb{B} : F_1(x_1) \neq 0\}$ and $B_\sigma = \{x \in C^{\mathbb{B}}[a,b] : F_\sigma(x) \neq 0\}$. Since B_σ is a scale-invariant $w^{\rho,\beta,\mathbb{B}}_{a,b;\varphi_a}$-null set, we have for $\rho_1 > 0$

$$\begin{aligned} 0 &= \frac{1}{\varphi_a(\mathbb{B})} w^{\rho,\beta,\mathbb{B}}_{a,b;\varphi_a}(\rho[\beta(t) - \beta(a)]^{\frac{1}{2}} \rho_1^{-1} B_\sigma) \\ &= \frac{1}{\varphi_a(\mathbb{B})} \int_{C^{\mathbb{B}}[a,b]} \chi_{\rho[\beta(t)-\beta(a)]^{\frac{1}{2}}\rho_1^{-1}B_\sigma}(x) dw^{\rho,\beta,\mathbb{B}}_{a,b;\varphi_a}(x) \end{aligned}$$

$$= \frac{1}{\varphi_a(\mathbb{B})}\int_{C^{\mathbb{B}}[a,b]}\chi_{\rho[\beta(t)-\beta(a)]^{\frac{1}{2}}\rho_1^{-1}B_1}(W_t(x)-W_a(x))dw^{\rho,\beta,\mathbb{B}}_{a,b;\varphi_a}(x)$$
$$= \frac{1}{\varphi_a(\mathbb{B})}\int_{\mathbb{B}}\chi_{\rho[\beta(t)-\beta(a)]^{\frac{1}{2}}\rho_1^{-1}B_1}(x_1)dm_{X_t}(x_1)=\mu_{\rho_1}(B_1)$$

by Lemma 2.4. Now the proof is complete as desired. □

3.2 The $\mathbf{L_1}$-analytic Feynman integrals

Let $1 \le p \le \infty$, let r be a fixed positive integer, but arbitrary, and let $\{h_1, h_2, \dots, h_r\}$ be an orthonormal subset of $\mathcal{H}$. Let $\mathcal{A}_t^{(p)}$ be the space of all cylinder functions F_r having the form

$$F_r(x) = f_r((h_1, W_t(x)-W_a(x))^{\sim}, \dots, (h_r, W_t(x)-W_a(x))^{\sim}) \tag{9}$$

for $w^{\rho,\beta,\mathbb{B}}_{a,b;\varphi_a}$-a.e $x \in C^{\mathbb{B}}[a,b]$, where $f_r \in L_p(\mathbb{R}^r)$. Without loss of generality, we can take f_r to be Borel measurable. For $h \in \mathcal{H}$, let $c_j(h) = (h, h_j)_1$ for $j = 1, \dots, r$, let $\vec{c}_r(h) = (c_1(h), \dots, c_r(h))$, let $\langle\cdot,\cdot\rangle_r$ be the dot product with the norm $\|\cdot\|_r$ on $\mathbb{R}^r$ and let

$$\Psi_t(\lambda_1,\lambda_2,\vec{u}_r) = \left[\frac{\lambda_1}{2\pi\rho^2[\beta(t)-\beta(a)]}\right]^{\frac{r}{2}}\exp\left\{-\frac{\lambda_2}{2\rho^2[\beta(t)-\beta(a)]}\|\vec{u}_r\|_r^2\right\}$$

for $\vec{u}_r \in \mathbb{R}^r$ and for $\lambda_j \in \mathbb{C}$, $j = 1, 2$. Now, we have the following theorem.

Theorem 3.2 *Let $\Phi(x) = F_\sigma(x)F_r(x)$ for $w^{\rho,\beta,\mathbb{B}}_{a,b;\varphi_a}$-a.e $x \in C^{\mathbb{B}}[a,b]$, where F_σ and $F_r \in \mathcal{A}_t^{(p)}(1 \le p \le \infty)$ are defined as in (7) and (9), respectively. Then we have for $\lambda \in \mathbb{C}+$*

$$GE^{anw_\lambda}[\Phi] = \varphi_a(\mathbb{B})\int_{\mathcal{H}}\int_{\mathbb{R}^r} f_r(\vec{u}_r)\exp\left\{i\langle\vec{c}_r(h),\vec{u}_r\rangle_r - \frac{\rho^2}{2\lambda}[\beta(t)-\beta(a)][|h|_1^2 - \|\vec{c}_r(h)\|_r^2]\right\}\Psi_t(\lambda,\lambda,\vec{u}_r)d\vec{u}_r d\sigma(h).$$

In particular, if $F_r \in \mathcal{A}_t^{(1)}$, then for a nonzero real q, $GE^{anf_q}[\Phi]$ exists and is given by the right-hand side of the equality just stated replacing λ with $-iq$.

Proof. For $\lambda > 0$, let I_Φ^λ be given by the right-hand side of (6) replacing F with Φ. First, we will prove that I_Φ^λ is well-defined, that is, $\Phi(\lambda^{-\frac{1}{2}}\cdot)$ is integrable over $C^{\mathbb{B}}[a,b]$. Take $p' \in [1,\infty]$ with $\frac{1}{p}+\frac{1}{p'}=1$. Since F_σ is bounded and $w^{\rho,\beta,\mathbb{B}}_{a,b;\varphi_a}$ is a finite measure, we have $F_\sigma \in L_{p'}(C^{\mathbb{B}}[a,b], w^{\rho,\beta,\mathbb{B}}_{a,b;\varphi_a})$. We also have by Lemma 2.4 and the change of variables theorem

$$\int_{C^{\mathbb{B}}[a,b]}|F_r(\lambda^{-\frac{1}{2}}x)|^p dw^{\rho,\beta,\mathbb{B}}_{a,b;\varphi_a}(x)$$
$$= \int_{C^{\mathbb{B}}[a,b]}|f_r(\lambda^{-\frac{1}{2}}((h_1, W_t(x)-W_a(x))^{\sim}, \dots, (h_r, W_t(x)-W_a(x))^{\sim}))|^p$$

$$
\begin{aligned}
& dw^{\rho,\beta,\mathbb{B}}_{a,b;\varphi_a}(x) \\
= & \int_{\mathbb{B}} |f_r(\lambda^{-\frac{1}{2}}((h_1,x_1)^\sim,\dots,(h_r,x_1)^\sim))|^p dm_{X_t}(x_1) \\
= & \varphi_a(\mathbb{B}) \int_{\mathbb{B}} |f_r(\lambda^{-\frac{1}{2}}\rho\sqrt{\beta(t)-\beta(a)}((h_1,x_1)^\sim,\dots,(h_r,x_1)^\sim))|^p d\mu(x_1).
\end{aligned}
$$

Since $\{h_1,h_2,\dots,h_r\}$ is an orthonormal subset of $\mathcal{H}$ and each $(h_j,\cdot)^\sim$ is Gaussian, we have by the change of variables theorem

$$
\begin{aligned}
& \int_{C^{\mathbb{B}}[a,b]} |F_r(\lambda^{-\frac{1}{2}}x)|^p dw^{\rho,\beta,\mathbb{B}}_{a,b;\varphi_a}(x) \\
= & \varphi_a(\mathbb{B})\left(\frac{1}{2\pi}\right)^{\frac{r}{2}} \int_{\mathbb{R}^r} |f_r(\lambda^{-\frac{1}{2}}\rho\sqrt{\beta(t)-\beta(a)}\vec{u}_r)|^p \exp\left\{-\frac{1}{2}\|\vec{u}_r\|_r^2\right\} d\vec{u}_r \\
\le & \varphi_a(\mathbb{B})\left[\frac{\lambda}{2\pi\rho^2[\beta(t)-\beta(a)]}\right]^{\frac{r}{2}} \int_{\mathbb{R}^r} |f_r(\vec{u}_r)|^p d\vec{u}_r < \infty
\end{aligned}
$$

since $f_r \in L_p(\mathbb{R}^r)$. By Hölder's inequality, we have $I^\lambda_\Phi \in L_1(C^{\mathbb{B}}[a,b], w^{\rho,\beta,\mathbb{B}}_{a,b;\varphi_a})$ so that I^λ_Φ exists. Now, we have by Lemma 2.4 and the Fubini theorem

$$
\begin{aligned}
I^\lambda_\Phi = & \int_{C^{\mathbb{B}}[a,b]} F_r(\lambda^{-\frac{1}{2}}x)F_\sigma(\lambda^{-\frac{1}{2}}x) dw^{\rho,\beta,\mathbb{B}}_{a,b;\varphi_a}(x) \\
= & \int_{C^{\mathbb{B}}[a,b]} \int_{\mathcal{H}} f_r(\lambda^{-\frac{1}{2}}((h_1,W_t(x)-W_a(x))^\sim,\dots,(h_r,W_t(x)-W_a(x))^\sim)) \\
& \times \exp\{i\lambda^{-\frac{1}{2}}(h,W_t(x)-W_a(x))^\sim\} d\sigma(h) dw^{\rho,\beta,\mathbb{B}}_{a,b;\varphi_a}(x) \\
= & \int_{\mathcal{H}} \int_{\mathbb{B}} f_r(\lambda^{-\frac{1}{2}}((h_1,x_1)^\sim,\dots,(h_r,x_1)^\sim)) \exp\{i\lambda^{-\frac{1}{2}}(h,x_1)^\sim\} dm_{X_t}(x_1) \\
& d\sigma(h).
\end{aligned}
$$

For $h\in\mathcal{H}$, let $c_{r+1}(h) = \sqrt{|h|_1^2 - \|\vec{c}_r(h)\|_r^2}$ and let

$$
h_{r+1} = \begin{cases} \frac{1}{c_{r+1}(h)}[h - \sum_{j=1}^r c_j(h)h_j] & \text{if} \quad c_{r+1}(h)\neq 0 \\ 0 & \text{if} \quad c_{r+1}(h) = 0. \end{cases}
$$

By the Gram-Schmidt orthonormalization process, $\{h_1,\dots,h_r,h_{r+1}\}$ is an orthonormal subset of $\mathcal{H}$ if $c_{r+1}(h)\neq 0$ and $h = \sum_{j=1}^{r+1} c_j(h)h_j$ for all $h\in\mathcal{H}$. We have by the change of variables theorem

$$
\begin{aligned}
I^\lambda_\Phi = & \varphi_a(\mathbb{B}) \int_{\mathcal{H}} \int_{\mathbb{B}} f_r(\lambda^{-\frac{1}{2}}\rho\sqrt{\beta(t)-\beta(a)}((h_1,x_1)^\sim,\dots,(h_r,x_1)^\sim)) \\
& \times \exp\left\{i\lambda^{-\frac{1}{2}}\rho\sqrt{\beta(t)-\beta(a)} \sum_{j=1}^{r+1} c_j(h)(h_j,x_1)^\sim\right\} d\mu(x_1) d\sigma(h).
\end{aligned}
$$

Since $(h_{r+1},\cdot)^\sim$ is a (degenerate) mean zero Gaussian random variable with variance $|h_{r+1}|_1^2$ on $(\mathbb{B},\mathcal{B}(\mathbb{B}),\mu)$ by [9, Lemma 2.1], we have

$$\int_{\mathbb{B}} \exp\{i\alpha(h_{r+1},x_1)^\sim\}d\mu(x_1) = \exp\left\{-\frac{\alpha^2}{2}|h_{r+1}|_1^2\right\} \tag{10}$$

for $\alpha\in\mathbb{R}$, which is the characteristic functional of $(h_{r+1},\cdot)^\sim$. Using (10), we have

$$\begin{aligned} I_\Phi^\lambda &= \varphi_a(\mathbb{B})\left(\frac{1}{2\pi}\right)^{\frac{r}{2}}\int_{\mathcal{H}}\int_{\mathbb{R}^r} f_r(\lambda^{-\frac12}\rho\sqrt{\beta(t)-\beta(a)}\vec{u}_r)\exp\Big\{i\lambda^{-\frac12}\rho \\ &\quad \times\sqrt{\beta(t)-\beta(a)}\langle\vec{c}_r(h),\vec{u}_r\rangle_r - \frac{\rho^2}{2\lambda}[\beta(t)-\beta(a)][c_{r+1}(h)]^2 - \frac12\|\vec{u}_r\|_r^2\Big\} \\ &\quad d\vec{u}_r d\sigma(h). \end{aligned}$$

Letting $\vec{v}_r=\lambda^{-\frac12}\rho\sqrt{\beta(t)-\beta(a)}\vec{u}_r$, we have by the change of variables theorem

$$\begin{aligned} I_\Phi^\lambda &= \varphi_a(\mathbb{B})\int_{\mathcal{H}}\int_{\mathbb{R}^r} f_r(\vec{v}_r)\exp\left\{i\langle\vec{c}_r(h),\vec{v}_r\rangle_r - \frac{\rho^2}{2\lambda}[\beta(t)-\beta(a)][c_{r+1}(h)]^2\right\} \\ &\quad \times\Psi_t(\lambda,\lambda,\vec{v}_r)d\vec{v}_r d\sigma(h). \end{aligned}$$

For $\lambda\in\mathbb{C}_+^\sim-\{0\}$, let $G(\lambda)$ be given by the right-hand side of the last equality except for $\varphi_a(\mathbb{B})$ if it exists. We note that for $\lambda\in\mathbb{C}_+^\sim-\{0\}$

$$\left|\exp\left\{-\frac{\rho^2}{2\lambda}[\beta(t)-\beta(a)][c_{r+1}(h)]^2\right\}\right| = \exp\left\{-\frac{\rho^2\operatorname{Re}\lambda}{2|\lambda|^2}[\beta(t)-\beta(a)][c_{r+1}(h)]^2\right\}$$

which is dominated by 1, since $\beta(t)-\beta(a)\ge 0$ and $\operatorname{Re}\lambda\ge 0$. We will show that using Morera's theorem (see [6, p.169]), G is analytic on $\mathbb{C}_+$. Let $\lambda_0\in\mathbb{C}_+$ and take a sequence $\{\lambda_n\}_{n=1}^\infty$ in $\mathbb{C}_+$ such that $\lim_{n\to\infty}\lambda_n=\lambda_0$. Let $M_1=\sup\{|\lambda_j|:j=0,1,\ldots\}$ and $M_2=\inf\{\operatorname{Re}\lambda_j:j=0,1,\ldots\}$. Then we have $M_2>0$, so that for $j=0,1,\ldots$, we have by Hölder's inequality

$$\begin{aligned} &\int_{\mathcal{H}}\int_{\mathbb{R}^r}\bigg|f_r(\vec{v}_r)\exp\left\{i\langle\vec{c}_r(h),\vec{v}_r\rangle_r - \frac{\rho^2}{2\lambda_j}[\beta(t)-\beta(a)][c_{r+1}(h)]^2\right\} \\ &\quad \times\Psi_t(\lambda_j,\lambda_j,\vec{v}_r)\bigg|d\vec{v}_r d|\sigma|(h) \\ &\le |\sigma|(\mathcal{H})\int_{\mathbb{R}^r}|f_r(\vec{v}_r)\Psi_t(M_1,M_2,\vec{v}_r)|d\vec{v}_r \\ &\le |\sigma|(\mathcal{H})\|f_r\|_p\|\Psi_t(M_1,M_2,\cdot)\|_{p'}, \end{aligned}$$

where $\|\cdot\|_p$ and $\|\cdot\|_{p'}$ denote the norms on $L_p(\mathbb{R}^r)$ and $L_{p'}(\mathbb{R}^r)$, respectively. Now, G is defined on $\mathbb{C}_+$ since λ_0 is arbitrary. By the dominated convergence theorem, we have $\lim_{n\to\infty} G(\lambda_n) = G(\lambda_0)$ so that G is continuous on $\mathbb{C}_+$. Let Δ be a closed contour lying in $\mathbb{C}_+$, let $\lambda(s)(\alpha \le s \le \beta)$ be a parametrization of Δ and let L be the length of Δ. Let $M_3 = \sup\{|\lambda(s)| : s \in [\alpha,\beta]\}$ and $M_4 = \inf\{\operatorname{Re}\lambda(s) : s \in [\alpha,\beta]\}$. Then $M_4 > 0$ and we have by the ML-inequality (see (5) in [6, p.138])

$$\begin{aligned}
&\int_\alpha^\beta \int_{\mathcal{H}} \int_{\mathbb{R}^r} \bigg| f_r(\vec{v}_r) \exp\bigg\{ i\langle \vec{c}_r(h), \vec{v}_r\rangle_r - \frac{\rho^2}{2\lambda(s)}[\beta(t)-\beta(a)][c_{r+1}(h)]^2 \bigg\} \\
&\times \Psi_t(\lambda(s),\lambda(s),\vec{v}_r)\lambda'(s) \bigg| d\vec{v}_r d|\sigma|(h) ds \\
\le\ & L|\sigma|(\mathcal{H})| \int_{\mathbb{R}^r} |f_r(\vec{v}_r)\Psi_t(M_3,M_4,\vec{v}_r)| d\vec{v}_r \\
\le\ & L|\sigma|(\mathcal{H}) \|f_r\|_p \|\Psi_t(M_3,M_4,\cdot)\|_{p'}.
\end{aligned}$$

We have by the Fubini theorem and Cauchy-Goursat theorem (see [6, p.151])

$$\begin{aligned}
\int_\Delta G(\lambda) d\lambda &= \int_\alpha^\beta \bigg[\int_{\mathcal{H}} \int_{\mathbb{R}^r} f_r(\vec{v}_r) \exp\bigg\{ i\langle \vec{c}_r(h), \vec{v}_r\rangle_r - \frac{\rho^2}{2\lambda(s)}[\beta(t)-\beta(a)] \\
&\quad \times [c_{r+1}(h)]^2 \bigg\} \Psi_t(\lambda(s),\lambda(s),\vec{v}_r) d\vec{v}_r d\sigma(h) \bigg] \lambda'(s) ds \\
&= \int_{\mathcal{H}} \int_{\mathbb{R}^r} \int_\Delta f_r(\vec{v}_r) \exp\bigg\{ i\langle \vec{c}_r(h), \vec{v}_r\rangle_r - \frac{\rho^2}{2\lambda}[\beta(t)-\beta(a)] \\
&\quad \times [c_{r+1}(h)]^2 \bigg\} \Psi_t(\lambda,\lambda,\vec{v}_r) d\lambda d\vec{v}_r d\sigma(h) = 0
\end{aligned}$$

since the exponential functions in the last integral are analytic on $\mathbb{C}_+$. Now, by Morera's theorem, G is analytic on $\mathbb{C}_+$ and $GE^{anw_\lambda}[\Phi] = \varphi_a(\mathbb{B})G(\lambda)$ for $\lambda \in \mathbb{C}_+$ by the uniqueness of analytic extension, which completes the proof of the equality in this theorem. To prove the remaining part, suppose that $F_r \in \mathcal{A}_t^{(1)}$. In this case, it is obvious that $G(-iq)$ exists for any nonzero real q. Take a sequence $\{\lambda_n\}_{n=1}^\infty$ in $\mathbb{C}_+$ such that $\lim_{n\to\infty}\lambda_n = -iq$. Let $M_5 = \sup\{|\lambda_j| : j = 1,2,\ldots\}$ and $M_6 = \max\{M_5, |q|\}$. Then we have $|\Psi_t(\lambda,\lambda,\vec{v}_r)| \le \Psi_t(M_6,0,\vec{0})$ for $\vec{v}_r \in \mathbb{R}^r$ and for $\lambda \in \{-iq\} \cup \{\lambda_j : j = 1,2,\ldots\}$. Hence we have for $\lambda \in \{-iq\} \cup \{\lambda_j : j = 1,2,\ldots\}$

$$\begin{aligned}
&\int_{\mathcal{H}} \int_{\mathbb{R}^r} \bigg| f_r(\vec{v}_r) \exp\bigg\{ i\langle \vec{c}_r(h), \vec{v}_r\rangle_r - \frac{\rho^2}{2\lambda}[\beta(t)-\beta(a)][c_{r+1}(h)]^2 \bigg\} \Psi_t(\lambda,\lambda,\vec{v}_r) \bigg| \\
&d\vec{v}_r d|\sigma|(h) \\
\le\ & |\sigma|(\mathcal{H}) \Psi_t(M_6,0,\vec{0}) \int_{\mathbb{R}^r} |f_r(\vec{v}_r)| d\vec{v}_r \\
=\ & |\sigma|(\mathcal{H}) \Psi_t(M_6,0,\vec{0}) \|f_r\|_1.
\end{aligned}$$

By the dominated convergence theorem, we have

$$GE^{anf_q}[\Phi] = \lim_{n\to\infty} GE^{anw_{\lambda_n}}[\Phi] = \varphi_a(\mathbb{B}) \lim_{n\to\infty} G(\lambda_n) = \varphi_a(\mathbb{B})G(-iq).$$

Now, the proof is complete. □

Let $\mathcal{M}(\mathbb{R}^r)$ be the space of all complex Borel measures on $\mathbb{R}^r$. For $\nu \in \mathcal{M}(\mathbb{R}^r)$, let $\hat{\nu}$ be the Fourier transform of ν given by

$$\hat{\nu}(\vec{u}_r) = \int_{\mathbb{R}^r} \exp\{i\langle \vec{u}_r, \vec{v}_r\rangle_r\} d\nu(\vec{v}_r)$$

for $\vec{u}_r \in \mathbb{R}^r$. For $w^{\rho,\beta,\mathbb{B}}_{a,b;\varphi_a}$-a.e $x \in C^{\mathbb{B}}[a,b]$, let G_r be defined by

$$G_r(x) = \hat{\nu}((h_1, W_t(x) - W_a(x))^{\sim}, \ldots, (h_r, W_t(x) - W_a(x))^{\sim}). \tag{11}$$

Then $G_r \in \mathcal{A}_t^{(\infty)}$ and we have the following corollary.

Corollary 3.3 *Let $\Lambda(x) = F_\sigma(x)G_r(x)$ for $w^{\rho,\beta,\mathbb{B}}_{a,b;\varphi_a}$-a.e $x \in C^{\mathbb{B}}[a,b]$, where F_σ and G_r are defined as in* (7) *and* (11)*, respectively. Then we have for $\lambda \in \mathbb{C}+$*

$$\begin{aligned} GE^{anw_\lambda}[\Lambda] &= \varphi_a(\mathbb{B}) \int_{\mathcal{H}} \int_{\mathbb{R}^r} \exp\bigg\{ -\frac{\rho^2}{2\lambda}[\beta(t)-\beta(a)][|h|_1^2 + \|\vec{v}_r\|_r^2 \\ &\quad + 2\langle \vec{c}_r(h), \vec{v}_r\rangle_r] \bigg\} d\nu(\vec{v}_r) d\sigma(h) \end{aligned}$$

and for a nonzero real q, $GE^{anf_q}[\Lambda]$ is given by the right-hand side of the equality just stated replacing λ with $-iq$.

Proof. Since $G_r \in \mathcal{A}_t^{(\infty)}$, for $\lambda > 0$, we have by Theorem 3.2 and the Fubini theorem

$$\begin{aligned} I_\Lambda^\lambda &= \varphi_a(\mathbb{B}) \int_{\mathcal{H}} \int_{\mathbb{R}^r} \hat{\nu}(\vec{u}_r) \exp\bigg\{ i\langle \vec{c}_r(h), \vec{u}_r\rangle_r - \frac{\rho^2}{2\lambda}[\beta(t)-\beta(a)][|h|_1^2 \\ &\quad - \|\vec{c}_r(h)\|_r^2] \bigg\} \Psi_t(\lambda,\lambda,\vec{u}_r) d\vec{u}_r d\sigma(h) \\ &= \varphi_a(\mathbb{B}) \int_{\mathcal{H}} \int_{\mathbb{R}^r} \int_{\mathbb{R}^r} \exp\bigg\{ i\langle \vec{c}_r(h) + \vec{v}_r, \vec{u}_r\rangle_r - \frac{\rho^2}{2\lambda}[\beta(t)-\beta(a)][|h|_1^2 \\ &\quad - \|\vec{c}_r(h)\|_r^2] \bigg\} \Psi_t(\lambda,\lambda,\vec{u}_r) d\vec{u}_r d\nu(\vec{v}_r) d\sigma(h) \\ &= \varphi_a(\mathbb{B}) \int_{\mathcal{H}} \int_{\mathbb{R}^r} \exp\bigg\{ -\frac{\rho^2}{2\lambda}[\beta(t)-\beta(a)][|h|_1^2 + \|\vec{c}_r(h) + \vec{v}_r\|_r^2 - \|\vec{c}_r(h)\|_r^2] \bigg\} \\ &\quad d\nu(\vec{v}_r) d\sigma(h) \\ &= \varphi_a(\mathbb{B}) \int_{\mathcal{H}} \int_{\mathbb{R}^r} \exp\bigg\{ -\frac{\rho^2}{2\lambda}[\beta(t)-\beta(a)][|h|_1^2 + \|\vec{v}_r\|_r^2 + 2\langle \vec{c}_r(h), \vec{v}_r\rangle_r] \bigg\} \\ &\quad d\nu(\vec{v}_r) d\sigma(h). \end{aligned}$$

Now, by an analytic continuation and the dominated convergence theorem, we have the corollary. □

3.3 Examples

In this subsection, we will evaluate the L_1-analytic Feynman integrals of various functions defined on $C^{\mathbb{B}}[a,b]$. To do this, let q be any nonzero real number.

Example 3.4 (Fourier transform of the normal distribution) Let F_σ be defined as in (7) and let ν be the multivariate standard normal distribution on $\mathbb{R}^r$. Then $\hat{\nu}$ is the characteristic functional of the distribution so that $\hat{\nu}(\vec{u}_r)=\exp\{-\frac{1}{2}\|\vec{u}_r\|_r^2\}$ for $\vec{u}_r\in\mathbb{R}^r$. Hence for $\lambda>0$, we have by Theorem 3.2

$$\begin{aligned}
I^\lambda_{\hat{\nu}F_\sigma} &= \varphi_a(\mathbb{B})\int_{\mathcal{H}}\int_{\mathbb{R}^r}\exp\bigg\{i\langle\vec{c}_r(h),\vec{u}_r\rangle_r-\frac{\rho^2}{2\lambda}[\beta(t)-\beta(a)][|h|_1^2-\|\vec{c}_r(h)\|_r^2] \\
&\quad -\frac{1}{2}\|\vec{u}_r\|_r^2\bigg\}\Psi_t(\lambda,\lambda,\vec{u}_r)d\vec{u}_r d\sigma(h) \\
&= \varphi_a(\mathbb{B})\bigg[\frac{\lambda}{2\pi\rho^2[\beta(t)-\beta(a)]}\bigg]^{\frac{r}{2}}\int_{\mathcal{H}}\int_{\mathbb{R}^r}\exp\bigg\{i\langle\vec{c}_r(h),\vec{u}_r\rangle_r-\frac{\rho^2}{2\lambda}[\beta(t)- \\
&\quad \beta(a)][|h|_1^2-\|\vec{c}_r(h)\|_r^2]-\frac{1}{2}\|\vec{u}_r\|_r^2-\frac{\lambda}{2\rho^2[\beta(t)-\beta(a)]}\|\vec{u}_r\|_r^2\bigg\}d\vec{u}_r d\sigma(h).
\end{aligned}$$

We note that

$$\begin{aligned}
&\bigg[\frac{\lambda}{2\pi\rho^2[\beta(t)-\beta(a)]}\bigg]^{\frac{r}{2}}\int_{\mathbb{R}^r}\exp\bigg\{i\langle\vec{c}_r(h),\vec{u}_r\rangle_r-\frac{\lambda}{2\rho^2[\beta(t)-\beta(a)]}\|\vec{u}_r\|_r^2 \\
&\quad -\frac{1}{2}\|\vec{u}_r\|_r^2\bigg\}d\vec{u}_r \\
&= \bigg[\frac{\lambda}{\lambda+\rho^2[\beta(t)-\beta(a)]}\bigg]^{\frac{r}{2}}\bigg[\frac{\lambda+\rho^2[\beta(t)-\beta(a)]}{2\pi\rho^2[\beta(t)-\beta(a)]}\bigg]^{\frac{r}{2}}\int_{\mathbb{R}^r}\exp\bigg\{i\langle\vec{c}_r(h),\vec{u}_r\rangle_r \\
&\quad -\frac{\lambda+\rho^2[\beta(t)-\beta(a)]}{2\rho^2[\beta(t)-\beta(a)]}\|\vec{u}_r\|_r^2\bigg\}d\vec{u}_r \\
&= \bigg[\frac{\lambda}{\lambda+\rho^2[\beta(t)-\beta(a)]}\bigg]^{\frac{r}{2}}\exp\bigg\{-\frac{\rho^2[\beta(t)-\beta(a)]}{2[\lambda+\rho^2[\beta(t)-\beta(a)]]}\|\vec{c}_r(h)\|_r^2\bigg\}
\end{aligned}$$

so that we have

$$\begin{aligned}
I^\lambda_{\hat{\nu}F_\sigma} &= \varphi_a(\mathbb{B})\bigg[\frac{\lambda}{\lambda+\rho^2[\beta(t)-\beta(a)]}\bigg]^{\frac{r}{2}}\int_{\mathcal{H}}\exp\bigg\{-\frac{\rho^2}{2\lambda}[\beta(t)-\beta(a)]|h|_1^2 \\
&\quad -\frac{\rho^2}{2}[\beta(t)-\beta(a)]\|\vec{c}_r(h)\|_r^2\bigg[\frac{1}{\lambda+\rho^2[\beta(t)-\beta(a)]}-\frac{1}{\lambda}\bigg]\bigg\}d\sigma(h) \\
&= \varphi_a(\mathbb{B})\bigg[\frac{\lambda}{\lambda+\rho^2[\beta(t)-\beta(a)]}\bigg]^{\frac{r}{2}}\int_{\mathcal{H}}\exp\bigg\{-\frac{\rho^2}{2\lambda}[\beta(t)-\beta(a)]|h|_1^2
\end{aligned}$$

$$+\frac{[\rho^2[\beta(t)-\beta(a)]\|\vec{c}_r(h)\|_r]^2}{2\lambda[\lambda+\rho^2[\beta(t)-\beta(a)]]}\Big\}d\sigma(h).$$

Now, by an analytic continuation, $GE^{anw_\lambda}[\hat{\nu}F_\sigma]$ is given by the right-hand side of the last equality for $\lambda \in \mathbb{C}_+$. Moreover, by the dominated convergence theorem, $GE^{anf_q}[\hat{\nu}F_\sigma]$ is given by the same formula replacing λ with $-iq$.

Example 3.5 (Functions in the Banach algebra) Let $\delta_{\vec{0}}$ be the Dirac measure at $\vec{0} \in \mathbb{R}^r$. Letting $\hat{\nu} = \delta_{\vec{0}}$ in Corollary 3.3, we have for $\lambda \in \mathbb{C}_+$

$$GE^{anw_\lambda}[F_\sigma] = \varphi_a(\mathbb{B})\int_{\mathcal{H}} \exp\Big\{-\frac{\rho^2|h|_1^2}{2\lambda}[\beta(t)-\beta(a)]\Big\}d\sigma(h).$$

Moreover, $GE^{anf_q}[F_\sigma]$ is given by the same formula replacing λ with $-iq$.

Example 3.6 (Cylinder functions) Let δ_0 be the Dirac measure at $0 \in \mathcal{H}$. Letting $\sigma = \delta_0$ in Theorem 3.2, we have for $\lambda \in \mathbb{C}_+$

$$\begin{aligned} GE^{anw_\lambda}[F_r] &= \varphi_a(\mathbb{B})\int_{\mathbb{R}^r} f_r(\vec{u}_r)\Psi_t(\lambda,\lambda,\vec{u}_r)d\vec{u}_r \\ &= \varphi_a(\mathbb{B})(f_r * \Psi_t(\lambda,\lambda,\cdot))(\vec{0}). \end{aligned}$$

Hence we have the following:

1. If $f_r \in L_1(\mathbb{R}^r)$, then $GE^{anf_q}[F_r]$ is given by the formula just stated above replacing λ with $-iq$.

2. Letting $f_r \equiv c \in L_\infty(\mathbb{R}^r)$, where c is any constant, we have

$$GE^{anw_\lambda}[c] = GE^{anf_q}[c] = c\varphi_a(\mathbb{B}).$$

Example 3.7 (Fourier transforms) Let δ_0 be the Dirac measure at $0 \in \mathcal{H}$. Letting $\sigma = \delta_0$ in Corollary 3.3, we have for $\lambda \in \mathbb{C}_+$

$$GE^{anw_\lambda}[G_r] = \varphi_a(\mathbb{B})\int_{\mathbb{R}^r} \exp\Big\{-\frac{\rho^2}{2\lambda}[\beta(t)-\beta(a)]\|\vec{v}_r\|_r^2\Big\}d\nu(\vec{v}_r).$$

Moreover, $GE^{anf_q}[G_r]$ is given by the same formula replacing λ with $-iq$. In addition, if ν is the multivariate standard normal distribution on $\mathbb{R}^r$, then we have by Example 3.4

$$GE^{anw_\lambda}[G_r] = \varphi_a(\mathbb{B})\Big[\frac{\lambda}{\lambda+\rho^2[\beta(t)-\beta(a)]}\Big]^{\frac{r}{2}}.$$

Moreover, $GE^{anf_q}[G_r]$ is given by the formula just stated above replacing λ with $-iq$.

Remark 3.8 We note the following:

1. We note that the L_1-analytic Feynman integrals of the functions in this paper have the scale $\rho\sqrt{\beta(t)-\beta(a)}$ and the initial weight φ_a. Letting $\rho = 1$, $\varphi_a(\mathbb{B}) = 1$ and $\beta(s) = s$ for $s \in [a, b]$, we can obtain the analytic Feynman integrals in [1, 2, 3, 4, 9, 14]. In other words, this paper generalizes the related works in the references just stated.

2. Let $C_0^{\mathbb{B}}[a, b]$ be the space of $\mathbb{B}$-valued continuous functions x on $[a, b]$ with $x(a) = 0$. Suppose that $\varphi_a = \delta_0$ which is the Dirac measure concentrated at $0(\in \mathbb{B})$. Then we can show by Theorem 2.2 and (5) that

$$w_{a,b;\varphi_a}^{\rho,\beta,\mathbb{B}}(C_0^{\mathbb{B}}[a, b]) = w_{a,b;\varphi_a}^{\rho,\beta,\mathbb{B}}(C^{\mathbb{B}}[a, b]).$$

   Hence, if $h \in \mathbb{B}^*$, then we have $(W_a, h)_2 \equiv 0$ $w_{a,b;\varphi_a}^{\rho,\beta,\mathbb{B}}$-a.e. on $C^{\mathbb{B}}[a, b]$. If $h \neq 0$ and $a \le s_1 < s_2 \le b$, then one can show that $(W_{s_2} - W_{s_1}, h)_2$ is Gaussian on $C^{\mathbb{B}}[a, b]$ with mean 0 and variance $\rho^2|h|_1^2[\beta(s_2) - \beta(s_1)]$. Hence the stochastic process $\{(W_t, h)_2 : a \le t \le b\}$ for any $h \in \mathbb{B}^* - \{0\}$ is a generalized Brownian motion process (see [14] for its definition). We note that it is an independent process and a stationary process if β is linear. We also note that this process is a generalized Brownian motion process only for $\varphi_a = \delta_0$. This is due to the fact that φ_a can have a weight other than 0.

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